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平面刚架以单元截面特性为变量的解析解*

隋允康① 王文军①

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摘 要

取平面刚架单元截面面积与抗弯惯矩为变量,采用符号计算软件求解含变量的总刚度阵之逆,从而得到这种意义下的解析解• 在微机上编制了通用的程序,计算出了相应的算例•

关键词 平面刚架 符号计算 含参矩阵求逆 解析解
中图分类号 O342

§1. 引 言

文[1]以桁架结构为例,视杆截面积为变量,提出了基于有限元概念解析解的思想,给出了计算含参数总刚度阵逆阵的定理,并且推导了简单桁架的相应解析解• 文[2]也在这方面开展了研究• 我们引入符号计算软件[3]对桁架结构含参数总刚度阵求逆,在机上实现了杆面积为变量的解析解的计算,本文把这一思想应用于平面刚架单元截面特性为变量的解析解的计算•

§2. 平面刚架以单元截面特性为变量的解析问题的建立与求解

组成平面刚架的梁单元较组成桁架的杆单元,除了截面积 A_i 表达单元刚度,还多了一个抗弯惯矩 J_i,记 $\boldsymbol{v}_i = [A_i, J_i]^T$, $\boldsymbol{V} = [\boldsymbol{v}_1, \cdots, \boldsymbol{v}_m]^T$• 因梁单元刚度阵为 $\boldsymbol{v}_i$ 的张性函数,故平面刚架总刚度阵为 $\boldsymbol{V}$ 的线性函数• 此时总位移方程可表示为:

$$\boldsymbol{K}(\boldsymbol{V})\boldsymbol{u} = \boldsymbol{p} \tag{2.1}$$

这就是以截面特性为变量的解析问题• 若 $\boldsymbol{V}$ 取为数值,则式(2.1)退回为数值平面刚架表达式,因而称含参数 $\boldsymbol{V}$ 的式(2.1)为基于有限元概念的解析问题• 由于矩阵含参数,以往数值求解线性方程组的算法均失效,为此,提出两种解法•

一种是从线性代数方程组的 Cramer 定理出发

$$\boldsymbol{u} = \begin{bmatrix} |\boldsymbol{K}_{11}| & \cdots & |\boldsymbol{K}_{1n}| \\ \vdots & & \vdots \\ |\boldsymbol{K}_{n1}| & \cdots & |\boldsymbol{K}_{nn}| \end{bmatrix} \boldsymbol{p} / |\boldsymbol{K}| \tag{2.2}$$

其中 $|\boldsymbol{K}|$ 为 $\boldsymbol{K}$ 的行列式,$|\boldsymbol{K}_{ij}|$ 为 $\boldsymbol{K}$ 阵元素 k_{ij} 的代数余子式• 从单刚组装总刚的概念看,k_{ij} 可以看成每个单元对总刚贡献之和

* 国家自然科学基金资助课题
① 大连理工大学工程力学研究所,大连 116023

$$k_{ij} = \sum_{l=1}^{m} k_{ijl}^{0} v_l \tag{2.3}$$

由行列式定义

$$|\boldsymbol{K}| = \sum_{(P_1 \cdots P_n)} (-1)^{[P_1 \cdots P_n]} k_{1P_1} \cdots k_{nP_n} \tag{2.4}$$

将(2.3)代入(2.4)式得

$$|\boldsymbol{K}| = \sum_{(P_1 \cdots P_n)} (-1)^{[P_1 \cdots P_n]} \sum_{l_1=1}^{m} \cdots \sum_{l_n=1}^{m} k_{1P_1 l_1}^{0} \cdots k_{nP_n l_n}^{0} v_{l_1} \cdots v_{l_n} \tag{2.5}$$

其中 $P_1 \cdots P_n$ 是数码 1, …, n 的一个全排列, $[P_1 \cdots P_n]$ 为排列 $P_1 \cdots P_n$ 的逆序总数, $\sum_{(P_1 \cdots P_n)}$ 对 1, …, n 的一切排列求和•

由(2.5)式可知, $|\boldsymbol{K}|$ 为 $2m$ 个截面特性 $(A_1, J_1, \cdots, A_m, J_m)$ 的 n 次齐次函数, 同样可知, $|\boldsymbol{K}_{ij}|$ 为 $2m$ 个截面特性的 $n-1$ 次齐次函数• 由式(2.2), $\boldsymbol{u}$ 为截面特性的有理函数•

第二种解法是利用符号计算软件求出含参数的总刚度阵的逆阵, 本文就是按这种途径进行的• 在求逆前须先组装出含参数总刚, 这又有两种做法, 一种是直接用符号计算软件形成含参数总刚度阵, 一种是在数值刚架程序的基础上形成含参数总刚度阵• 本文按后一途径, 数值平面刚架的程序只将总刚度阵由 2 维增广为 3 维, 如图 1 所示•

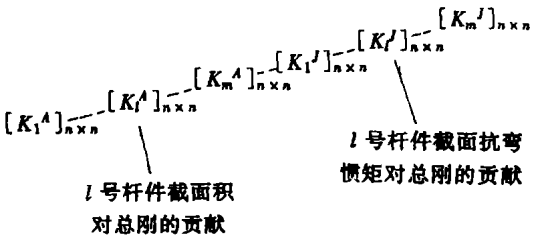

图 **1** 增维法扩展总刚度阵 $\boldsymbol{K}$

接着用符号计算软件组成含截面特性参数的 2 维总刚度阵, 按单元循环时, 将面积和抗弯惯矩对总刚度阵的贡献由 3 维数组调出来, 分别乘以 A_l 与 J_l, 累加进含参数的总刚度阵之中• 本文采用 Mathematica 软件完成了这一工作, 最后的求解是用其 Inverse 函数求出了含参数的逆阵•

在微机上编程时, 还进行了设计变量连接, 删除理论上为零而数值很小的系数和输出结果等工作• 程序的使用同数值平面刚架程序完全一样, 输入的 $\boldsymbol{v}_l^0$ 可以是 $[1.0 \quad 1.0]^T$ 也可以是具体的数值, 真实的截面性能为 $[A_l A_l^0 \quad J_l J_l^0]^T$, 亦即 $\boldsymbol{v}_l$ 为无量纲截面性能•

§ 3. 算 例

调通后的程序计算了很多算例, 限于篇幅只列举部分结果•

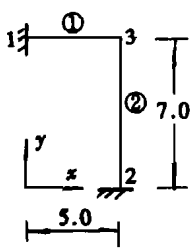

图 **2** 平面刚架算例 **1** 图示

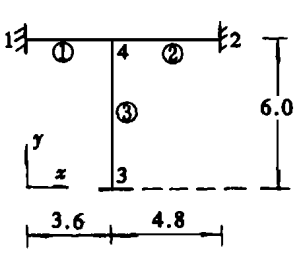

图 **3** 平面刚架算例 **2** 图示

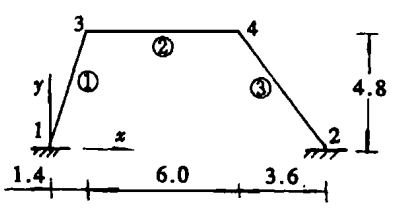

图 **4** 平面刚架算例 **3** 图示

例 1　2 单元 4 变量 3 自由度刚架•

例 2　3 单元 6 变量 3 自由度刚架•

各单元截面性能初始值见表 1•

表 1　　单元截面性能初始值

单元号 i	1	2	3
$E_i^0A_i^0$	1. 4238E4	1. 4238E4	2. 8476E4
$E_i^0J_i^0$	1. 7481E3	1. 7481E4	1. 3985E3

总刚度阵 $\boldsymbol{K}$ 的上三角元素为:

$k_{11}= 3955a_1+ 2966.25a_2+ 77.6944b_3,\qquad k_{12}= 0,\qquad k_{13}= -233.083b_3$

$k_{22}= 4746a_3+ 449.614b_1+ 1896.81b_2,\qquad k_{23}= 809.306b_1- 4552.34b_2$

$k_{33}= 1942.33b_1+ 14567.5b_2+ 932.333b_3$

求逆用 0. 16 秒, $\boldsymbol{K}^{-1}$的元素为 $g_{ij}/r(i= 1, \cdots, m; j= 1, \cdots, m)$:

$$g_{11}= 9.21831a_3b_1+ 0.218325b_1^2+ 69.1374a_3b_2+ 17.6025b_1b_2+ 6.90794b_2^2 + 4.42485a_3b_3+ 0.41919b_1b_3+ 1.76846b_2b_3$$

$$g_{12}= -0.188636b_1b_3+ 1.06108b_2b_3$$

$$g_{13}= 1.10621a_3b_3+ 0.104798b_1b_3+ 0.442115b_2b_3$$

$$g_{22}= 7.68193a_1b_1+ 5.76145a_2b_1+ 57.6145a_1b_2+ 43.2108a_2b_2+ 3.68738a_1b_3 + 2.76553a_2b_3+ 0.150908b_1b_3+ 1.13181b_2b_3+ 0.0181093b_3^2$$

$$g_{23}= -3.2008a_1b_1- 2.4006a_2b_1+ 18.0045a_1b_2+ 13.5034a_2b_2- 0.0628785b_1b_3 + 0.353692b_2b_3$$

$$g_{33}= 18.7704a_1a_3+ 14.0778a_2a_3+ 1.77822a_1b_1+ 1.33367a_2b_1+ 7.50188a_1b_2 + 5.62641a_2b_2+ 0.368738a_3b_3+ 0.0349325b_1b_3+ 0.147372b_2b_3$$

$$r= 36458.4a_1a_3b_1+ 27343.8a_2a_3b_1+ 863.475a_1b_1^2+ 647.606a_2b_1^2+ 273438a_1a_3b_2 + 205079a_2a_3b_2+ 69617.8a_1b_1b_2+ 52213.3a_2b_1b_2+ 27320.9a_1b_2^2+ 20490.7a_2b_2^2 + 17500.3a_1a_3b_3+ 13125.2a_2a_3b_3+ 1657.9a_1b_1b_3+ 1243.42a_2b_1b_3+ 716.212a_3b_1b_3 + 16.9626b_1^2b_3+ 6994.26a_1b_2b_3+ 5245.69a_2b_2b_3+ 5371.59a_3b_2b_3+ 1367.61b_1b_2b_3 + 536.709b_2^2b_3+ 85.9467a_3b_3^2+ 8.14219b_1b_3^2+ 34.3499b_2b_3^2$$

例 **3**　3 单元 6 变量 6 自由度刚架•

各单元截面性能初始值见表 2•

表 2　　单元截面性能初始值

单元号 i	1	2	3
$E_i^0A_i^0$	1. 4238E4	2. 8476E4	1. 4238E4
$E_i^0J_i^0$	1. 7481E3	1. 3985E4	1. 7481E3

总刚度阵 $\boldsymbol{K}$ 的上三角元素为:

$k_{11}=223.252a_1+4746a_2+154.661b_1,\quad k_{12}=765.435a_1-45.1094b_1,\quad k_{13}=-402.762b_1$

$k_{14}=-4746a_2,\quad k_{15}=k_{16}=k_{24}=k_{34}=0,\quad k_{22}=2624.35a_1+13.1569b_1+776.944b_2$

$k_{23}=117.472b_1-2330.83b_2,\quad k_{25}=-776.944b_2,\quad k_{26}=-k_{35}=-2330.83b_2$

$k_{33}=1398.48b_1+9323.33b_2,\quad k_{36}=4661.67b_2,\quad k_{44}=4746a_2+854.28a_3+62.1547b_3$

$k_{45}=-1139.04a_3+46.616b_3,\quad k_{46}=-233.08b_3,\quad k_{55}=1518.72a_3+776.944b_2+34.962b_3$

$k_{56}=2330.83b_2-174.81b_3,\quad k_{66}=9323.33b_2+1165.4b_3$

求逆用 11.37秒, $\boldsymbol{K}^{-1}$的元素为 $g_{ij}/r(i=1,\cdots,m;j=1,\cdots,m)$:

$$\begin{aligned}
g_{11}=&\,24663.5a_1a_2a_3b_1b_2+30.912a_2a_3b_1^2b_2+123319a_1a_2a_3b_2^2+3154.33a_1a_2b_1b_2^2\\
&+567.779a_1a_3b_1b_2^2+4283.69a_2a_3b_1b_2^2+3.95348a_2b_1^2b_2^2+0.711626a_3b_1^2b_2^2\\
&+3082.89a_1a_2a_3b_1b_3+3.86394a_2a_3b_1^2b_3+20552.9a_1a_2a_3b_2b_3+3564.34a_1a_2b_1b_2b_3\\
&+1782.17a_1a_3b_1b_2b_3+1475.74a_2a_3b_1b_2b_3+4.46736a_2b_1^2b_2b_3+2.23368a_3b_1^2b_2b_3\\
&+10199a_1a_2b_2^2b_3+6781.8a_1a_3b_2^2b_3+1521.19a_2a_3b_2^2b_3+41.3098a_1b_1b_2^2b_3\\
&+170.811a_2b_1b_2^2b_3+168.845a_3b_1b_2^2b_3+17.7426a_1a_2b_1b_3^2+24.6425a_1a_3b_1b_3^2\\
&+118.285a_1a_2b_2b_3^2+164.285a_1a_3b_2b_3^2+5.16365a_1b_1b_2b_3^2+8.49311a_2b_1b_2b_3^2\\
&+11.796a_3b_1b_2b_3^2+8.60621a_1b_2^2b_3^2+8.75468a_2b_2^2b_3^2+12.1593a_3b_2^2b_3^2\\
g_{12}=&-7193.52a_1a_2a_3b_1b_2+105.984a_2a_3b_1^2b_2-35968.1a_1a_2a_3b_2^2-920.013a_1a_2b_1b_2^2\\
&-165.602a_1a_3b_1b_2^2+7333.11a_2a_3b_1b_2^2+13.5548a_2b_1^2b_2^2+2.43986a_3b_1^2b_2^2\\
&-899.177a_1a_2a_3b_1b_3+13.2478a_2a_3b_1^2b_3-5994.6a_1a_2a_3b_2b_3-1039.6a_1a_2b_1b_2b_3\\
&-519.799a_1a_3b_1b_2b_3+2998.28a_2a_3b_1b_2b_3+15.3167a_2b_1^2b_2b_3+7.65834a_3b_1^2b_2b_3\\
&-2974.71a_1a_2b_2^2b_3-1978.03a_1a_3b_2^2b_3+3993.11a_2a_3b_2^2b_3-12.0487a_1b_1b_2^2b_3\\
&+200.634a_2b_1b_2^2b_3+247.681a_3b_1b_2^2b_3+0.177516b_1^2b_2^2b_3-5.17491a_1a_2b_1b_3^2\\
&-7.18738a_1a_3b_1b_3^2+0.0762433a_2b_1^2b_3^2+0.105894a_3b_1^2b_3^2-34.4999a_1a_2b_2b_3^2\\
&-47.9166a_1a_3b_2b_3^2-1.50606a_1b_1b_2b_3^2-2.19702a_2b_1b_2b_3^2+9.12714a_3b_1b_2b_3^2\\
&-2.51015a_1b_2^2b_3^2-11.6729a_2b_2^2b_3^2+0.14793b_1b_2^2b_3^2\\
g_{13}=&\,7707.34a_1a_2a_3b_1b_2-21409.6a_1a_2a_3b_2^2+985.728a_1a_2b_1b_2^2+177.431a_1a_3b_1b_2^2\\
&+686.723a_2a_3b_1b_2^2+963.404a_1a_2a_3b_1b_3-8563.71a_1a_2a_3b_2b_3+1113.86a_1a_2b_1b_2b_3\\
&+556.928a_1a_3b_1b_2b_3+186.382a_2a_3b_1b_2b_3+903.583a_1a_2b_2^2b_3-249.17a_1a_3b_2^2b_3\\
&+475.369a_2a_3b_2^2b_3+12.9093a_1b_1b_2^2b_3+37.5176a_2b_1b_2^2b_3+31.4933a_3b_1b_2^2b_3\\
&+5.54455a_1a_2b_1b_3^2+7.70077a_1a_3b_1b_3^2+30.8035a_1a_2b_2b_3^2-11.9791a_1a_3b_2b_3^2\\
&+1.61364a_1b_1b_2b_3^2+2.80706a_2b_1b_2b_3^2+2.80706a_3b_1b_2b_3^2\\
g_{14}=&\,24663.5a_1a_2a_3b_1b_2+30.912a_2a_3b_1^2b_2+123319a_1a_2a_3b_2^2+3154.33a_1a_2b_1b_2^2\\
&-1640.25a_1a_3b_1b_2^2+4283.69a_2a_3b_1b_2^2+3.95348a_2b_1^2b_2^2+3.25315a_3b_1^2b_2^2\\
&+3082.89a_1a_2a_3b_1b_3+3.86394a_2a_3b_1^2b_3+20552.9a_1a_2a_3b_2b_3+3564.34a_1a_2b_1b_2b_3\\
&-898.97a_1a_3b_1b_2b_3+1475.74a_2a_3b_1b_2b_3+4.46736a_2b_1^2b_2b_3+3.65974a_3b_1^2b_2b_3\\
&+10199a_1a_2b_2^2b_3-644.008a_1a_3b_2^2b_3+1521.19a_2a_3b_2^2b_3+115.538a_1b_1b_2^2b_3\\
&+170.811a_2b_1b_2^2b_3+78.2949a_3b_1b_2^2b_3-0.133137b_1^2b_2^2b_3+17.7426a_1a_2b_1b_3^2\\
&+118.285a_1a_2b_2b_3^2+4.76023a_1b_1b_2b_3^2+8.49311a_2b_1b_2b_3^2+1.88261a_1b_2^2b_3^2
\end{aligned}$$

$$+\ 8.75468a_2b_2^2b_3^2$$

$$g_{15}=18497.6a_1a_2a_3b_1b_2+23.184a_2a_3b_1^2b_2+92489.4a_1a_2a_3b_2^2-6834.37a_1a_2b_1b_2^2$$
$$-1230.19a_1a_3b_1b_2^2+3212.77a_2a_3b_1b_2^2+13.5548a_2b_1^2b_2^2+2.43986a_3b_1^2b_2^2$$
$$+2312.17a_1a_2a_3b_1b_3+2.89796a_2a_3b_1^2b_3+15414.7a_1a_2a_3b_2b_3-4292.45a_1a_2b_1b_2b_3$$
$$-674.228a_1a_3b_1b_2b_3+1106.8a_2a_3b_1b_2b_3+7.69222a_2b_1^2b_2b_3+2.7448a_3b_1^2b_2b_3$$
$$-8396.2a_1a_2b_2^2b_3-483.006a_1a_3b_2^2b_3+1140.89a_2a_3b_2^2b_3-89.5045a_1b_1b_2^2b_3$$
$$-24.4721a_2b_1b_2^2b_3+58.7211a_3b_1b_2^2b_3+0.177516b_1^2b_2^2b_3-23.6568a_1a_2b_1b_3^2$$
$$-157.714a_1a_2b_2b_3^2-6.34698a_1b_1b_2b_3^2-11.3242a_2b_1b_2b_3^2-2.51014a_1b_2^2b_3^2$$
$$-11.6729a_2b_2^2b_3^2+0.14793b_1b_2^2b_3^2$$

$$g_{16}=-10276.5a_1a_2a_3b_1b_2+20.7a_2a_3b_1^2b_2-21409.6a_1a_2a_3b_2^2+985.726a_1a_2b_1b_2^2$$
$$+177.431a_1a_3b_1b_2^2+686.723a_2a_3b_1b_2^2+963.404a_1a_2a_3b_1b_3+1.20748a_2a_3b_1^2b_3$$
$$+6422.78a_1a_2a_3b_2b_3+83.7857a_1a_2b_1b_2b_3-326.107a_1a_3b_1b_2b_3$$
$$+461.168a_2a_3b_1b_2b_3+2.2591a_2b_1^2b_2b_3+1.35546a_3b_1^2b_2b_3+903.584a_1a_2b_2^2b_3$$
$$-249.17a_1a_3b_2^2b_3+475.37a_2a_3b_2^2b_3+12.9093a_1b_1b_2^2b_3+37.5176a_2b_1b_2^2b_3$$
$$+31.4933a_3b_1b_2^2b_3$$

$$g_{22}=2098.11a_1a_2a_3b_1b_2+363.373a_2a_3b_1^2b_2+10490.7a_1a_2a_3b_2^2+268.337a_1a_2b_1b_2^2$$
$$+48.3007a_1a_3b_1b_2^2+13025.9a_2a_3b_1b_2^2+46.4736a_2b_1^2b_2^2+8.36524a_3b_1^2b_2^2$$
$$+262.26a_1a_2a_3b_1b_3+45.421a_2a_3b_1^2b_3+1748.42a_1a_2a_3b_2b_3+303.216a_1a_2b_1b_2b_3$$
$$+151.608a_1a_3b_1b_2b_3+6602.77a_2a_3b_1b_2b_3+52.5143a_2b_1^2b_2b_3+26.2571a_3b_1^2b_2b_3$$
$$+867.622a_1a_2b_2^2b_3+576.924a_1a_3b_2^2b_3+12264.5a_2a_3b_2^2b_3+3.5142a_1b_1b_2^2b_3$$
$$+320.758a_2b_1b_2^2b_3+399.672a_3b_1b_2^2b_3+0.608628b_1^2b_2^2b_3+1.50935a_1a_2b_1b_3^2$$
$$+2.09632a_1a_3b_1b_3^2+44.5646a_2a_3b_1b_3^2+0.261406a_2b_1^2b_3^2+0.363064a_3b_1^2b_3^2$$
$$+10.0625a_1a_2b_2b_3^2+13.9757a_1a_3b_2b_3^2+297.102a_2a_3b_2b_3^2+0.439268a_1b_1b_2b_3^2$$
$$+4.20682a_2b_1b_2b_3^2+9.68183a_3b_1b_2b_3^2+0.0760773b_1^2b_2b_3^2+0.732126a_1b_2^2b_3^2$$
$$+15.5639a_2b_2^2b_3^2+0.50719b_1b_2^2b_3^2$$

$$g_{23}=-2247.97a_1a_2a_3b_1b_2+6244.46a_1a_2a_3b_2^2-287.504a_1a_2b_1b_2^2-51.7507a_1a_3b_1b_2^2$$
$$+1254.34a_2a_3b_1b_2^2-280.993a_1a_2a_3b_1b_3+2497.75a_1a_2a_3b_2b_3-324.874a_1a_2b_1b_2b_3$$
$$-162.437a_1a_3b_1b_2b_3+354.213a_2a_3b_1b_2b_3-263.545a_1a_2b_2^2b_3+72.6745a_1a_3b_2^2b_3$$
$$+1544.95a_2a_3b_2^2b_3-3.76522a_1b_1b_2^2b_3+48.5891a_2b_1b_2^2b_3+50.3462a_3b_1b_2^2b_3$$
$$-1.61716a_1a_2b_1b_3^2-2.24606a_1a_3b_1b_3^2-3.74343a_2a_3b_1b_3^2-8.98436a_1a_2b_2b_3^2$$
$$+3.49392a_1a_3b_2b_3^2+74.2754a_2a_3b_2b_3^2-0.470645a_1b_1b_2b_3^2-0.380999a_2b_1b_2b_3^2$$
$$+2.42046a_3b_1b_2b_3^2$$

$$g_{24}=-7193.52a_1a_2a_3b_1b_2+105.984a_2a_3b_1^2b_2-35968.1a_1a_2a_3b_2^2-920.013a_1a_2b_1b_2^2$$
$$+478.406a_1a_3b_1b_2^2+7333.11a_2a_3b_1b_2^2+13.5548a_2b_1^2b_2^2+11.1537a_3b_1^2b_2^2$$
$$-899.177a_1a_2a_3b_1b_3+13.2478a_2a_3b_1^2b_3-5994.6a_1a_2a_3b_2b_3-1039.6a_1a_2b_1b_2b_3$$
$$+262.2a_1a_3b_1b_2b_3+2998.28a_2a_3b_1b_2b_3+15.3167a_2b_1^2b_2b_3+12.5477a_3b_1^2b_2b_3$$
$$-2974.71a_1a_2b_2^2b_3+187.836a_1a_3b_2^2b_3+3993.11a_2a_3b_2^2b_3-33.6986a_1b_1b_2^2b_3$$

$$
\begin{aligned}
&+200.634a_2b_1b_2^2b_3+130.126a_3b_1b_2^2b_3-0.456471b_1^2b_2^2b_3-5.17491a_1a_2b_1b_3^2\\
&+0.0762433a_2b_1^2b_3^2-34.4999a_1a_2b_2b_3^2-1.3884a_1b_1b_2b_3^2-2.19702a_2b_1b_2b_3^2\\
&-0.057058b_1^2b_2b_3^2-0.549094a_1b_2^2b_3^2-11.6729a_2b_2^2b_3^2-0.380392b_1b_2^2b_3^2\\
g_{25}=&-5395.14a_1a_2a_3b_1b_2+79.4879a_2a_3b_1^2b_2-26976.1a_1a_2a_3b_2^2+1993.36a_1a_2b_1b_2^2\\
&+358.805a_1a_3b_1b_2^2+5499.83a_2a_3b_1b_2^2+46.4735a_2b_1^2b_2^2+8.36524a_3b_1^2b_2^2\\
&-674.382a_1a_2a_3b_1b_3+9.93585a_2a_3b_1^2b_3-4495.95a_1a_2a_3b_2b_3+1251.96a_1a_2b_1b_2b_3\\
&+196.65a_1a_3b_1b_2b_3+2248.71a_2a_3b_1b_2b_3+26.3733a_2b_1^2b_2b_3+9.41075a_3b_1^2b_2b_3\\
&+2448.89a_1a_2b_2^2b_3+140.877a_1a_3b_2^2b_3+2994.83a_2a_3b_2^2b_3+26.1055a_1b_1b_2^2b_3\\
&+29.2233a_2b_1b_2^2b_3+97.5943a_3b_1b_2^2b_3+0.608628b_1^2b_2^2b_3+6.89989a_1a_2b_1b_3^2\\
&-0.101658a_2b_1^2b_3^2+45.9999a_1a_2b_2b_3^2+1.8512a_1b_1b_2b_3^2+2.92936a_2b_1b_2b_3^2\\
&+0.0760773b_1^2b_2b_3^2+0.732125a_1b_2^2b_3^2+15.5639a_2b_2^2b_3^2+0.50719b_1b_2^2b_3^2\\
g_{26}=&2997.3a_1a_2a_3b_1b_2+70.9714a_2a_3b_1^2b_2+6244.46a_1a_2a_3b_2^2-287.503a_1a_2b_1b_2^2\\
&-51.7506a_1a_3b_1b_2^2+1254.34a_2a_3b_1b_2^2-280.993a_1a_2a_3b_1b_3+4.13994a_2a_3b_1^2b_3\\
&-1873.31a_1a_2a_3b_2b_3-24.4375a_1a_2b_1b_2b_3+95.1144a_1a_3b_1b_2b_3\\
&+1041.07a_2a_3b_1b_2b_3+7.74547a_2b_1^2b_2b_3+4.64728a_3b_1^2b_2b_3-263.545a_1a_2b_2^2b_3\\
&+72.6745a_1a_3b_2^2b_3+1544.95a_2a_3b_2^2b_3-3.76521a_1b_1b_2^2b_3+48.5892a_2b_1b_2^2b_3\\
&+50.3463a_3b_1b_2^2b_3\\
g_{33}=&3211.39a_1a_2a_3b_1b_2+3716.94a_1a_2a_3b_2^2+410.72a_1a_2b_1b_2^2+73.9296a_1a_3b_1b_2^2\\
&+123.216a_2a_3b_1b_2^2+401.418a_1a_2a_3b_1b_3+7879.8a_1a_2a_3b_2b_3+464.107a_1a_2b_1b_2b_3\\
&+232.053a_1a_3b_1b_2b_3+31.8988a_2a_3b_1b_2b_3+85.5667a_1a_2b_2^2b_3+9.31726a_1a_3b_2^2b_3\\
&+198.071a_2a_3b_2^2b_3+5.37888a_1b_1b_2^2b_3+8.9648a_2b_1b_2^2b_3+6.45465a_3b_1b_2^2b_3\\
&+83.6287a_1a_2a_3b_3^2+2.31023a_1a_2b_1b_3^2+3.20865a_1a_3b_1b_3^2+0.419264a_2a_3b_1b_3^2\\
&+10.6957a_1a_2b_2b_3^2+1.16464a_1a_3b_2b_3^2+24.7585a_2a_3b_2b_3^2+0.67235a_1b_1b_2b_3^2\\
&+1.12058a_2b_1b_2b_3^2+0.806819a_3b_1b_2b_3^2\\
g_{34}=&7707.34a_1a_2a_3b_1b_2-21409.6a_1a_2a_3b_2^2+985.728a_1a_2b_1b_2^2-660.437a_1a_3b_1b_2^2\\
&+686.723a_2a_3b_1b_2^2+963.404a_1a_2a_3b_1b_3-8563.71a_1a_2a_3b_2b_3+1113.86a_1a_2b_1b_2b_3\\
&-348.696a_1a_3b_1b_2b_3+186.382a_2a_3b_1b_2b_3+903.583a_1a_2b_2^2b_3+22.3614a_1a_3b_2^2b_3\\
&+475.369a_2a_3b_2^2b_3+47.1996a_1b_1b_2^2b_3+37.5176a_2b_1b_2^2b_3+15.4911a_3b_1b_2^2b_3\\
&+5.54455a_1a_2b_1b_3^2+30.8035a_1a_2b_2b_3^2+1.86577a_1b_1b_2b_3^2+2.80706a_2b_1b_2b_3^2\\
g_{35}=&5780.5a_1a_2a_3b_1b_2-16057.2a_1a_2a_3b_2^2-2751.82a_1a_2b_1b_2^2-495.328a_1a_3b_1b_2^2\\
&+515.043a_2a_3b_1b_2^2+722.553a_1a_2a_3b_1b_3-6422.78a_1a_2a_3b_2b_3-1541.61a_1a_2b_1b_2b_3\\
&-261.522a_1a_3b_1b_2b_3+139.786a_2a_3b_1b_2b_3-776.944a_1a_2b_2^2b_3+16.771a_1a_3b_2^2b_3\\
&+356.527a_2a_3b_2^2b_3-36.0385a_1b_1b_2^2b_3-5.19962a_2b_1b_2^2b_3+11.6184a_3b_1b_2^2b_3\\
&-7.39274a_1a_2b_1b_3^2-41.0714a_1a_2b_2b_3^2-2.48769a_1b_1b_2b_3^2-3.74274a_2b_1b_2b_3^2\\
g_{36}=&-3612.82a_1a_2a_3b_1b_2+3716.94a_1a_2a_3b_2^2+410.719a_1a_2b_1b_2^2+73.9295a_1a_3b_1b_2^2\\
&+123.216a_2a_3b_1b_2^2+301.064a_1a_2a_3b_1b_3-3010.68a_1a_2a_3b_2b_3-6.1607a_1a_2b_1b_2b_3\\
&-126.294a_1a_3b_1b_2b_3+49.0803a_2a_3b_1b_2b_3+85.5665a_1a_2b_2^2b_3+9.31724a_1a_3b_2^2b_3
\end{aligned}
$$

$$+\ 198.071a_2a_3b_2^2b_3+\ 5.37887a_1b_1b_2^2b_3+\ 8.96478a_2b_1b_2^2b_3+\ 6.45464a_3b_1b_2^2b_3$$

$$g_{44}=\ 24663.5a_1a_2a_3b_1b_2+\ 236.571a_1a_3b_1^2b_2+\ 30.912a_2a_3b_1^2b_2+\ 123319a_1a_2a_3b_2^2$$
$$+\ 3154.33a_1a_2b_1b_2^2+\ 5921.37a_1a_3b_1b_2^2+\ 4283.69a_2a_3b_1b_2^2+\ 30.2562a_1b_1^2b_2^2$$
$$+\ 3.95348a_2b_1^2b_2^2+\ 14.8715a_3b_1^2b_2^2+\ 3082.89a_1a_2a_3b_1b_3+\ 29.571a_1a_3b_1^2b_3$$
$$+\ 3.86394a_2a_3b_1^2b_3+\ 20552.9a_1a_2a_3b_2b_3+\ 3564.34a_1a_2b_1b_2b_3+\ 1107.5a_1a_3b_1b_2b_3$$
$$+\ 1475.74a_2a_3b_1b_2b_3+\ 34.189a_1b_1^2b_2b_3+\ 4.46736a_2b_1^2b_2b_3+\ 7.43565a_3b_1^2b_2b_3$$
$$+\ 10199a_1a_2b_2^2b_3+\ 71.5565a_1a_3b_2^2b_3+\ 1521.19a_2a_3b_2^2b_3+\ 439.884a_1b_1b_2^2b_3$$
$$+\ 170.811a_2b_1b_2^2b_3+\ 49.5718a_3b_1b_2^2b_3+\ 0.342353b_1^2b_2^2b_3+\ 17.7426a_1a_2b_1b_3^2$$
$$+\ 0.170186a_1b_1^2b_3^2+\ 118.285a_1a_2b_2b_3^2+\ 6.37387a_1b_1b_2b_3^2+\ 8.49311a_2b_1b_2b_3^2$$
$$+\ 0.41182a_1b_2^2b_3^2+\ 8.75468a_2b_2^2b_3^2+\ 0.285294b_1b_2^2b_3^2$$

$$g_{45}=\ 18497.6a_1a_2a_3b_1b_2+\ 177.428a_1a_3b_1^2b_2+\ 23.184a_2a_3b_1^2b_2+\ 92489.4a_1a_2a_3b_2^2$$
$$-\ 6834.37a_1a_2b_1b_2^2+\ 4441.03a_1a_3b_1b_2^2+\ 3212.77a_2a_3b_1b_2^2+\ 13.5548a_2b_1^2b_2^2$$
$$+\ 11.1537a_3b_1^2b_2^2+\ 2312.17a_1a_2a_3b_1b_3+\ 22.1782a_1a_3b_1^2b_3+\ 2.89796a_2a_3b_1^2b_3$$
$$+\ 15414.7a_1a_2a_3b_2b_3-\ 4292.45a_1a_2b_1b_2b_3+\ 830.627a_1a_3b_1b_2b_3$$
$$+\ 1106.8a_2a_3b_1b_2b_3-\ 16.3381a_1b_1^2b_2b_3+\ 7.69222a_2b_1^2b_2b_3+\ 5.57674a_3b_1^2b_2b_3$$
$$-\ 8396.2a_1a_2b_2^2b_3+\ 53.6674a_1a_3b_2^2b_3+\ 1140.89a_2a_3b_2^2b_3-\ 316.896a_1b_1b_2^2b_3$$
$$-\ 24.4721a_2b_1b_2^2b_3+\ 37.1788a_3b_1b_2^2b_3-\ 0.456471b_1^2b_2^2b_3-\ 23.6568a_1a_2b_1b_3^2$$
$$-\ 0.226915a_1b_1^2b_3^2-\ 157.714a_1a_2b_2b_3^2-\ 8.49849a_1b_1b_2b_3^2-\ 11.3242a_2b_1b_2b_3^2$$
$$-\ 0.549093a_1b_2^2b_3^2-\ 11.6729a_2b_2^2b_3^2-\ 0.380392b_1b_2^2b_3^2$$

$$g_{46}=-\ 10276.5a_1a_2a_3b_1b_2-\ 44.3571a_1a_3b_1^2b_2+\ 20.7a_2a_3b_1^2b_2-\ 21409.6a_1a_2a_3b_2^2$$
$$+\ 985.726a_1a_2b_1b_2^2-\ 660.437a_1a_3b_1b_2^2+\ 686.723a_2a_3b_1b_2^2+\ 963.404a_1a_2a_3b_1b_3$$
$$+\ 9.24093a_1a_3b_1^2b_3+\ 1.20748a_2a_3b_1^2b_3+\ 6422.78a_1a_2a_3b_2b_3+\ 83.7857a_1a_2b_1b_2b_3$$
$$+\ 346.095a_1a_3b_1b_2b_3+\ 461.168a_2a_3b_1b_2b_3+\ 4.84092a_1b_1^2b_2b_3+\ 2.2591a_2b_1^2b_2b_3$$
$$+\ 2.32364a_3b_1^2b_2b_3+\ 903.584a_1a_2b_2^2b_3+\ 22.3614a_1a_3b_2^2b_3+\ 475.37a_2a_3b_2^2b_3$$
$$+\ 47.1996a_1b_1b_2^2b_3+\ 37.5176a_2b_1b_2^2b_3+\ 15.4912a_3b_1b_2^2b_3$$

$$g_{55}=\ 13873.2a_1a_2a_3b_1b_2+\ 739.285a_1a_2b_1^2b_2+\ 133.071a_1a_3b_1^2b_2+\ 17.388a_2a_3b_1^2b_2$$
$$+\ 69367a_1a_2a_3b_2^2+\ 18504.3a_1a_2b_1b_2^2+\ 3330.77a_1a_3b_1b_2^2+\ 2409.58a_2a_3b_1b_2^2$$
$$+\ 46.4735a_2b_1^2b_2^2+\ 8.36524a_3b_1^2b_2^2+\ 1734.13a_1a_2a_3b_1b_3+\ 92.4093a_1a_2b_1^2b_3$$
$$+\ 16.6337a_1a_3b_1^2b_3+\ 2.17347a_2a_3b_1^2b_3+\ 11561a_1a_2a_3b_2b_3+\ 6205.17a_1a_2b_1b_2b_3$$
$$+\ 622.971a_1a_3b_1b_2b_3+\ 830.102a_2a_3b_1b_2b_3+\ 9.68184a_1b_1^2b_2b_3+\ 16.3688a_2b_1^2b_2b_3$$
$$+\ 4.18255a_3b_1^2b_2b_3+\ 7110.56a_1a_2b_2^2b_3+\ 40.2505a_1a_3b_2^2b_3+\ 855.667a_2a_3b_2^2b_3$$
$$+\ 242.336a_1b_1b_2^2b_3+\ 60.4207a_2b_1b_2^2b_3+\ 27.8841a_3b_1b_2^2b_3+\ 0.608628b_1^2b_2^2b_3$$
$$+\ 31.5424a_1a_2b_1b_3^2+\ 0.302553a_1b_1^2b_3^2+\ 210.285a_1a_2b_2b_3^2+\ 11.3313a_1b_1b_2b_3^2$$
$$+\ 15.0989a_2b_1b_2b_3^2+\ 0.0760773b_1^2b_2b_3^2+\ 0.732125a_1b_2^2b_3^2+\ 15.5639a_2b_2^2b_3^2$$
$$+\ 0.507189b_1b_2^2b_3^2$$

$$g_{56}=-\ 7707.34a_1a_2a_3b_1b_2-\ 184.821a_1a_2b_1^2b_2-\ 33.2678a_1a_3b_1^2b_2+\ 15.525a_2a_3b_1^2b_2$$
$$-\ 16057.2a_1a_2a_3b_2^2-\ 2751.82a_1a_2b_1b_2^2-\ 495.328a_1a_3b_1b_2^2+\ 515.042a_2a_3b_1b_2^2$$

$$+722.553a_1a_2a_3b_1b_3+13.8614a_1a_2b_1^2b_3+6.9307a_1a_3b_1^2b_3+0.905612a_2a_3b_1^2b_3$$
$$+4817.09a_1a_2a_3b_2b_3+52.5714a_1a_2b_1b_2b_3+259.571a_1a_3b_1b_2b_3$$
$$+345.876a_2a_3b_1b_2b_3-2.42046a_1b_1^2b_2b_3+3.71137a_2b_1^2b_2b_3+1.74273a_3b_1^2b_2b_3$$
$$-776.945a_1a_2b_2^2b_3+16.7711a_1a_3b_2^2b_3+356.528a_2a_3b_2^2b_3-36.0385a_1b_1b_2^2b_3$$
$$-5.19951a_2b_1b_2^2b_3+11.6184a_3b_1b_2^2b_3$$

$$g_{66}=120.426a_1a_2a_3b_1^2+9455.76a_1a_2a_3b_1b_2+61.6071a_1a_2b_1^2b_2+11.0893a_1a_3b_1^2b_2$$
$$+18.4821a_2a_3b_1^2b_2+3716.94a_1a_2a_3b_2^2+410.72a_1a_2b_1b_2^2+73.9296a_1a_3b_1b_2^2$$
$$+123.216a_2a_3b_1b_2^2+401.418a_1a_2a_3b_1b_3+2.77228a_1a_2b_1^2b_3+3.85039a_1a_3b_1^2b_3$$
$$+0.503118a_2a_3b_1^2b_3+2676.16a_1a_2a_3b_2b_3+32.8571a_1a_2b_1b_2b_3$$
$$+144.206a_1a_3b_1b_2b_3+192.153a_2a_3b_1b_2b_3+0.80682a_1b_1^2b_2b_3+1.3447a_2b_1^2b_2b_3$$
$$+0.968184a_3b_1^2b_2b_3+85.5667a_1a_2b_2^2b_3+9.31726a_1a_3b_2^2b_3+198.071a_2a_3b_2^2b_3$$
$$+5.37888a_1b_1b_2^2b_3+8.9648a_2b_1b_2^2b_3+6.45465a_3b_1b_2^2b_3$$

$$r=1.12277\times10^6a_1a_2a_3b_1^2b_2+4.63668\times10^7a_1a_2a_3b_1b_2^2+143596a_1a_2b_1^2b_2^2$$
$$+25847.3a_1a_3b_1^2b_2^2+43078.8a_2a_3b_1^2b_2^2+140344a_1a_2a_3b_1^2b_3$$
$$+2.22474\times10^7a_1a_2a_3b_1b_2b_3+162261a_1a_2b_1^2b_2b_3+81130.5a_1a_3b_1^2b_2b_3$$
$$+11152.5a_2a_3b_1^2b_2b_3+3.8639\times10^7a_1a_2a_3b_2^2b_3+1.18706\times10^6a_1a_2b_1b_2^2b_3$$
$$+1.46574\times10^6a_1a_3b_1b_2^2b_3+293432a_2a_3b_1b_2^2b_3+116953a_1a_2a_3b_1b_3^2$$
$$+779699a_1a_2a_3b_2b_3^2+9572.93a_1a_2b_1b_2b_3^2+42014.4a_1a_3b_1b_2b_3^2$$
$$+55983.8a_2a_3b_1b_2b_3^2+24929.9a_1a_2b_2^2b_3^2+57707.9a_2a_3b_2^2b_3^2$$

例 **4** 5 单元 6 变量自由度刚架•

变量连接情况为: 1 号、2 号单元独立; 而 3、4、5 单元为同一截面性能•

例 **5** 4 单元 8 变量 9 自由度刚架•

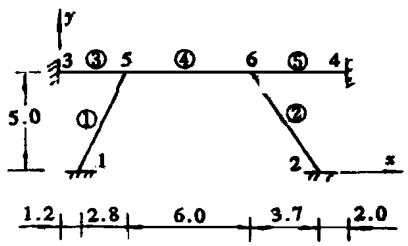

图 **5** 平面钢架算例 **4** 图示

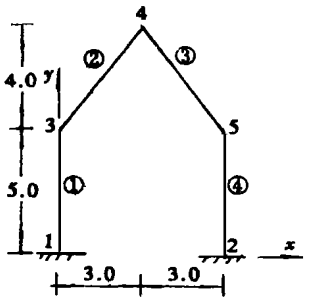

图 **6** 平面刚架算例 **5** 图示

例 **6** 5 单元 6 变量 9 自由度刚架•

变量连接情况为: 1 号、2 号单元独立; 而 3、4、5 单元为同一截面性能•

例 **7** 6 单元 6 变量 9 自由度刚架•

变量连接情况为: 1 号、2 号单元为同一截面性能; 3、6 号单元为同一截面性能; 4、5 号单元为同一截面性能•

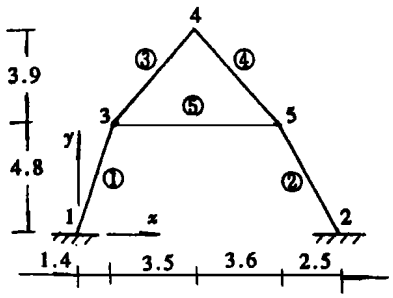

图 7 平面刚架算例 6 图示

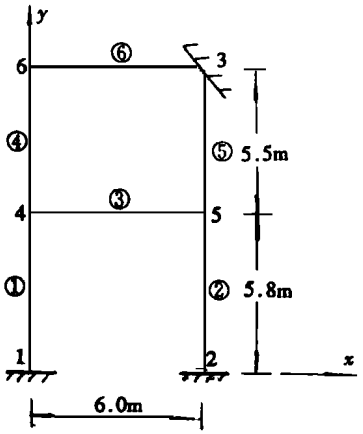

图 8 平面刚架算例 7 图示

§4. 结 语

工作表明, 基于有限元概念的解析解可以向刚架结构推广, 运用符号计算软件进行求解也是可行的• 这一工作建立了结构响应的精确显函数, 可以探讨在结构优化设计中用之取代结构重分析与敏度重分析, 有利于提高优化的效率• 还有进一步的工作待进行, 一是 3 维存贮总刚度阵的方式还可以改进, 使之更节省, 或直接用符号计算软件组装 2 维总刚度阵; 二是应探求效率更高的含参数矩阵求逆算法; 三是新解析解有待近似简化以便用于结构优化之中•

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The Analytical Solution with Respect to Characteristics of Elements' Cross Section as Variables of the Plane Frame

Sui Yunkang Wang Wenjun
(Research Institute of Engineering Mechanics, Dalian University of Technology, Dalian 116023, P.R.China)

Abstract

Taking the sectional area and the bending moment of inertia as variables for each beam element, the plane frame will possess a stiffness matrix containing parameters. In terms of the symbolic computation software, the inverse matrix is solved to obtain the new analytical solution with respect to characteristics of elements cross section. The general program is coded in the micro computer and corresponding examples are computed.

Key words plane frame, symbolic computation, inverse matrix containing parameters, analytical solution