

# Reissner 厚板弹性弯曲的一般解析解

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## 摘 要

针对大型工程建设中的 Reissner 厚板弹性弯曲问题, 本文采用复级数方法求解相应的常系数偏微分方程组的边值问题, 并首次得到了任意边界条件下的一般解析解。该解形式简单, 计算方便、可靠。以四边简支和三边固支一边自由两种支撑条件下厚板承受均布载荷为例进行了分析验算, 与已有的计算结果相比, 计算结果相当满意。同时本文还着重对解的收敛速度、正确性(合理性)及边界满足情况进行了考察。

关键词 Reissner 厚板弯曲 复级数方法 一般解析解

## § 1. 矩形厚板的一般解析解

### 1. 基本方程和边界条件

如图 1 所示, 工程中常用的 Reissner 厚板理论具有很好的计算精度, 其基本假设为:

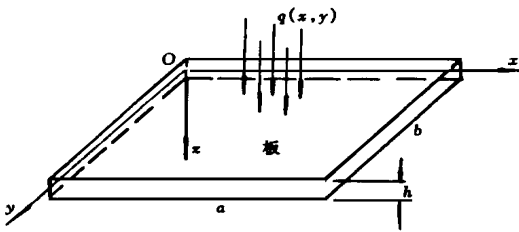


图 1 矩形弹性厚板

(a)、考虑具有弹性常数  $E, \mu, G$  的矩形弹性板, 其长、宽、厚分别为  $a, b, h$ , 且其受力状态和变形可由中面位移  $W$ , 中面法线沿  $x$  轴,  $y$  轴的平均转角  $\Phi_x, \Phi_y$  描述。

(b)、厚板上表面受横向载荷  $q(x, y)$  作用, 且其边界条件可取简支、固支和自由的任意形式。

则广义位移表示的平衡方程为<sup>[1]</sup>:

$$\left. \begin{aligned} \Delta^2 \Phi_x + b_2 \frac{\partial}{\partial y} \left( \frac{\partial \Phi_x}{\partial x} - \frac{\partial \Phi_x}{\partial y} \right) - b_1 \left( \Phi_x + \frac{\partial W}{\partial x} \right) + b_3 \frac{\partial q}{\partial x} &= \mu_0 \\ \Delta^2 \Phi_y + b_2 \frac{\partial}{\partial x} \left( \frac{\partial \Phi_y}{\partial y} - \frac{\partial \Phi_y}{\partial x} \right) - b_1 \left( \Phi_y + \frac{\partial W}{\partial y} \right) + b_3 \frac{\partial q}{\partial y} &= 0 \\ \Delta^2 W + b_4 q &= 0 \end{aligned} \right\} \text{件,} \quad (1.1)$$

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$$\text{方程中 } b_1 = \frac{5Gh}{6D}, b_2 = \frac{1+\mu}{2}, b_3 = \frac{6\mu(1+\mu)}{5Eh}, b_4 = \frac{6}{5Gh}, D = \frac{Eh^3}{12(1-\mu^2)}$$

矩形板的齐次边界条件(以板边  $x = 0$  为例):

- (1)、板边固支  $W = 0, \Phi_x = 0, \Phi_y = 0;$   
 (2)、板边简支  $W = 0, \Phi_y = 0, M_x = 0;$   
 (3)、板边自由  $Q_x = 0, M_x = 0, M_{xy} = 0$

## 2 齐次解的建立

假设方程(1.1)有如下齐次解:

$$\left. \begin{aligned} W &= \sum_{m=-\infty}^{+\infty} f_m(y) \exp[\xi x] \\ \Phi_x &= \sum_{m=-\infty}^{+\infty} g_m(y) \exp[\xi x] \\ \Phi_y &= \sum_{m=-\infty}^{+\infty} h_m(y) \exp[\xi x] \end{aligned} \right\} \quad (1.2)$$

其中  $\xi = i\alpha = i \frac{m\pi}{a}$ .

对任一  $\xi$  得一关于  $f_m, g_m, h_m$  的二阶常微分方程组, 若设

$$f_m(y) = A_m \exp[\eta y], \quad g_m(y) = B_m \exp[\eta y], \quad h_m(y) = C_m \exp[\eta y]$$

将(1.2)代入(1.1), 可得特征方程

$$\begin{vmatrix} -b_1\xi & \xi^2 + (1-b_2)\eta^2 - b_1 & b_2\xi\eta \\ -b_1\eta & b_2\xi\eta & (1-b_2)\xi^2 + \eta^2 - b_1 \\ \xi^2 + \eta^2 & \xi & \eta \end{vmatrix} = 0 \quad (1.3)$$

解之, 得:

$$\eta_{1,2} = \pm s_{1m} = \pm \sqrt{\frac{b_1}{1-b_2} + \alpha^2} \quad (1.4a)$$

$$\text{和 } \eta_{3,4,5,6} = \pm s_{2m} = \pm \alpha \quad (1.4b)$$

则  $W, \Phi_x, \Phi_y$  通解的实数形式为:

$$\begin{aligned} W &= \sum_{m=1}^{+\infty} \left\{ A_{1m} \frac{\text{sh } s_{1m}y}{\text{sh } s_{1m}b} + A_{2m} \frac{\text{sh } s_{1m}(b-y)}{\text{sh } s_{1m}b} + A_{3m} \frac{\text{sh } s_{2m}y}{\text{sh } s_{2m}b} + A_{4m} \frac{s_{2m}y \text{ sh } s_{2m}y}{\text{sh } s_{2m}b} \right. \\ &\quad \left. + A_{5m} \frac{\text{sh } s_{2m}(b-y)}{\text{sh } s_{2m}b} + A_{6m} \frac{s_{2m}(b-y) \text{ sh } s_{2m}(b-y)}{\text{sh } s_{2m}b} \right\} \sin \alpha x \\ &\quad + \sum_{m=1}^{+\infty} \left\{ A_{7m} \frac{\text{sh } s_{1m}y}{\text{sh } s_{1m}b} + A_{8m} \frac{\text{sh } s_{1m}(b-y)}{\text{sh } s_{1m}b} + A_{9m} \frac{\text{sh } s_{2m}y}{\text{sh } s_{2m}b} + A_{10m} \frac{s_{2m}y \text{ sh } s_{2m}y}{\text{sh } s_{2m}b} \right. \\ &\quad \left. + A_{11m} \frac{\text{sh } s_{2m}(b-y)}{\text{sh } s_{2m}b} + A_{12m} \frac{s_{2m}(b-y) \text{ sh } s_{2m}(b-y)}{\text{sh } s_{2m}b} \right\} \cos \alpha x \end{aligned} \quad (1.5a)$$

$$\begin{aligned} \Phi_x &= \sum_{m=1}^{+\infty} \left\{ B_{1m} \frac{\text{sh } s_{1m}y}{\text{sh } s_{1m}b} + B_{2m} \frac{\text{sh } s_{1m}(b-y)}{\text{sh } s_{1m}b} + B_{3m} \frac{\text{sh } s_{2m}y}{\text{sh } s_{2m}b} + B_{4m} \frac{s_{2m}y \text{ sh } s_{2m}y}{\text{sh } s_{2m}b} \right. \\ &\quad \left. + B_{5m} \frac{\text{sh } s_{2m}(b-y)}{\text{sh } s_{2m}b} + B_{6m} \frac{s_{2m}(b-y) \text{ sh } s_{2m}(b-y)}{\text{sh } s_{2m}b} \right\} \cos \alpha x \\ &\quad + \sum_{m=1}^{+\infty} \left\{ B_{7m} \frac{\text{sh } s_{1m}y}{\text{sh } s_{1m}b} + B_{8m} \frac{\text{sh } s_{1m}(b-y)}{\text{sh } s_{1m}b} + B_{9m} \frac{\text{sh } s_{2m}y}{\text{sh } s_{2m}b} + B_{10m} \frac{s_{2m}y \text{ sh } s_{2m}y}{\text{sh } s_{2m}b} \right. \end{aligned}$$

$$\begin{aligned}
& + B_{11m} \frac{\text{sh}s_{2m}(b-y)}{\text{sh}s_{2m}b} + B_{12m} \frac{s_{2m}(b-y)\text{sh}s_{2m}(b-y)}{\text{sh}s_{2m}b} \left. \vphantom{B_{11m}} \right\} \sin \alpha x \\
& + B_{10} \frac{\text{sh}s_{10}y}{\text{sh}s_{10}b} + B_{20} \frac{\text{sh}s_{10}(b-y)}{\text{sh}s_{10}b} \quad (1.5b) \\
\Phi = & \sum_{m=1}^{+\infty} \left\{ C_{1m} \frac{\text{sh}s_{1m}y}{\text{sh}s_{1m}b} + C_{2m} \frac{\text{sh}s_{1m}(b-y)}{\text{sh}s_{1m}b} + C_{3m} \frac{\text{sh}s_{2m}y}{\text{sh}s_{2m}b} + C_{4m} \frac{s_{2m}y\text{sh}s_{2m}y}{\text{sh}s_{2m}b} \right. \\
& + C_{5m} \frac{\text{sh}s_{2m}(b-y)}{\text{sh}s_{2m}b} + C_{6m} \frac{s_{2m}(b-y)\text{sh}s_{2m}(b-y)}{\text{sh}s_{2m}b} \left. \vphantom{C_{1m}} \right\} \sin \alpha x \\
& + \sum_{m=1}^{+\infty} \left\{ C_{7m} \frac{\text{sh}s_{1m}y}{\text{sh}s_{1m}b} + C_{8m} \frac{\text{sh}s_{1m}(b-y)}{\text{sh}s_{1m}b} + C_{9m} \frac{\text{sh}s_{2m}y}{\text{sh}s_{2m}b} + C_{10m} \frac{s_{2m}y\text{sh}s_{2m}y}{\text{sh}s_{2m}b} \right. \\
& + C_{11m} \frac{\text{sh}s_{2m}(b-y)}{\text{sh}s_{2m}b} + C_{12m} \frac{s_{2m}(b-y)\text{sh}s_{2m}(b-y)}{\text{sh}s_{2m}b} \left. \vphantom{C_{7m}} \right\} \cos \alpha x \quad (1.5c)
\end{aligned}$$

将上述通解代入方程(1.1)的齐次方程,可得待定系数  $A_{km}, B_{km}, C_{km}$  ( $k=1, 2, \dots, 12$ ) 满足的相容方程,它是变量  $y$  的一组恒等式;则  $y$  的各独立函数基前系数应为零,得待定系数之间的转换关系.若以  $C_{km}$  ( $k=1, 2, \dots, 6$ ) 为已知常数 ( $k=7, 8, \dots, 12$  类似),则有:

$$A_{km} = \sum_{j=1}^6 G_{kj}^m C_{jm}, \quad B_{km} = \sum_{j=1}^6 H_{kj}^m C_{jm}$$

矩阵  $G_{6 \times 6}, H_{6 \times 6}$  中  $G_{1j}^m = G_{2j}^m = G_{7j}^m = G_{8j}^m = 0$  ( $1 \leq j \leq 6$ ). 这表明  $A_{1m} = A_{2m} = A_{7m} = A_{8m} = 0$ , 即位移  $W$  中有四项空缺.

### 3 补充解的建立

若直接用上述级数解去满足边界条件,则对一些具体问题(如悬壁矩形厚板等),效果并不理想.类似于文献[2],我们选取如下补充解形式

$$\begin{aligned}
W^s &= \sum_{i=0}^1 \sum_{j=0}^3 a_{ij} \frac{x^i y^j}{a^i b^j} + \sum_{i=2}^3 \sum_{j=0}^1 a_{ij} \frac{x^i y^j}{a^i b^j}, \\
\Phi_x &= \sum_{j=1}^3 b_{0j} \frac{y^j}{b^j} + \sum_{i=0}^2 \sum_{j=0}^1 b_{ij} \frac{x^i y^j}{a^i b^j}, \\
\Phi_y &= \sum_{i=1}^3 c_{i0} \frac{x^i}{a^i} + \sum_{i=0}^1 \sum_{j=0}^2 c_{ij} \frac{x^i y^j}{a^i b^j},
\end{aligned}$$

其中,待定系数  $b_{ij}, c_{ij}$  与  $a_{ij}$  之间的转换关系由方程(1.1)的齐次形式确定,或更方便地由其等价方程  $Q_x = -D \frac{\partial}{\partial x} (\dots W), Q_y = -D \frac{\partial}{\partial y} (\dots W)$  确定<sup>[1]</sup>.

### 4 特解的建立

将载荷展成双正弦级数

$$q = \sum_{m=1}^{+\infty} \sum_{n=1}^{+\infty} q_{mn} \sin \alpha x \sin \beta y$$

相应特解可取为

$$\begin{aligned}
W^p &= \sum_{m=1}^{+\infty} \sum_{n=1}^{+\infty} W_{mn} \sin \alpha x \sin \beta y \\
\Phi_x^p &= \sum_{m=1}^{+\infty} \sum_{n=1}^{+\infty} \Phi_{xmn} \cos \alpha x \sin \beta y \\
\Phi_y^p &= \sum_{m=1}^{+\infty} \sum_{n=1}^{+\infty} \Phi_{ymn} \sin \alpha x \cos \beta y
\end{aligned}$$

其中,  $W_{mn}$ ,  $\Phi_{xmn}$ ,  $\Phi_{ymn}$  由下式给出

$$\begin{bmatrix} b_1 \alpha & \alpha^2 + (1 - b_2) \beta^2 + b_1 & b_2 \alpha \beta \\ b_1 \beta & b_2 \alpha \beta & (1 - b_2) \alpha^2 + \beta^2 + b_1 \\ \alpha^2 + \beta^2 & \alpha & \beta \end{bmatrix} \begin{cases} W_{mn} \\ \Phi_{xmn} \\ \Phi_{ymn} \end{cases} = \begin{cases} b_3 \alpha \\ b_3 \beta \\ b_4 \end{cases} q_{mn} \quad (1.6)$$

可以指出的是, 上述特解即为承受任意载荷的四边简支矩形厚板的解。

### 5 一般解析解形式

如果将  $W$ ,  $\Phi_x$ ,  $\Phi_y$  先按  $y$  方向展开成复数形式的福氏级数, 按照步骤 2, 可得  $W$ ,  $\Phi_x$ ,  $\Phi_y$  的另一组齐次解。根据线性微分方程组解的可迭加性, 适当选取其通解、补充解、特解, 即构成厚板横向弯曲问题的一般解析解。考虑到实际计算中板长  $a$ 、板宽  $b$  比较接近, 将系数进行适当变换, 其一般解可选为关于  $x, y$  轴对称的解形式。

$$\begin{aligned} W = & \sum_{m=1}^{+\infty} \left\{ A_{3m} \frac{\text{sh } s_{2m} y}{\text{sh } s_{2m} b} + A_{4m} \frac{s_{2m} y \text{ sh } s_{2m} y}{\text{sh } s_{2m} b} + A_{5m} \frac{\text{sh } s_{2m} (b - y)}{\text{sh } s_{2m} b} \right. \\ & \left. + A_{6m} \frac{s_{2m} (b - y) \text{ sh } s_{2m} (b - y)}{\text{sh } s_{2m} b} \right\} \sin \alpha x + \sum_{n=1}^{+\infty} \left\{ A_{9n} \frac{\text{sh } s_{2n} x}{\text{sh } s_{2n} a} + A_{10n} \frac{s_{2n} x \text{ sh } s_{2n} x}{\text{sh } s_{2n} a} \right. \\ & \left. + A_{11n} \frac{\text{sh } s_{2n} (a - x)}{\text{sh } s_{2n} a} + A_{12n} \frac{s_{2n} (a - x) \text{ sh } s_{2n} (a - x)}{\text{sh } s_{2n} a} \right\} \sin \beta y \\ & + \sum_{i=0}^1 \sum_{j=0}^3 a_{ij} \frac{x^i y^j}{a^i b^j} + \sum_{i=2}^3 \sum_{j=0}^1 a_{ij} \frac{x^i y^j}{a^i b^j} + \sum_{m=1}^{+\infty} \sum_{n=1}^{+\infty} W_{mn} \sin \alpha x \sin \beta y \end{aligned} \quad (1.7a)$$

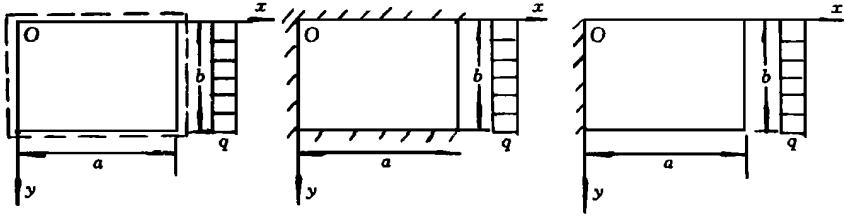
$$\begin{aligned} \Phi_x = & \sum_{m=1}^{+\infty} \left\{ B_{1m} \frac{\text{sh } s_{1m} y}{\text{sh } s_{1m} b} + B_{2m} \frac{\text{sh } s_{1m} (b - y)}{\text{sh } s_{1m} b} + B_{3m} \frac{\text{sh } s_{2m} y}{\text{sh } s_{2m} b} + B_{4m} \frac{s_{2m} y \text{ sh } s_{2m} y}{\text{sh } s_{2m} b} \right. \\ & \left. + B_{5m} \frac{\text{sh } s_{2m} (b - y)}{\text{sh } s_{2m} b} + B_{6m} \frac{s_{2m} (b - y) \text{ sh } s_{2m} (b - y)}{\text{sh } s_{2m} b} \right\} \cos \alpha x \\ & + B_{10} \frac{\text{sh } s_{10} y}{\text{sh } s_{10} b} + B_{20} \frac{\text{sh } s_{10} (b - y)}{\text{sh } s_{10} b} + \sum_{n=1}^{+\infty} \left\{ B_{7n} \frac{\text{sh } s_{1n} x}{\text{sh } s_{1n} a} + B_{8n} \frac{\text{sh } s_{1n} (a - x)}{\text{sh } s_{1n} a} \right. \\ & \left. + B_{9n} \frac{\text{sh } s_{2n} x}{\text{sh } s_{2n} a} + B_{10n} \frac{s_{2n} x \text{ sh } s_{2n} x}{\text{sh } s_{2n} a} + B_{11n} \frac{\text{sh } s_{2n} (a - x)}{\text{sh } s_{2n} a} \right. \\ & \left. + B_{12n} \frac{s_{2n} (a - x) \text{ sh } s_{2n} (a - x)}{\text{sh } s_{2n} a} \right\} \sin \beta y + \sum_{j=1}^3 b_{0j} \frac{y^j}{b^j} \\ & + \sum_{i=0}^2 \sum_{j=0}^1 b_{ij} \frac{x^i y^j}{a^i b^j} + \sum_{m=1}^{+\infty} \sum_{n=1}^{+\infty} \Phi_{xmn} \cos \alpha x \sin \beta y \end{aligned} \quad (1.7b)$$

$$\begin{aligned} \Phi_y = & \sum_{m=1}^{+\infty} \left\{ C_{1m} \frac{\text{sh } s_{1m} y}{\text{sh } s_{1m} b} + C_{2m} \frac{\text{sh } s_{1m} (b - y)}{\text{sh } s_{1m} b} + C_{3m} \frac{\text{sh } s_{2m} y}{\text{sh } s_{2m} b} + C_{4m} \frac{s_{2m} y \text{ sh } s_{2m} y}{\text{sh } s_{2m} b} \right. \\ & \left. + C_{5m} \frac{\text{sh } s_{2m} (b - y)}{\text{sh } s_{2m} b} + C_{6m} \frac{s_{2m} (b - y) \text{ sh } s_{2m} (b - y)}{\text{sh } s_{2m} b} \right\} \sin \alpha x \\ & + \sum_{n=1}^{+\infty} \left\{ C_{7n} \frac{\text{sh } s_{1n} x}{\text{sh } s_{1n} a} + C_{8n} \frac{\text{sh } s_{1n} (a - x)}{\text{sh } s_{1n} a} + C_{9n} \frac{\text{sh } s_{2n} x}{\text{sh } s_{2n} a} + C_{10n} \frac{s_{2n} x \text{ sh } s_{2n} x}{\text{sh } s_{2n} a} \right. \\ & \left. + C_{11n} \frac{\text{sh } s_{2n} (a - x)}{\text{sh } s_{2n} a} + C_{12n} \frac{s_{2n} (a - x) \text{ sh } s_{2n} (a - x)}{\text{sh } s_{2n} a} \right\} \cos \beta y \\ & + C_{70} \frac{\text{sh } s_{10} x}{\text{sh } s_{10} a} + C_{80} \frac{\text{sh } s_{10} (a - x)}{\text{sh } s_{10} a} + \sum_{i=1}^3 c_{i0} \frac{x^i}{a^i} \\ & + \sum_{i=0}^1 \sum_{j=0}^2 c_{ij} \frac{x^i y^j}{a^i b^j} + \sum_{m=1}^{+\infty} \sum_{n=1}^{+\infty} \Phi_{ymn} \sin \alpha x \cos \beta y \end{aligned} \quad (1.7c)$$

## 6 角点条件

在所得解析解中,引入多项式作为其补充解,从而保证了我们能有效地解决各种复杂边界条件问题。注意到某函数的正弦级数展开式在展开区间端点处不收敛于原函数,因此我们将一般解展成正弦级数并去满足相应边界条件时,还应附加由解析解直接满足端点(角点)值所得的补充方程。通常可称之为角点条件。下面给出了一些特定边界条件下的角点条件。

a)、四边简支(见图 2a)



(a) 四边简支

(b) 三边固定一边自由

(c) 悬壁

图 2 几种典型的边界条件

$$\begin{aligned} W|_{(0,0)} = 0, & \quad W|_{(a,0)} = 0, & \quad W|_{(0,b)} = 0, & \quad W|_{(a,b)} = 0, \\ M_x|_{(0,0)} = 0, & \quad M_x|_{(a,0)} = 0, & \quad M_x|_{(0,b)} = 0, & \quad M_x|_{(a,b)} = 0, \\ M_y|_{(0,0)} = 0, & \quad M_y|_{(a,0)} = 0, & \quad M_y|_{(0,b)} = 0, & \quad M_y|_{(a,b)} = 0, \end{aligned}$$

b)、三边固定一边自由(见图 2b)

$$\begin{aligned} W|_{(0,0)} = 0, & \quad W|_{(a,0)} = 0, & \quad W|_{(0,b)} = 0, & \quad W|_{(a,b)} = 0, \\ \beta_x|_{(0,0)} = 0, & \quad \beta_x|_{(a,b)} = 0, & \quad Q_x|_{(a,0)} = 0, & \quad Q_x|_{(a,b)} = 0, \\ \beta_y|_{(0,0)} = 0, & \quad \beta_y|_{(a,0)} = 0, & \quad \beta_y|_{(0,b)} = 0, & \quad \beta_y|_{(a,b)} = 0, \\ M_x|_{(a,0)} = 0, & \quad M_x|_{(a,b)} = 0 \end{aligned}$$

c)、悬壁(见图 2c)

$$\begin{aligned} W|_{(0,0)} = 0, & \quad W|_{(0,b)} = 0, & \quad \beta_x|_{(0,0)} = 0, & \quad \beta_x|_{(0,b)} = 0, \\ Q_x|_{(a,0)} = 0, & \quad Q_x|_{(a,b)} = 0, & \quad M_x|_{(a,0)} = 0, & \quad M_x|_{(a,b)} = 0, \\ Q_y|_{(0,0)} = 0, & \quad Q_y|_{(a,0)} = 0, & \quad Q_y|_{(0,b)} = 0, & \quad Q_y|_{(a,b)} = 0, \\ M_y|_{(0,0)} = 0, & \quad M_y|_{(a,0)} = 0, & \quad M_y|_{(0,b)} = 0, & \quad M_y|_{(a,b)} = 0 \end{aligned}$$

在厚板弯曲问题中,补充方程个数为 12 至 16,但求解待定系数时只需 12 个。鉴于混合边界条件时,广义位移比内力收敛快,我们将优先考虑位移角点条件。

通过物理方程,可方便地将弯矩和剪力  $M_x, M_y, M_{xy}, Q_x, Q_y$  用  $A_{im}, B_{im}, C_{im} (1 \leq i \leq 6), A_{in}, B_{in}, C_{in} (7 \leq i \leq 12)$  及  $a_{ij}, b_{ij}, c_{ij}$  表示。若级数解均取前  $M, N$  项,则基本求解变量个数为  $6M + 6N + 16$ 。由四条边 12 个边界条件给出的方程数  $4M + 2(M + 1) + 4N + 2(N + 1) = 6M + 6N + 4$ ,由角点条件给出的补充方程数为 12,基本求解变量和方程数目相等,可直接求出  $A_{im}, B_{im}, C_{im} (1 \leq i \leq 6), A_{in}, B_{in}, C_{in} (7 \leq i \leq 12), a_{ij}, b_{ij}, c_{ij}$  各系数,从而求得该边值问题。

## § 2. 算例分析

笔者针对四边简支(SSSS)和三边固定一边自由(CFCC)两种边界条件,在微机上进行了

计算验证• 下面将上述的解析解作三方面的讨论:

1、解的收敛速度; 2、解的正确性; 3、边界条件满足情况•

1. 收敛性验证

如图 2(a、b), 考虑两种不同支撑情况下厚板承受均布载时的弯曲, 计算参数为  $E = 200\text{GPa}$ ,  $\mu = 0.30$ ,  $b/a = 1.0$ ,  $h/a = 0.30$ • 从下表可以看出, 两种边界下(SSSS 和 CFCC), 最大挠度和沿  $y$  方向的最大弯矩都收敛很快, 这表明本文所取级数解收敛性良好•

表 1 厚板最大挠度  $W = 10^{-3}\theta_1 qa^4/D$  和最大弯矩  $M_x = 10^{-2}\theta_2 qa^2$  收敛情况

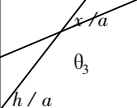
		$M = N = 4$	$M = N = 6$	$M = N = 8$	$M = N = 10$	$M = N = 12$	$M = N = 14$	$M = N = 16$
W ( $\theta_1$ )	SSSS	5. 63330	5. 68552	5. 66731	5. 67543	5. 67100	5. 67363	5. 67192
	CFCC	5. 95567	6. 20444	6. 40750	6. 50565	6. 59173	6. 64422	6. 69160
$M_x$ ( $\theta_2$ )	SSSS	4. 78971	4. 98775	4. 88853	4. 94926	4. 90518	4. 93913	4. 91117
	CFCC	- 4. 61833	- 4. 75487	- 4. 66729	- 4. 72492	- 4. 68025	- 4. 71195	- 4. 68562

表 2 四边简支板中点的挠度  $W$ , 弯矩  $M_x$  和  $M_y$

$b/a$	$h/a$	$W (10^{-3}qa^4/D)$		$M_x (10^{-2}qa^2)$		$M_y (10^{-2}qa^2)$	
		CFSM	TRM	CFSM	TRM	CFSM	TRM
1. 0	0.005	4. 06280	4. 437	4. 78999	4. 789	4. 78999	4. 789
	0.010	4. 06414	4. 438	4. 78911	4. 789	4. 78911	4. 789
	0.050	4. 10709	4. 485	4. 79286	4. 792	4. 79286	4. 792
	0.100	4. 24128	4. 631	4. 80459	4. 804	4. 80459	4. 804
	0.200	4. 77807	5. 218	4. 85152	4. 848	4. 85152	4. 849
	0.300	5. 67270	6. 194	4. 92973	4. 923	4. 92973	4. 924
	0.500	8. 53555	9. 320	5. 18002	5. 161	5. 18002	5. 164
	薄板解		4. 06235		4. 78851		4. 78851
2. 0	0.005	10. 1294	11. 061	10. 1681	10. 168	4. 63466	4. 635
	0.010	10. 1314	11. 063	10. 1681	10. 168	4. 63468	4. 635
	0.050	10. 1978	11. 124	10. 1688	10. 169	4. 64116	3. 642
	0.100	10. 4052	11. 316	10. 1709	10. 172	4. 66084	4. 662
	0.200	11. 2348	12. 084	10. 1794	10. 181	4. 73954	4. 742
	0.300	12. 6175	13. 363	10. 1934	10. 197	4. 88079	4. 875
	0.500	17. 0420	17. 456	10. 2385	10. 248	5. 29049	5. 303
	薄板解		10. 1287		10. 1681		4. 63460
5. 0	0.005	12. 9716	14. 164	12. 4616	12. 462	3. 77206	3. 775
	0.010	12. 9738	14. 167	12. 4616	12. 462	3. 77228	3. 775
	0.050	13. 0466	14. 247	12. 4615	12. 462	3. 77936	3. 782
	0.100	13. 2741	14. 494	12. 4612	12. 462	3. 80149	3. 804
	0.200	14. 1839	15. 485	12. 4598	12. 462	3. 88999	3. 894
	0.300	15. 7002	17. 135	12. 4576	12. 462	4. 03749	4. 044
	0.500	20. 5524	22. 419	12. 4504	12. 462	4. 50950	4. 524
	薄板解		12. 9708		12. 4616		3. 77199

表 3

图 2(b) 中沿  $y = b/2$  的挠度  $W = 10^{-4} \theta_3 q a^4 / D$ 

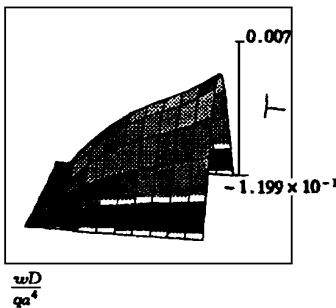
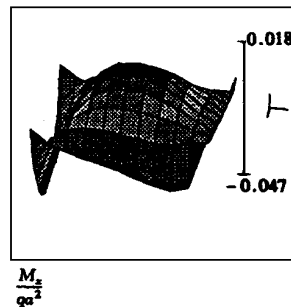
方 法		0.00	0.20	0.40	0.60	0.80	1.00
	$\theta_3$						
CFSM	0.010	0.00000	6.55554	15.4589	21.2355	24.5391	28.1293
	0.050	0.00000	6.97237	16.1028	22.0514	25.5652	29.5578
	0.100	0.00000	8.22497	18.0462	24.4502	28.3700	33.0218
	0.200	0.00000	12.9441	25.4479	33.5676	38.9509	45.8283
	0.300	0.00000	20.5314	37.4128	48.3423	56.2470	66.9160
	0.500	0.00000	44.4473	72.1544	95.0206	112.284	134.568
TRM	0.010	0.000	6.563	15.536	21.547	25.437	30.227
	0.050	0.0000	7.002	16.213	22.400	26.499	31.699
	0.100	0.0000	8.823	18.265	24.937	29.466	35.351
	0.200	0.0000	13.318	26.146	34.703	40.829	48.974
	0.300	0.0000	21.408	38.986	50.621	59.362	71.031
	0.500	0.000	47.169	79.779	100.942	117.679	139.823

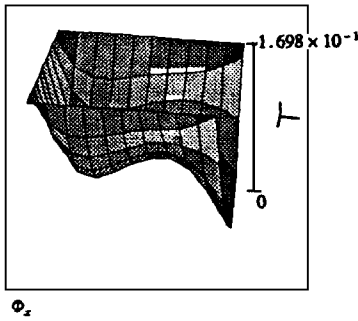
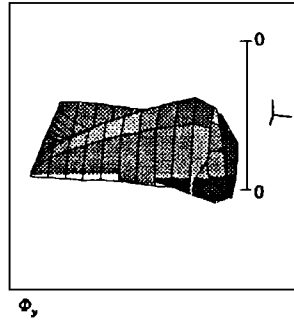
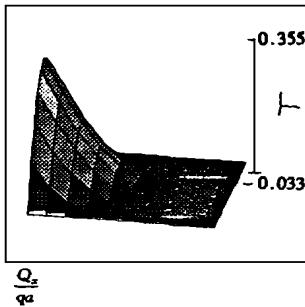
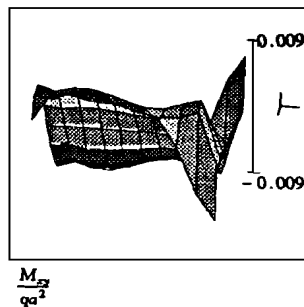
## 2 正确性验证

对于受力情况如图 2, 材料常数选为  $E = 200\text{GPa}$ ,  $\nu = 0.30$ , 计算结果如表 2~ 3 所示 (表中 CFSM 及 TRM 分别指本文使用的复级数方法和文献[3]使用的功的互等定理法)。表 2 中, 当长宽比  $b/a$  及厚长比  $h/a$  变化时, 本文所得板的中心处弯矩与已有结果十分接近; 虽然两者位移有一定差距, 但  $h/a$  很小时, 本文结果更接近薄板解。表 3 中, 固定长宽比  $b/a = 1$ , 当厚宽比  $h/a$  变化时, 本文所得的沿特定方向的位移分布亦与文献[3]吻合。

## 3 边界条件的满足情况

计算方案的实用性很大程度上取决于内力的收敛性及边界条件的满足情况, 我们采用算例 1 的计算参数, 对于三边固定一边自由的情形进行计算。图 3~ 8 分别为位移、转角、弯矩、扭矩、剪力的计算结果。如图所示, 它们均具有良好的对称性, 并很好地满足了边界条件。

图 3 厚板挠度  $W$  分布图 4 厚板弯矩  $M_x$  分布

图 5 厚板转角  $\Phi_x$  分布图 6 厚板转角  $\Phi_y$  分布图 7 厚板剪力  $Q_x$  分布图 8 厚板扭矩  $M_{xy}$  分布

### § 3. 简单结论

a) 本文第一次建立了 Reissner 厚板横向弯曲的一般解析解, 该解具有良好的收敛速度, 不仅对于位移和转角有良好的求解精度, 很好地满足了边界条件, 而且对于弯矩、扭矩、剪力等高阶量也有良好的求解精度, 并很好地满足了边界条件。

b) 本文利用所得解析解针对厚板的一些实际情况(如承受均布载荷)进行了大量的计算验证, 得出了一些有益的结论, 如对厚板而言, 标准化后的位移随板厚比  $h/a$  增大而明显增大, 而弯矩基本不变等。

c) 本文提出的求解常系数偏微分方程组的复级数展开方法, 还可推广运用于钢筋混凝土场坪等问题的各向异性厚板的弯曲、振动及稳定的分析研究。

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# General Analytic Solution for Elastic Bending of Reissner Plates

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## Abstract

In this paper, by developing the complex Fourier series method to solve the boundary value problem of a system of partial differential equations with constant coefficients, for the first time a general analytic solution satisfying an arbitrary boundary condition is presented for the elastic bending of thick Reissner plates in engineering. The solution is simple and convenient to programming. Analysis and computation are performed for the uniformly loaded plates under two different supporting conditions (four simply supported edges or three clamped and one free edges), the results of which are fairly satisfactory in comparison with those available for reference. And at the same time the analytic solution has been investigated mainly in three aspects: a) speed of convergence; b) reliability(rationality); c) fitting of boundary conditions.

**Key words** bending of Reissner plates, complex Fourier series method, general analytic solution