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剪切变形对直线型正交异性层合圆板大幅度受迫振动的影响*

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摘 要

本文研究了计及横向剪切变形的直线型正交异性层合圆板在简谐载荷 $q_0\cos\omega t$ 作用下的非线性受迫振动问题• 采用伽辽金方法得到强振频率与振幅关系的解析解• 最后, 分析了横向剪切对板振动的影响, 并给出了板的非线性自由振动的非线性周期对线性周期的比值•

关键词 大幅度受迫振动 直线型正交异性 层合圆板 横向剪切

§ 1. 引 言

复合材料层合板壳是许多现代工程的重要结构元件, 对这类板壳非线性问题的分析已引起了极大的研究兴趣[1~11], 尤其在层合圆板的非线性研究方面已取得了一些成果[12~15]• 但是, 由于非线性数学的困难, 计及横向剪切影响的层合圆板的非线性振动问题还无人进行探讨• 然而, 横向剪切对复合材料层合板的影响比对各向同性板的影响要大得多• 因此在本文中, 我们研究了一个更复杂更重要的问题, 即计及横向剪切影响的对称直线型正交异性层合圆板在简谐载荷下的大幅度受迫振动问题, 应用伽辽金方法获得了解析解•

§ 2. 基 本 方 程

考虑图 1 所示对称直线型正交异性层合圆板, 半径为 a, 厚度为 h, 受简谐力 $q_0\cos\omega t$ 作用•这里, ω 是横向载荷的频率, q_0 是均布横向压力• 选取直角坐标系 xyz, 原点置于板的中心, x 和 y 轴分别平行于材料的对称轴•

通过在计及横向剪切影响的对称直线型正交异性层合圆板的大挠度基本方程[13] 中增加横向惯性项 $m\partial^2 w/\partial t^2$, 我们得到如下的层合圆板的运动方程

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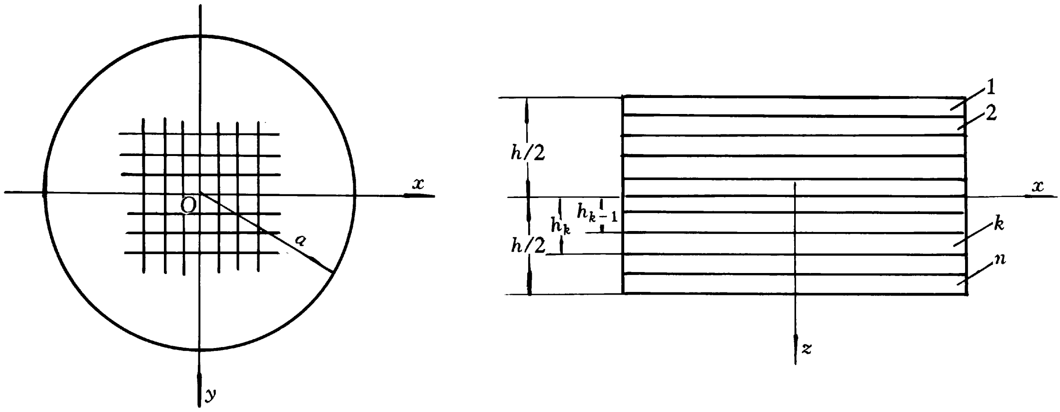

(a)　　　　　　　　　　(b)

图 **1**　对称直线型正交异性层合圆板

$$A_{11}\frac{\partial u_0}{\partial x^2} + A_{66}\frac{\partial^2 u_0}{\partial y^2} + (A_{12}+A_{66})\frac{\partial^2 v_0}{\partial x\partial y}$$

$$+\left(A_{11}\frac{\partial^2 w}{\partial x^2} + A_{66}\frac{\partial^2 w}{\partial y^2}\right)\frac{\partial w}{\partial x} + (A_{12}\text{非线}+A_{66})\frac{\partial^2 w}{\partial x\partial y}\frac{\partial w}{\partial y} = 0$$

$$(A_{12}+A_{66})\frac{\partial^2 u_0}{\partial x\partial y} + A_{66}\frac{\partial^2 v_0}{\partial x^2} + A_{22}\frac{\partial^2 v_0}{\partial y^2}$$

$$+\left(A_{66}\frac{\partial^2 w}{\partial x^2} + A_{22}\frac{\partial^2 w}{\partial y^2}\right)\frac{\partial w}{\partial y} + (A_{12}+A_{66})\frac{\partial^2 w}{\partial x\partial y}\frac{\partial w}{\partial x} = 0$$

$$D_{11}\frac{\partial^3 \phi_x}{\partial x^3} + (D_{12}\text{m}+2D_{66})\left(\frac{\partial^3 \phi_x^{\text{wn}}}{\partial x\partial y^2} + \frac{\partial^3 \phi_y}{\partial x^2\partial y}\right) + D\text{e}_{22}\frac{\partial^3 \phi_y}{\partial y^3} + \left(\frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right)$$

论 分

$$\cdot\left(A_{11}\frac{\partial^2 w}{\partial x^2} + A_{12}\frac{\partial^2 w}{\partial y^2}\text{板动}\right) + \left(\frac{\partial v_0}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right)$$

$$A_{44}=4h^2/9\sum_{k=1}^{n}\frac{1}{G_{44}^{(k)}}\left[h_k-h_{k-1}-\frac{8}{3h^2}(h_k^3-h_{k-1}^3)+\frac{16}{5h^4}(h_k^5-h_{k-1}^5)\right]$$

$$A_{55}=4h^2/9\sum_{k=1}^{n}\frac{1}{G_{55}^{(k)}}\left(h_k-h_{k-1}-\frac{8}{3h^2}(h_k^3-h_{k-1}^3)+\frac{16}{5h^4}(h_k^5-h_{k-1}^5)\right)$$

$$D_{ij}=\frac{1}{3}\sum_{k=1}^{n}Q_{ij}^{(k)}(h_k^3-h_{k-1}^3)\qquad(i,j=1,2,6)$$

$$Q_{11}^{(k)}=\frac{E_x^{(k)}}{1-\nu_{xy}^{(k)}\nu_{yx}^{(k)}},\qquad Q_{12}^{(k)}=\frac{\nu_{xy}^{(k)}E_y^{(k)}}{1-\nu_{xy}^{(k)}\nu_{yx}^{(k)}}$$

$$Q_{22}^{(k)}=\frac{E_y^{(k)}}{1-\nu_{xy}^{(k)}\nu_{yx}^{(k)}},\qquad Q_{44}^{(k)}=G_{yz}^{(k)}$$

$$Q_{55}^{(k)}=G_{zx}^{(k)},\qquad Q_{66}^{(k)}=G^{(k)}\tag{2.2}$$

考虑板的边界条件为夹紧固定情况:

$$\text{当 } x^2+y^2-a^2=0 \text{ 时}, u_0=v_0=w=\phi_x=\phi_y=0\tag{2.3}$$

为使最后的解更一般化, 引入以下无量纲参量:

$$\zeta=\frac{x}{a},\quad \eta=\frac{y}{a},\quad U=\frac{a}{h^2}u_0,\quad V=\frac{a}{h^2}v_0,\quad W=\frac{w}{h}$$

$$\Psi_\zeta=\frac{a}{h}\phi_x,\quad \Psi_\eta=\frac{a}{h}\phi_y,\quad Q_0=\frac{a^4}{D_{11}h}q_0,\quad \Omega=\sqrt{\frac{ma^4}{D_{11}}}\,\omega$$

$$\tau=\sqrt{\frac{D_{11}}{ma^4}}\,t,\quad a_{ij}=\frac{A_{ij}h^2}{D_{11}},\quad d_{ij}=\frac{D_{ij}}{D_{11}},\quad \lambda=\frac{a}{h}\tag{2.4}$$

于是, 方程(2.1)和边界条件(2.3)变成

$$L_1(U,V,W,\Psi_\zeta,\Psi_\eta)=a_{11}\frac{\partial^2U}{\partial\zeta^2}+a_{66}\frac{\partial^2U}{\partial\eta^2}+(a_{12}+a_{66})\frac{\partial^2V}{\partial\zeta\partial\eta}$$
$$+\left(a_{11}\frac{\partial^2W}{\partial\zeta^2}+a_{66}\frac{\partial^2W}{\partial\eta^2}\right)\frac{\partial W}{\partial\zeta}+(a_{12}+a_{66})\frac{\partial^2W}{\partial\zeta\partial\eta}\frac{\partial W}{\partial\eta}=0$$

$$L_2(U,V,W,\Psi_\zeta,\Psi_\eta)=(a_{12}+a_{66})\frac{\partial^2U}{\partial\zeta\partial\eta}+a_{66}\frac{\partial^2V}{\partial\zeta^2}+a_{22}\frac{\partial^2V}{\partial\eta^2}$$
$$+\left(a_{66}\frac{\partial^2W}{\partial\zeta^2}+a_{22}\frac{\partial^2W}{\partial\eta^2}\right)\frac{\partial W}{\partial\eta}+(a_{12}+a_{66})\frac{\partial^2W}{\partial\zeta\partial\eta}\frac{\partial W}{\partial\zeta}=0$$

$$L_3(U,V,W,\Psi_\zeta,\Psi_\eta)=\frac{\partial^3\Psi_\zeta}{\partial\zeta^3}+(d_{12}+2d_{66})\left(\frac{\partial^3\Psi_\zeta}{\partial\zeta\partial\eta^2}+\frac{\partial^3\Psi_\eta}{\partial\zeta^2\partial\eta}\right)$$
$$+d_{22}\frac{\partial^3\Psi_\eta}{\partial\eta^3}+\left[\frac{\partial U}{\partial\zeta}+\frac{1}{2}\left(\frac{\partial W}{\partial\zeta}\right)^2\right]\left(a_{11}\frac{\partial^2W}{\partial\zeta^2}+a_{12}\frac{\partial^2W}{\partial\eta^2}\right)$$
$$+\left[\frac{\partial V}{\partial\eta}+\frac{1}{2}\left(\frac{\partial W}{\partial\eta}\right)^2\right]\left(a_{12}\frac{\partial^2W}{\partial\zeta^2}+a_{22}\frac{\partial^2W}{\partial\eta^2}\right)+2a_{66}\left(\frac{\partial U}{\partial\eta}+\frac{\partial V}{\partial\zeta}\right.$$
$$\left.+\frac{\partial W}{\partial\zeta}\frac{\partial W}{\partial\eta}\right)\frac{\partial^2W}{\partial\zeta\partial\eta}+\frac{\partial^2W}{\partial\tau^2}-Q_0\cos\Omega\tau=0$$

$$L_4(U,V,W,\Psi_\zeta,\Psi_\eta)=d_{11}\frac{\partial^2\Psi_\zeta}{\partial\zeta^2}+d_{66}\frac{\partial^2\Psi_\zeta}{\partial\eta^2}+(d_{12}+d_{66})\frac{\partial^2\Psi_\eta}{\partial\zeta\partial\eta}$$
$$-\lambda^2a_{55}\left(\frac{\partial W}{\partial\zeta}+\Psi_\zeta\right)=0$$

$$L_5(U,V,W,\Psi_\zeta,\Psi_\eta)=(d_{12}+d_{66})\frac{\partial^2\Psi_\zeta}{\partial\zeta\partial\eta}+d_{66}\frac{\partial^2\Psi_\eta}{\partial\zeta^2}+d_{22}\frac{\partial^2\Psi_\eta}{\partial\eta^2}$$
$$5-\lambda^2 a_{44}\left(\frac{\partial W}{\partial\eta}+\Psi_\eta\right)=0$$

(2.5)

当 $\zeta^2+\eta^2-1=0$ 时, $U=V=W=\Psi_\zeta=\Psi_\eta=0$ (2.6)

§3. 解析解

现在应用伽辽金方法对具有夹紧固定边界的对称直线型正交异性层合圆板的大幅度振动问题进行单模态分析• 选取满足边界条件(2.6)的位移和转角函数为如下的分离形式

$$U=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}b_{ij}(\tau)\zeta(1-\zeta^2-\eta^2)\zeta^{2i}\eta^{2j}$$
$$V=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}c_{ij}(\tau)\eta(1-\zeta^2-\eta^2)\zeta^{2i}\eta^{2j}$$
$$(W=\varphi(\tau)(1-\zeta^2-\eta^2)^2$$
$$\Psi_\zeta=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}e_{ij}(\tau)\zeta(1-\zeta^2-\eta^2)\zeta^{2i}\eta^{2j}$$
$$\Psi_\eta=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}f_{ij}(\tau)\eta(1-\zeta^2-\eta^2)\zeta^{2i}\eta^{2j}$$

(3.1)

其中 φ 是无量纲时间 τ 的函数, 其最大值为

$$\varphi_{\max}=W_m \tag{3.2}$$

这里, W_m 是板的无量纲最大挠度•

应用伽辽金方法于方程(2.5), 我们得到如下的关于 U, V, W, Ψ_ζ 和 Ψ_η 的条件

$$\int_0^1\int_0^{\sqrt{1-\eta^2}}L_1(U,V,W,\Psi_\zeta,\Psi_\eta)\frac{\partial U}{\partial b_{ij}}d\eta d\zeta=0$$
$$\int_0^1\int_0^{\sqrt{1-\eta^2}}L_2(U,V,W,\Psi_\zeta,\Psi_\eta)\frac{\partial V}{\partial c_{ij}}d\eta d\zeta=0$$
$$\int_0^1\int_0^{\sqrt{1-\eta^2}}L_3(U,V,W,\Psi_\zeta,\Psi_\eta)\frac{\partial W}{\partial \varphi}d\eta d\zeta=0$$
$$\int_0^1\int_0^{\sqrt{1-\eta^2}}L_4(U,V,W,\Psi_\zeta,\Psi_\eta)\frac{\partial \Psi_\zeta}{\partial e_{ij}}d\eta d\zeta=0$$
$$\int_0^1\int_0^{\sqrt{1-\eta^2}}L_5(U,V,W,\Psi_\zeta,\Psi_\eta)\frac{\partial \Psi_\eta}{\partial f_{ij}}d\eta d\zeta=0$$

(3.3)

经过代入和积分, 首先从方程(3.3)的前两式和后两式中解出以 φ 表示的未知函数 b_{ij}, c_{ij}, e_{ij} 和 f_{ij}, 然后再代入方程(3.3)的第三式, 便得如下的关于时间函数 φ 的非线性常微分方程

$$\alpha_3\frac{d^2\varphi}{d\tau^2}+\alpha_2\varphi^3+\alpha_1\varphi=\alpha_4 Q_0\cos\Omega\tau \tag{3.4}$$

其中

$$\alpha_1=\int_0^1\int_0^{\sqrt{1-\zeta^2}}\left[\frac{\partial^3\Psi_\zeta}{\partial\zeta^3}+(d_{12}+2d_{66})\left(\frac{\partial^3\Psi_\zeta}{\partial\zeta\partial\eta^2}+\frac{\partial^3\Psi_\eta}{\partial\zeta^2\partial\eta}\right)+d_{22}\frac{\partial^3\Psi_\eta}{\partial\eta^3}\right](1-\zeta^2-\eta^2)^2d\eta d\zeta$$

$$\begin{aligned}\alpha_2=\int_0^1\int_0^{\sqrt{1-\eta^2}}\Bigg\{&\left[\frac{\partial U}{\partial\zeta}+\frac{1}{2}\left(\frac{\partial W}{\partial\zeta}\right)^2\right]\left(a_{11}\frac{\partial^2W}{\partial\zeta^2}+a_{12}\frac{\partial^2W}{\partial\eta^2}\right)\\&+\left[\frac{\partial V}{\partial\eta}+\frac{1}{2}\left(\frac{\partial W}{\partial\eta}\right)^2\right]\left(a_{12}\frac{\partial^2W}{\partial\zeta^2}+a_{22}\frac{\partial^2W}{\partial\eta^2}\right)+2a_{66}\left(\frac{\partial U}{\partial\eta}+\frac{\partial V}{\partial\zeta}\right.\\&\left.+\frac{\partial W}{\partial\zeta}\frac{\partial W}{\partial\eta}\right)\frac{\partial^2W}{\partial\zeta\partial\eta}\Bigg\}(1-\zeta^2-\eta^2)^2d\eta d\zeta\end{aligned}$$

$$\alpha_3=\int_0^1\int_0^{\sqrt{1-\eta^2}}\left[(1-\zeta^2-\eta^2)^2\right]^2d\eta d\zeta$$

$$\alpha_4=\int_0^1\int_0^{\sqrt{1-\eta^2}}(1-\zeta^2-\eta^2)^2d\eta d\zeta \tag{3.5}$$

方程(3.4)可以写为如下形式

$$\frac{d^2\varphi}{d\tau^2}+\Omega_0^2\varphi+\beta^2\varphi^3=Q\cos\Omega\tau \tag{3.6}$$

其中 Ω_0 是无量纲固有频率,

$$\Omega_0=\sqrt{\frac{\alpha_1}{\alpha_3}},\quad \beta=\sqrt{\frac{\alpha_2}{\alpha_3}},\quad Q=\frac{\alpha_4}{\alpha_3}Q_0 \tag{3.7}$$

方程(3.6)是著名的 Duffing 方程, 它的前两项近似解为

$$\varphi(\tau)=W_m\cos\Omega\tau+\frac{\beta^2W_m^3}{32(\Omega_0^2+3\beta^2W_m^2/4-Q/W_m)}\cos\Omega\tau \tag{3.8}$$

于是, 我们得到以无量纲振幅 W_m 表示的无量纲圆频率 Ω 的表达式

$$\frac{\Omega^2}{\Omega_0^2}=1+\frac{3}{4}\left(\frac{\beta}{\Omega_0}W_m\right)^2-\frac{Q}{W_m\Omega_0^2} \tag{3.9}$$

特别地, 如果在方程(3.6)中令 $q_0=0$, 便得这种板的非线性自由振动方程如下:

$$\frac{d^2\varphi}{d\tau^2}+\Omega_0^2\tau+\beta^2\varphi^3=0 \tag{3.10}$$

相应的初始条件为

$$当\ \tau=0\ 时,\quad \varphi=W_m,\quad \frac{d\varphi}{d\tau}=0 \tag{3.11}$$

应用条件(3.11), 非线性二阶常微分方程(3.10)的解可以表示成如下椭圆余弦 cn 的形式

$$\varphi=W_m\mathrm{cn}(p\tau,k) \tag{3.12}$$

其中

$$p=\sqrt{\Omega_0^2+\beta^2W_m^2},\quad k=\sqrt{\frac{\beta^2W_m^2}{2(\Omega_0^2+\beta^2W_m^2)}} \tag{3.13}$$

由此给出非线性周期 T

$$T=\frac{4K(k)}{p} \tag{3.14}$$

其中 K 为第一类完全椭圆积分•

特别地,在 $W_m \to 0$ 的情况下,有

$$K(k) \to \frac{\pi}{2}, \quad p = \Omega_0 \tag{3.15}$$

于是,由式(3.14),我们得到对应的线性周期如下

$$T_0 = \frac{2\pi}{\Omega_0} \tag{3.16}$$

因此,非线性周期 T 与线性周期 T_0 的比值为

$$\frac{T}{T_0} = \frac{2\Omega_0 K(k)}{\pi\sqrt{\Omega_0^2 + \beta^2 W_m^2}} \tag{3.17}$$

§4. 算　　例

为简化计算,作为一个特例,我们考虑一个对称直线型正交异性层合石墨/环氧圆板,它的每一层具有相同的厚度和弹性常数• 复合材料的弹性常数[18]和几何参数为

$$\frac{E_x}{E_y} = 15, \quad \frac{G}{E_y} = 0.429, \quad \nu_{xy} = 0.4$$

$$G_{yz} = G_{zx} = G_s, \quad \frac{a}{h} = 10$$

在此板计算时,对 U 和 V 的级数取 4 项,对 Ψ_ξ 和 Ψ_η 的级数取一项•

图2 给出了这一板的层数 $n = 3$ 和 $G_s/E_y = 0.343$ 时在承受简谐力作用下大幅度受迫振动的响应曲线• $Q = 0$ 的曲线是大幅度自由振动曲线,称为主干线• 主干线右边对应的无量纲振幅 W_m 是负值,主干线左边对应的为正值• 这意味着,对于特定的振幅,如果强振频率 ω 大于自振频率 ω_0,则振动的板与驱动力是异相;反之,如果强振频率小于自振频率,则振动的板与驱动力是同相• 实线对应于稳态运动,虚线对应于不稳态运动•

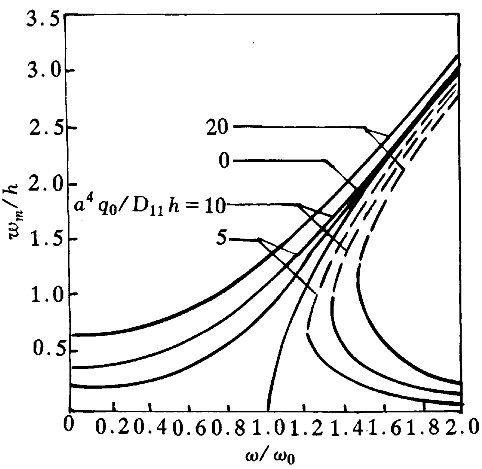

图 2 强振频率与振幅的之间的关系

对于不同层数和 $G_s/E_y = 0.343$ 的情况,对称层合圆板自由振动的非线性周期 T 与线性周期 T_0 的比值和无量纲振幅的关系绘于图 3 中• 此图表明,板的非线性周期随着层数的增加而增加,但影响轻微• 此图还表明,非线性周期随着振幅的增加而减少,且呈硬化型的非线性特性•

对于层数 $n = 3$ 时不同的 G_s/E_y 值情况下的类似结果由图 4 给出• 由图看到,非线性周期随着横向剪切模量的增加而增加• 显然,横向剪切变形对层合圆板振动的影响是很重要的• 图中的虚线表示不计横向剪切变形影响时的曲线•

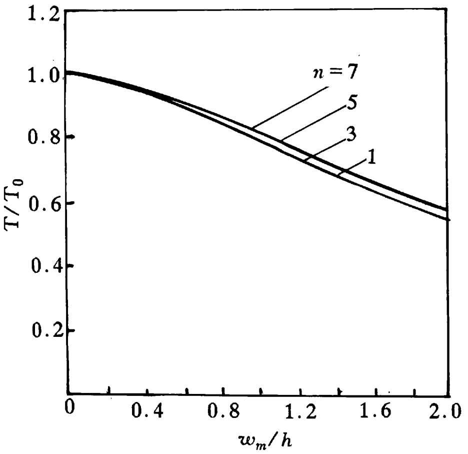

图 3　不同层数下对称层合圆板的振动周期

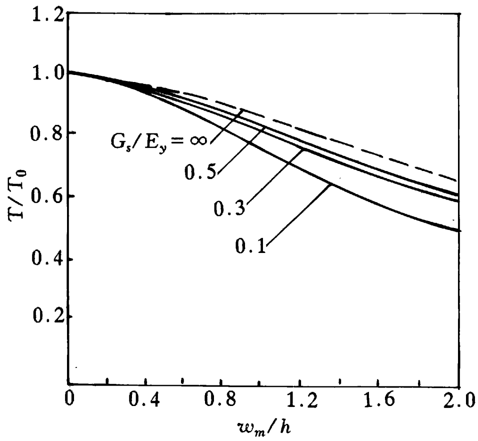

图 4　横向剪切模量对自由振动周期的影响

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Shear Effects on Large Amplitude Forced Vibration of Symmetrically Laminated Rectilinearly Orthotropic Circular Plates

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Abstract

In this paper, nonlinear forced vibration of symmetrically laminated rectilinearly orthotropic circular plates excited by a harmonic force $q_0\cos\omega t$ including effects of transverse shear deformation is discussed. The analytical solution for the relationship between forcing frequency and amplitude of vibration is obtained by Galerkin's method. Finally, the paper analyses the effect of the transverse shear on the vibration of the plate and gives the ratio of nonlinear period to linear period for nonlinear free vibration of the plate.

Key words large amplitude forced vibration, rectilinear orthotropy, laminated circular plate, transverse shear