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圆柱壳开孔的应力集中
——非圆孔问题的一般解

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摘　要

本文从Donnell型圆柱壳的基本方程出发，利用复变函数方法和保角映射技术，对圆柱壳开非圆形孔的问题进行了研究。首先给出了逼近具有非圆形孔的圆柱壳开孔问题一般解的完备函数序列，构造出了问题的一般解；其次利用有关圆柱壳开小孔的假设概念，给出了圆柱壳开非圆孔时边界条件的一般表达式。进而利用正交函数展开的方法，将待解的问题归结为一组无穷代数方程组的求解问题，并进行直接求解。在本文最后，对圆柱壳开圆孔，椭圆孔附近的应力集中问题进行了数值计算，给出了分析结果。

关键词　圆柱壳，柱壳开非圆孔，复变函数方法，保角映射技术

一、前　　言

由于圆柱壳开孔问题在工程结构，特别是在航天器、舰船及压力容器等结构设计中最具普遍性与重要性，以及求解圆柱壳开孔应力集中问题理论上所遇到的困难，则使它已成为薄壳理论研究中的一个重要课题。从50年代起，国内外已投入了大量的人力和物力，从事这方面的分析，实验和计算研究工作[8~11]。

自А. И. Лурье[1]于1946年首次给出圆柱壳开小圆孔问题的渐近解答至今的50年中，曾有大批学者进行这方面的研究工作，并使其得以发展。此时，所谓开小圆孔在数学上是由条件$\frac{r_0}{\sqrt{Rh}}\ll 1$所限定的圆柱壳开孔问题。其中$R$为圆柱壳半径；$h$为壳体厚度；$r_0$为圆孔半径。A.C.Eringen和A.K.Naghdi[3],P.Van.Dyke[4]，J.G.Lekkerkeker[2]，钱令希[5]，龙驭球[6]，Steel.C.R等[8,9]及张丕辛和黄克智等[10]都对圆柱壳开孔问题给出了很好的解答。有关以上工作详细的评述可参考徐秉汉的专著[10]和薛明德的文献[12]。

作者认为，以上大部分研究工作，虽对Лурье的工作有重要改进，但均没突破有关小孔这一重要假设。因为他们在处理边界条件时，多采取了用圆柱壳切平面上开孔边界线替代

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了其在展开面上的开孔边界线，以近似地处理边界条件．众所周知，要保证这个替代条件成立，则只能限制圆柱壳上开一个小孔．80年代Steel[8,9]要突破这一限制，他也遇到了数学上的困难，并没获得全面的成功[12]．

本文采用有关圆柱壳开小孔的假定，对圆柱壳开非圆孔的应力集中问题进行了研究．作者认为，关于小孔的假设，在力学上可理解为：非圆孔在圆柱壳的切平面上与其在展开面上开孔线差别是甚小的．虽然采用了这一重要假设，但由于求解非圆开孔问题在数学上的困难，使得非圆开孔问题迟迟没有获得很好的解决[9]．

本文从Donnell型圆柱壳基本控制方程出发，构造圆柱壳开非圆孔应力集中问题的一般解．首先，使Donnell型圆柱壳控制方程降阶，因为降阶后的圆柱壳方程与弹性波动方程皆属于Helmholtz型，所以可直接利用复变函数与保角映射技术对其进行了求解[1]，并可在映射平面上给出逼近问题的一般解的完备函数序列，构造出一般解；另外，在映射平面上，给出了孔边边界条件的一般表达式．进一步利用这个一般解和边界条件表达式，则可将圆柱壳开非圆孔问题化为一组无穷代数方程组的求解．这样就给出了在不同的边界条件下，求解圆柱壳开任意形状孔应力集中问题的统一的分析和计算方法．

二、圆柱壳的控制方程及其一般解

1 Donnell型圆柱壳的控制方程

设φ_0和w_0为圆柱壳开孔前的膜应力函数和法向位移函数，它们是已知的．而圆柱壳开孔后开孔引起的附加膜应力函数φ和法向位移函数w则是待定的，它们应满足Donnell方程[9]．

$$D\nabla^2\nabla^2 w+\frac{1}{R}\frac{\partial^2\varphi}{\partial x^2}=0 \tag{2.1}$$

$$\frac{1}{Eh}\nabla^2\nabla^2\varphi-\frac{1}{R}\frac{\partial^2 w}{\partial x^2}=0 \tag{2.2}$$

式中 D为圆柱壳的抗弯刚度，$D=\dfrac{Eh^3}{12(1-\nu^2)}$；

E，ν分别为壳体的弹性模量和泊松比；

R，h分别为壳体的曲率半径和厚度；

x为沿柱壳母线之坐标线；y为沿柱壳周线之坐标线；

∇^2为拉普拉斯算子．

在x-y坐标系中，圆柱壳中的内力及内力矩可表示为

$$\left.\begin{aligned}&N_x=\frac{\partial^2\varphi}{\partial y^2},\quad N_y=\frac{\partial^2\varphi}{\partial x^2}\\&N_{xy}=-\frac{\partial^2\varphi}{\partial x\partial y}\\&M_x=-D\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right)\\&M_y=-D\left(\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}\right)\end{aligned}\right\} \tag{2.3}$$

$$\left.\begin{aligned} M_{xy}&=-D(1-\nu)\frac{\partial^2 w}{\partial x\partial y}\\ Q_x&=-D\frac{\partial}{\partial x}(\nabla^2 w),\quad Q_y=-D\frac{\partial}{\partial y}(\nabla^2 w)\end{aligned}\right\}$$

采用混合法求解壳体开孔应力集中问题，可引入复函数

$$\sigma(x,\ y)=w+\frac{i}{\sqrt{DEh}}\varphi \tag{2.4}$$

这样，(2.1)和(2.2)组成的控制方程组，可以归并为一个方程

$$\nabla^2\nabla^2\sigma-4\alpha^2\frac{\partial^2\sigma}{\partial x^2}=0 \tag{2.5}$$

式中 α为圆柱壳的曲率参数，

$$\alpha=\frac{1}{2}\frac{[12(1-\nu^2)]^{\frac{1}{4}}}{\sqrt{Rh}}\exp[i\frac{\pi}{4}]$$

方程(2.5)的解可表示成

$$\sigma=\sigma_1+\sigma_2 \tag{2.6}$$

其中 σ_1和σ_2分别满足下列方程式

$$\left.\begin{aligned}\nabla^2\sigma_1-2\alpha\frac{\partial\sigma_1}{\partial x}=0\\ \nabla^2\sigma_2+2\alpha\frac{\partial\sigma_2}{\partial x}=0\end{aligned}\right\} \tag{2.7}$$

设

$$\left.\begin{aligned}\sigma_1=\exp[\alpha x]u_1(x,y)\\ \sigma_2=\exp[-\alpha x]u_2(x,y)\end{aligned}\right\} \tag{2.8}$$

这样，函数$u_i(i=1,2)$应满足下列形式的方程

$$\nabla^2 u-\alpha^2 u=0 \tag{2.9}$$

用复变函数方法，引进复变量，$\zeta=x+iy$，$\bar{\zeta}=x-iy$，方程(2.9)可变成如下形式

$$\frac{\partial^2 u}{\partial\zeta\partial\bar{\zeta}}-\left(\frac{\alpha}{2}\right)^2 u=0 \tag{2.10}$$

2 圆柱壳控制方程的一般解

在求解薄圆柱壳中任意形状开孔附近的应力集中问题时，可使用保角映射方法．将ζ平面上非圆开孔之边界L的外域（或内域）映射为η平面上边界为S的一个单位圆之外域（或内域）．若令L和S的双方皆为无限，且无穷远点相对应．则映射函数应具有如下形式

$$\zeta=\omega(\eta)=C\eta+\text{全纯函数} \tag{2.11}$$

所谓函数在无限域内全纯，是指在此域内的任意有限部分全纯，同时，当$|\eta|$充分大时，式(2.11)可表达成如下的形式

$$m_0+\frac{m_1}{\eta}+\frac{m_2}{\eta^2}+\cdots$$

$m_i(i=1,2,\cdots)$为常数．为保证映射函数的单值性，在S域内$\omega'(\eta)$不能为零．

在η平面上，方程(2.10)可表示为

$$\frac{\partial^2 u}{\partial\eta\partial\bar{\eta}}-\left(\frac{\alpha}{2}\right)^2\omega'(\eta)\,\overline{\omega'(\eta)}\,u=0 \tag{2.12}$$

由文献[7]可知，方程(2.12)所决定薄圆柱壳控制方程的一般解可表示为如下形式

$$u=\sum_{-\infty}^{+\infty}E_n H_n^{(1)}(i\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^n i^{n+1} \tag{2.13}$$

其中 E_n为任意复常数；$H_n^{(1)}(\cdot)$为第一类n阶Hankel函数。

分别代式(2.13)至(2.8)，最后再代至式(2.6)，则有

$$\sigma=\sigma_1+\sigma_2=\exp\left[\frac{\alpha}{2}\left(\omega'(\eta)+\overline{\omega'(\eta)}\right)\right]\sum_{-\infty}^{+\infty}F_n H_n^{(1)}(i\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^n i^{n+1}$$

$$+\exp\left[-\frac{\alpha}{2}\left(\omega'(\eta)+\overline{\omega'(\eta)}\right)\right]\sum_{-\infty}^{+\infty}G_n H_n^{(1)}(i\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^n i^{n+1} \tag{2.14}$$

式中 F_n，G_n为任意复常数.

设

$$F_n=A_n+iB_n,\quad G_n=C_n+iD_n$$

$$\Phi_n^{(1)}=\exp\left[\frac{\alpha}{2}\omega(\eta)+\overline{\omega(\eta)}\right]H_n^{(1)}(i\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^n i^{n+1}=u_n^{(1)}+iu_n^{(2)}$$

$$\Phi_n^{(2)}=\exp\left[-\frac{\alpha}{2}\left(\omega(\eta)+\overline{\omega(\eta)}\right)\right]H_n^{(1)}(i\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^n i^{n+1}=u_n^{(3)}+iu_n^{(4)}$$

因此，可有如下式子

$$\sigma_1=\sum_{-\infty}^{+\infty}F_n\Phi_n^{(1)},\quad \sigma_2=\sum_{-\infty}^{+\infty}G_n\Phi_n^{(2)}$$

$$\sigma=\sigma_1+\sigma_2=\sum_{-\infty}^{+\infty}F_n\Phi_n^{(1)}+\sum_{-\infty}^{+\infty}G_n\Phi_n^{(2)}=\omega+\frac{i}{\sqrt{DEh}}\varphi$$

$$w=\sum_{-\infty}^{+\infty}w_n=\sum_{-\infty}^{+\infty}\left[A_n u_n^{(1)}-B_n u_n^{(2)}+C_n u_n^{(3)}-D_n u_n^{(4)}\right] \tag{2.15}$$

$$\varphi=\sqrt{DEh}\sum_{-\infty}^{+\infty}\varphi_n$$

$$=\sqrt{DEh}\sum_{-\infty}^{+\infty}\left[A_n u_n^{(2)}-B_n u_n^{(1)}+C_n u_n^{(4)}-D_n u_n^{(3)}\right] \tag{2.16}$$

式中 A_n，B_n，C_n，D_n为4个未知的实常数，而$u_n^{(i)}\,(i=1,2,3,4)$为4个实函数. 未知的实常数可用孔边的边界条件来确定.

三、内力、内力矩及边界条件

1 内力、内力矩的表达式

利用内力与应力函数及内力矩与法向位移函数的关系式，引进复变量ζ和$\bar{\zeta}$，在ζ平面上，可得内力及内力矩的复合形式如下

$$\left.\begin{aligned}
&N_x+N_y=4\frac{\partial^2\varphi}{\partial\zeta\partial\bar{\zeta}}=\nabla^2\varphi\\
&N_y-N_x+2iN_{xy}=4\frac{\partial^2\varphi}{\partial\zeta^2}\\
&M_x+M_y=-4D(1+\nu)\frac{\partial^2 w}{\partial\zeta\partial\bar{\zeta}}=-D(1+\nu)\nabla^2 w\\
&M_y-M_x+2iM_{xy}=4D(1-\nu)\frac{\partial^2 w}{\partial\zeta^2}\\
&Q_x-iQ_y=-8D\frac{\partial}{\partial\zeta}\left[\frac{\partial^2 w}{\partial\zeta\partial\zeta}\right]=-2D\frac{\partial(\nabla^2 w)}{\partial\zeta}
\end{aligned}\right\}\tag{3.1}$$

在映射平面η上，式(3.1)可有如下形式

$$\left.\begin{aligned}
&N_\rho+N_\theta=\frac{4}{\omega'(\eta)\overline{\omega'(\eta)}}\frac{\partial^2\varphi}{\partial\eta\partial\bar{\eta}}\\
&N_\rho-N_\theta+2iN_{\rho\theta}=\frac{4\eta^2}{\rho^2\overline{\omega'(\eta)}}\frac{\partial}{\partial\eta}\left[\frac{1}{\omega'(\eta)}\frac{\partial\varphi}{\partial\eta}\right]\\
&M_\rho+M_\theta=-\frac{4D(1+\nu)}{\omega'(\eta)\overline{\omega'(\eta)}}\frac{\partial^2 w}{\omega\eta\partial\bar{\eta}}\\
&M_\theta-M_\rho+2iM_{\rho\theta}=\frac{4(1-\nu)\eta^2}{\rho^2\overline{\omega'(\eta)}}\frac{\partial}{\partial\eta}\left[\frac{1}{\omega'(\eta)}\frac{\partial w}{\partial\eta}\right]\\
&Q_\rho-iQ_\theta=-\frac{2D\eta}{\rho|\omega'(\eta)|}\frac{\partial}{\partial\eta}(\nabla^2 w)
\end{aligned}\right\}\tag{3.2}$$

若将(3.2)中前两式相减，及第三式与第四式相减，则有

$$N_\rho-iN_{\rho\theta}=\frac{1}{2}\nabla^2\varphi-\frac{2\eta^2}{\rho^2\overline{\omega'(\eta)}}\frac{\partial}{\partial\eta}\left[\frac{1}{\omega'(\eta)}\frac{\partial\varphi}{\partial\eta}\right]\tag{3.3}$$

$$M_\rho-iM_{\rho\theta}=-\frac{1}{2}D(1+\nu)\nabla^2 w-\frac{2D(1-\nu)\eta^2}{\rho^2\overline{\omega'(\eta)}}\frac{\partial}{\partial\eta}\left[\frac{1}{\omega'(\eta)}\frac{\partial w}{\partial\eta}\right]\tag{3.4}$$

2 边界条件

现在讨论边界条件一般表达式，所有这些讨论都在η平面上进行。

当圆柱壳承受外部荷载作用时，在开孔边界上可给出力的控制条件。此时，由式(3.3)～(3.4)可得边界条件的表达式，即当$\eta=\exp[i\theta]$时，可有

$$\begin{aligned}
N_\rho=&\frac{2}{\omega'(\eta)\overline{\omega'(\eta)}}\frac{\partial^2\varphi}{\partial\eta\partial\bar{\eta}}-\frac{\eta^2}{\overline{\omega'(\eta)}}\frac{\partial}{\partial\eta}\left[\frac{1}{\omega'(\eta)}\frac{\partial\varphi}{\partial\eta}\right]\\
&-\frac{\bar{\eta}^2}{\omega'(\eta)}\frac{\partial}{\partial\bar{\eta}}\left[\frac{1}{\overline{\omega'(\eta)}}\frac{\partial\varphi}{\partial\bar{\eta}}\right]+N_\rho^0=F_1
\end{aligned}$$

$$\left.\begin{aligned} N_{\rho\theta}=&-i\frac{\eta^2}{\overline{\omega'(\eta)}}\frac{\partial}{\partial\eta}\left[\frac{1}{\omega'(\eta)}\frac{\partial\varphi}{\partial\eta}\right]\\ &+i\frac{\bar{\eta}^2}{\omega'(\eta)}\frac{\partial}{\partial\bar{\eta}}\left[\frac{1}{\overline{\omega'(\eta)}}\frac{\partial\varphi}{\partial\bar{\eta}}\right]+N^0_{\rho\theta}=F_2\\ M_\rho=&-\frac{2D(1+\nu)}{\omega'(\eta)\overline{\omega'(\eta)}}\frac{\partial^2 w}{\partial\eta\partial\bar{\eta}}-D(1-\nu)\frac{\eta^2}{\overline{\omega'(\eta)}}\frac{\partial}{\partial\eta}\left[\frac{1}{\omega'(\eta)}\frac{\partial w}{\partial\eta}\right]\\ &-D(1-\nu)\frac{\bar{\eta}^2}{\omega'(\eta)}\frac{\partial}{\partial\bar{\eta}}\left[\frac{1}{\overline{\omega'(\eta)}}\frac{\partial w}{\partial\bar{\eta}}\right]+M^0_\rho=F_3 \end{aligned}\right\}\tag{3.5}$$

和等效剪力V_ρ如下

$$\left.\begin{aligned} V_\rho=&Q_\rho+\frac{1}{|\omega'(\eta)|}\frac{\partial M_{\rho\theta}}{\partial\theta}+V^0_\rho\\ =&-D(5-\nu)\frac{\eta}{|\omega'(\eta)|}\frac{\partial}{\partial\eta}\left[\frac{1}{\omega'(\eta)\overline{\omega'(\eta)}}\frac{\partial^2 w}{\partial\eta\partial\bar{\eta}}\right]\\ &-D(5-\nu)\frac{\bar{\eta}}{|\omega'(\eta)|}\frac{\partial}{\partial\bar{\eta}}\left[\frac{1}{\omega'(\eta)\overline{\omega'(\eta)}}\frac{\partial w}{\partial\eta\partial\bar{\eta}}\right]\\ &+2D(1-\nu)\frac{\bar{\eta}^2}{|\omega'(\eta)|\overline{\omega'(\eta)}}\frac{\partial}{\partial\eta}\left[\frac{1}{\omega'(\eta)}\frac{\partial w}{\partial\eta}\right]\\ &+2D(1-\nu)\frac{\bar{\eta}^2}{|\omega'(\eta)|\omega'(\eta)}\frac{\partial}{\partial\bar{\eta}}\left[\frac{1}{\overline{\omega'(\eta)}}\frac{\partial w}{\partial\bar{\eta}}\right]\\ &+D(1-\nu)\frac{\bar{\eta}^3}{|\omega'(\eta)|\overline{\omega'(\eta)}}\frac{\partial^2}{\partial\eta^2}\left[\frac{1}{\omega'(\eta)}\frac{\partial w}{\partial\eta}\right]\\ &+D(1-\nu)\frac{\eta^3}{|\omega'(\eta)|\omega'(\eta)}\frac{\partial^2}{\partial\bar{\eta}^2}\left[\frac{1}{\overline{\omega'(\eta)}}\frac{\partial w}{\partial\bar{\eta}}\right]\\ &+D(1-\nu)\frac{\omega''(\eta)\bar{\eta}}{|\omega'(\eta)|[\omega'(\eta)]^2}\frac{\partial}{\partial\bar{\eta}}\left[\frac{1}{\overline{\omega'(\eta)}}\frac{\partial w}{\partial\bar{\eta}}\right]\\ &+D(1-\nu)\frac{\overline{\omega''(\eta)\eta}}{|\omega'(\eta)|[\overline{\omega'(\eta)}]^2}\frac{\partial}{\partial\eta}\left[\frac{1}{\omega'(\eta)}\frac{\partial w}{\partial\eta}\right]+V^0_\rho=F_4 \end{aligned}\right\}\tag{3.6}$$

式中，$F_i\,(i=1,2,3,4)$为圆柱壳开孔边界上给定的广义内力函数；N^0_ρ，$N^0_{\rho\theta}$，M^0_ρ，V^0_ρ为无开孔时的基本应力状态，即薄膜应力状态所对应的广义内力状态值。

力的边值问题其边界条件表达式如下

$$\begin{aligned} N_\rho=&-\frac{\sqrt{DEh}}{4}\sum_{-\infty}^{+\infty}(A_n\mathrm{Im}+B_n\mathrm{Re})\Big\{2\alpha^2[\Phi^{(1)}_{n-1}-2\Phi^{(1)}_n+\Phi^{(1)}_{n+1}]\\ &+\alpha^2\frac{\eta^2\omega'(\eta)}{\overline{\omega'(\eta)}}[\Phi^{(1)}_{n-2}-2\Phi^{(1)}_{n-1}+\Phi^{(1)}_n]+\alpha^2\frac{\eta^{-2}\overline{\omega'(\eta)}}{\omega'(\eta)}[\Phi^{(1)}_n-2\Phi^{(1)}_{n+1}+\Phi^{(1)}_{n+2}]\Big\}\\ &+\frac{\sqrt{DEh}}{4}\sum_{-\infty}^{+\infty}(C_nI_m+D_nR_e)\Big\{2\alpha^2[\Phi^{(2)}_{n-1}+2\Phi^{(2)}_{n+1}+\Phi^{(2)}_{n\pm 2}]\\ &-\alpha^2\frac{\eta^2\omega'(\eta)}{\overline{\omega'(\eta)}}[\Phi^{(2)}_{n-2}+2\Phi^{(2)}_{n-1}+\Phi^{(2)}_n] \end{aligned}$$

$$
\left.\begin{aligned}
&-\alpha^2\frac{\bar{\eta}^2\overline{\omega'(\eta)}}{\omega'(\eta)}[\Phi_n^{(2)}+2\Phi_{n+1}^{(2)}+\Phi_{n+2}^{(2)}]\Big\}+N_\rho^0=F_1\\
&N_{\rho\theta}=-\frac{\sqrt{DEh}}{4}\sum_{-\infty}^{+\infty}(A_n\mathrm{Re}-B_n\mathrm{Im})\Big\{\alpha^2\frac{\eta^2\omega'(\eta)}{\overline{\omega'(\eta)}}[\Phi_{n-2}^{(1)}-2\Phi_{n-1}^{(1)}+\Phi_n^{(1)}]\\
&\quad-\alpha^2\frac{\bar{\eta}^2\overline{\omega'(\eta)}}{\omega'(\eta)}[\Phi_n^{(1)}-2\Phi_{n+1}^{(1)}+\Phi_{n+2}^{(1)}]\Big\}\\
&\quad-\frac{\sqrt{DEh}}{4}\sum_{-\infty}^{+\infty}(C_n\mathrm{Re}-D_n\mathrm{Im})\Big\{\alpha^2\frac{\eta^2\omega'(\eta)}{\overline{\omega'(\eta)}}[\Phi_{n-2}^{(2)}+2\Phi_{n-1}^{(2)}+\Phi_n^{(2)}]\\
&\quad-\alpha^2\frac{\bar{\eta}^2\overline{\omega'(\eta)}}{\omega'(\eta)}[\Phi_n^{(2)}+2\Phi_{n+1}^{(2)}+\Phi_{n+2}^{(2)}]\Big\}+N_{\rho\theta}^0=F_2\\
&M_\rho=\frac{D}{4}\sum_{-\infty}^{+\infty}(A_n\mathrm{Re}-B_n\mathrm{Im})\Big\{2\alpha^2(1+\nu)[\Phi_{n-1}^{(1)}-2\Phi_n^{(1)}+\Phi_{n+1}^{(1)}]\\
&\quad-\alpha^2(1-\nu)\frac{\eta^2\omega'(\eta)}{\overline{\omega'(\eta)}}\Big[\Phi_{n-2}^{(1)}-2\Phi_{n-1}^{(1)}+\Phi_n^{(1)}\Big]\\
&\quad-\alpha^2(1-\nu)\frac{\bar{\eta}^2\overline{\omega'(\eta)}}{\omega'(\eta)}[\Phi_n^{(1)}-2\Phi_{n+1}^{(1)}+\Phi_{n+2}^{(1)}]\Big\}\\
&\quad-\frac{D}{4}\sum_{-\infty}^{+\infty}(C_n\mathrm{Re}-D_n\mathrm{Im})\Big\{2\alpha^2(1+\nu)[\Phi_{n-1}^{(2)}+2\Phi_n^{(2)}+\Phi_{n+1}^{(2)}]\\
&\quad+\alpha^2(1-\nu)\frac{\eta^2\overline{\omega'(\eta)}}{\omega'(\eta)}[\Phi_{n-2}^{(2)}+2\Phi_{n-1}^{(2)}+\Phi_n^{(2)}]\\
&\quad+\alpha^2(1-\nu)\frac{\bar{\eta}^2\overline{\omega'(\eta)}}{\omega'(\eta)}[\Phi_n^{(2)}+2\Phi_{n+1}^{(2)}+\Phi_{n+2}^{(2)}]\Big\}+M_\rho^0=F_3\\
&V_\rho=-\frac{D}{8|\omega'(\eta)|}\sum_{-\infty}^{+\infty}(A_n\mathrm{Re}-B_n\mathrm{Im})\Big\{\alpha^3(1-\nu)\frac{\eta^3\omega'(\eta)}{\overline{\omega'(\eta)}}\\
&\quad\cdot[\Phi_{n-3}^{(1)}-3\Phi_{n-2}^{(1)}+3\Phi_{n-1}^{(1)}-\Phi_n^{(1)}]\\
&\quad-\alpha^3(1-\nu)\frac{\bar{\eta}^3\overline{\omega'(\eta)}}{\omega'(\eta)}[\Phi_n^{(1)}-3\Phi_{n+1}^{(1)}+3\Phi_{n+2}^{(1)}-\Phi_{n+3}^{(1)}]\\
&\quad-\frac{4(1-\nu)\alpha^2\eta^2\mathrm{Re}\{\omega'(\eta)[\overline{\omega'(\eta)}+\bar{\eta}\,\overline{\omega'(\eta)}]\}}{[\overline{\omega'(\eta)}]^2\omega'(\eta)}[\Phi_{n-2}^{(1)}-2\Phi_{n-1}^{(1)}+\Phi_n^{(1)}]\\
&\quad-\frac{4(1-\nu)\alpha^2\bar{\eta}^2\mathrm{Re}\{\omega'(\eta)[\overline{\omega'(\eta)}+\bar{\eta}\overline{\omega'(\eta)}]\}}{[\omega'(\eta)]^2\overline{\omega'(\eta)}}[\Phi_n^{(1)}-2\Phi_{n+1}^{(1)}+\Phi_{n+2}^{(1)}]\\
&\quad+\alpha^3(5-\nu)\eta\omega'(\eta)[\Phi_{n-2}^{(1)}-3\Phi_{n-1}^{(1)}+3\Phi_n^{(1)}-\Phi_{n+1}^{(1)}]\\
&\quad-\alpha^3(5-\nu)\bar{\eta}\overline{\omega'(\eta)}[\Phi_{n-1}^{(1)}-3\Phi_n^{(1)}+3\Phi_{n+1}^{(1)}-\Phi_{n+2}^{(1)}]\Big\}\\
&\quad-\frac{D}{8|\omega'(\eta)|}\sum_{-\infty}^{+\infty}(C_n\mathrm{Re}-D_n\mathrm{Im})\Big\{\alpha^3(1-\nu)\frac{\eta^3\omega'(\eta)}{\overline{\omega'(\eta)}}\\
&\quad\cdot[\Phi_{n-3}^{(2)}+3\Phi_{n-2}^{(2)}+3\Phi_{n-1}^{(2)}+\Phi_n^{(2)}]\\
&\quad+\alpha^3(1-\nu)\frac{\bar{\eta}^3\overline{\omega'(\eta)}}{\omega'(\eta)}[\Phi_n^{(2)}+3\Phi_{n+1}^{(2)}+3\Phi_{n+2}^{(2)}+\Phi_{n+3}^{(2)}]
\end{aligned}\right\}\quad(3.7)
$$

$$
\left.\begin{aligned}
&-\frac{4(1-\nu)\alpha^2\eta^2\mathrm{Re}\{\omega'(\eta)[\overline{\omega'(\eta)}+\bar{\eta}\overline{\omega'(\eta)}]\}}{[\overline{\omega'(\eta)}]^2\omega'(\eta)}[\Phi_{n-2}^{(2)}+2\Phi_{n-1}^{(2)}+\Phi_n^{(2)}] \\
&-\frac{4(1-\nu)\alpha^2\bar{\eta}^2\mathrm{Re}\{\omega'(\eta)[\overline{\omega'(\eta)}+\bar{\eta}\overline{\omega'(\eta)}]\}}{[\omega'(\eta)]^2\overline{\omega'(\eta)}}[\Phi_n^{(2)}+2\Phi_{n+1}^{(2)}+\Phi_{n+2}^{(2)}] \\
&-\alpha^3(5-\nu)\eta\omega'(\eta)[\Phi_{n-2}^{(2)}+3\Phi_{n-1}^{(2)}+3\Phi_n^{(2)}+\Phi_{n+1}^{(2)}] \\
&-\alpha^3(5-\nu)\bar{\eta}\overline{\omega'(\eta)}[\Phi_{n-1}^{(2)}+3\Phi_n^{(2)}+3\Phi_{n+1}^{(2)}+\Phi_{n+2}^{(2)}]\Big\}+V_\rho^0=F_4
\end{aligned}\right\rbrace
$$

式(3.7)即为开孔边界条件的一般表达式

四、圆柱壳开孔附近的应力集中

1 荷载

不失一般性，研究弹性圆柱壳中承受均匀内压时的壳体开孔应力集中问题．未开孔时，其由内压而构成的基本应力状态φ_0和w_0可以写成$\varphi_0=\frac{N_0}{2}\left(x^2+\frac{1}{2}y^2\right)$，$w_0=\frac{N_0R}{2Eh}(2-\nu)$，此时它们对应的基本内力为

$$
\left.\begin{aligned}
&N_x^0+N_y^0=\frac{3}{2}N_0 \\
&N_y^0-N_x^0+2iN_{xy}=\frac{1}{2}N_0 \\
&M_x^0=M_y^0=M_{xy}^0=0 \\
&Q_x^0=Q_y^0=0
\end{aligned}\right\rbrace \tag{4.1}
$$

式中　N_0为环向内力，$N_0=PR$；P为介质压力．

在η平面上，由式(4.1)可有

$$
\left.\begin{aligned}
&N_\rho^0=\frac{N_0}{8}\left[6-\frac{\eta^2\omega'(\eta)}{\overline{\omega'(\eta)}}-\frac{\bar{\eta}^2\overline{\omega'(\eta)}}{\omega'(\eta)}\right] \\
&N_{\rho\theta}^0=-\frac{N_0}{8}i\left[\frac{\eta^2\omega'(\eta)}{\overline{\omega'(\eta)}}-\frac{\bar{\eta}^2\overline{\omega'(\eta)}}{\omega'(\eta)}\right] \\
&M_\rho^0=M_\theta^0=M_{\rho\theta}^0=0 \\
&Q_\rho^0=Q_\theta^0=0
\end{aligned}\right\rbrace \tag{4.2}
$$

(4.2)式即为无开孔时的基本应力状态．

2 开孔附近的应力集中

现在研究薄圆柱壳承受荷载作用时，任意形状开孔附近的应力集中问题．假设开孔的边界条件是自由的．这样，圆柱壳开孔附近的应力状态应为膜应力与开孔扰动应力状态的叠加，也就是说基本应力状态与开孔应力状态构成了总的应力场

$$
\left.\begin{aligned}
&\varphi=\varphi^1+\varphi^0 \\
&w=w^1+w^0
\end{aligned}\right\rbrace \tag{4.3}
$$

力的边界条件表达式为

$$
\left.\begin{aligned}
&N_\rho^1+N_\rho^0=0,\quad N_{\rho\theta}^1+N_\rho^0=0 \\
&M_\rho^1+M_\rho^0=0,\quad V_\rho^1+V_\rho^0=0
\end{aligned}\right\rbrace \tag{4.4}
$$

式中 N_{ρ}^{1}，$N_{\rho\theta}^{1}$，M_{ρ}^{1}，V_{ρ}^{1}表示开孔扰动应力状态所对应的广义内力.

由边界条件(4.4)及有关式子，可得

$$\sum_{j=1}^{4}\left(\sum_{-\infty}^{+\infty}\epsilon_{n}^{ij}X_{n}^{j}\right)=\epsilon_{i}\qquad(i=1,2.3.4)\tag{4.5}$$

其中

$$\begin{aligned}\epsilon_{n}^{11}=\mathrm{Im}\Big\{&2\alpha^{2}\Big[iH_{n-1}^{(1)}(i\alpha|\omega(\eta)|)\Big\{\frac{\omega(\eta)}{|\omega(\eta)|}\Big\}^{n-1}\\&+2H_{n}^{(1)}(i\alpha|\omega(\eta)|)\Big\{\frac{\omega(\eta)}{|\omega(\eta)|}\Big\}^{n}-iH_{n+1}^{(1)}(i\alpha|\omega(\eta)|)\Big\{\frac{\omega(\eta)}{|\omega(\eta)|}\Big\}^{n+1}\Big]\\&+\alpha^{2}\frac{\eta^{2}\omega'(\eta)}{\overline{\omega'(\eta)}}\Big[H_{n-1}^{(1)}(i\alpha|\omega(\eta)|)\Big\{\frac{\omega(\eta)}{|\omega(\eta)|}\Big\}^{n-2}\\&-2iH_{n-1}^{(1)}(i\alpha|\omega(\eta)|)\Big\{\frac{\omega(\eta)}{|\omega(\eta)|}\Big\}^{n-1}-H_{n}^{(1)}(i\alpha|\omega(\eta)|)\Big\{\frac{\omega(\eta)}{|\omega(\eta)|}\Big\}^{n}\Big]\\&-\alpha^{2}\frac{\overline{\eta}^{2}\overline{\omega'(\eta)}}{\omega'(\eta)}\Big[H_{n}^{(1)}(i\alpha|\omega(\eta)|)\Big\{\frac{\omega(\eta)}{|\omega(\eta)|}\Big\}^{n}\\&-2iH_{n+1}^{(1)}(i\alpha|\omega(\eta)|)\Big\{\frac{\omega(\eta)}{|\omega(\eta)|}\Big\}^{n+1}-H_{n+2}^{(1)}(i\alpha|\omega(\eta)|)\Big\{\frac{\omega(\eta)}{|\omega(\eta)|}\Big\}^{n+2}\Big]\Big\}\\&\cdot\exp\Big[\frac{\alpha}{2}(\omega(\eta)+\overline{\omega(\eta)})\Big]i^{n+1}=\mathrm{Im}\delta_{1}\end{aligned}$$

$$\epsilon_{n}^{12}=\mathrm{Re}\ \delta_{1}$$

$$\begin{aligned}\epsilon_{n}^{13}=-\mathrm{Im}\Big\{&2\alpha^{2}\Big[iH_{n-1}^{(1)}(i\alpha|\omega(\eta)|)\Big\{\frac{\omega(\eta)}{|\omega(\eta)|}\Big\}^{n-1}\\&-2H_{n}^{(1)}(i\alpha|\omega(\eta)|)\Big\{\frac{\omega(\eta)}{|(\omega(\eta)|}\Big\}^{n}-iH_{n+1}^{(1)}(i\alpha|\omega(\eta)|)\Big\{\frac{\omega(\eta)}{|\omega(\eta)|}\Big\}^{n+1}\Big]\\&-\alpha^{2}\frac{\eta^{2}\omega'(\eta)}{\overline{\omega'(\eta)}}\Big[H_{n-2}^{(1)}(i\alpha|\omega(\eta)|)\Big\{\frac{\omega(\eta)}{|\omega(\eta)|}\Big\}^{n-2}\\&+2iH_{n-1}^{(1)}(i\alpha|\omega(\eta)|)\Big\{\frac{\omega(\eta)}{|\omega(\eta)|}\Big\}^{n-1}-H_{n}^{(1)}(i\alpha|\omega(\eta)|)\Big\{\frac{\omega(\eta)}{|\omega(\eta)|}\Big\}^{n}\Big]\\&+\alpha^{2}\frac{\overline{\eta}^{2}\overline{\omega'(\eta)}}{\omega'(\eta)}\Big[H_{n}^{(1)}(i\alpha|\omega(\eta)|)\Big\{\frac{\omega(\eta)}{|\omega(\eta)|}\Big\}^{n}\\&+2iH_{n+1}^{(1)}(i\alpha|\omega(\eta)|)\Big\{\frac{\omega(\eta)}{|\omega(\eta)|}\Big\}^{n+1}-H_{n\pm2}^{(1)}(i\alpha|\omega(\eta)|)\Big\{\frac{\omega(\eta)}{|\omega(\eta)|}\Big\}^{n+2}\Big]\Big\}\\&\cdot\exp\Big[-\frac{\alpha}{2}(\omega(\eta)+\overline{\omega(\eta)})\Big]i^{n+1}=-\mathrm{Im}\ \delta_{2}\end{aligned}$$

$$\epsilon_{n}^{14}=-\mathrm{Re}\ \delta_{2}$$

$$\begin{aligned}\epsilon_{n}^{21}=\mathrm{Re}\Big\{&\alpha^{2}\frac{\eta^{2}\omega'(\eta)}{\overline{\omega'(\eta)}}\Big[H_{n-2}^{(1)}(i\alpha|\omega(\eta)|)\Big\{\frac{\omega(\eta)}{|\omega(\eta)|}\Big\}^{n-2}\\&-2iH_{n-1}^{(1)}(i\alpha|\omega(\eta)|)\Big\{\frac{\omega(\eta)}{|\omega(\eta)|}\Big\}^{n-1}-H_{n}^{(1)}(i\alpha|\omega(\eta)|)\Big\{\frac{\omega(\eta)}{|\omega(\eta)|}\Big\}^{n}\Big]\end{aligned}$$

$$+\alpha^2\frac{\bar{\eta}^2\omega'(\eta)}{\omega'(\eta)}\left[H_n^{(1)}(i\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^n\right.$$

$$\left.-2iH_{n+1}^{(1)}(i\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n+1}-H_{n+2}^{(1)}(i\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n+2}\right]\Bigg\}$$

$$\cdot\exp\left[\frac{\alpha}{2}(\omega(\eta)+\overline{\omega(\eta)})\right]i^{n+1}=\mathrm{Re}\ \delta_3$$

$$\epsilon_n^{22}=-\mathrm{Im}\ \delta_3$$

$$\epsilon_n^{23}=\mathrm{Re}\Bigg\{\alpha^2\frac{\eta^2\omega'(\eta)}{\overline{\omega'(\eta)}}\left[H_{n-2}^{(1)}(i\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n-2}\right.$$

$$\left.+2iH_{n-1}^{(1)}(i\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n-1}-H_n^{(1)}(i\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^n\right]$$

$$+\alpha^2\frac{\bar{\eta}^2\overline{\omega'(\eta)}}{\omega'(\eta)}\left[H_n^{(1)}(i\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^n\right.$$

$$\left.+2iH_{n+1}^{(1)}(i\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n+1}-H_{n+2}^{(1)}(i\alpha|\omega(\eta|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n+2}\right]\Bigg\}$$

$$\cdot\exp\left[-\frac{\alpha}{2}(\omega(\eta)+\overline{\omega(\eta)})\right]i^{n+1}=\mathrm{Re}\ \delta_4$$

$$\epsilon_n^{24}=-\mathrm{Im}\ \delta_4$$

$$\epsilon_n^{31}=-\mathrm{Re}\Bigg\{2\alpha^2(1+\nu)\left[iH_{n-1}^{(1)}(i\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n-1}\right.$$

$$\left.+2iH_n^{(1)}(i\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^n-iH_{n+1}^{(1)}(i\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n+1}\right]$$

$$-\alpha^2(-\nu)\frac{\eta^2\omega'(\eta)}{\overline{\omega'(\eta)}}\left[H_{n-2}^{(1)}(i\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{\omega|(\eta)|}\right\}^{n-2}\right.$$

$$\left.-2iH_{n-1}^{(1)}(i\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n-1}-H_n^{(1)}(i\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^n\right]$$

$$+\alpha^2(1-\nu)\frac{\bar{\eta}^2\overline{\omega'(\eta)}}{\omega'(\eta)}\left[H_n^{(1)}(i\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^n\right.$$

$$\left.-2iH_{n+1}^{(1)}(i\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n+1}-H_{n+2}^{(1)}(i\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n+2}\right]\Bigg\}$$

$$\cdot\exp\left[\frac{\alpha}{2}(\omega(\eta)+\overline{\omega(\eta)})\right]i^{n+1}=\mathrm{Re}\ \delta_5$$

$$\epsilon_n^{32}=\mathrm{Im}\ \delta_6$$

$$\epsilon_n^{33}=\mathrm{Re}\Bigg\{2\alpha^2(1+\nu)\left[iH_{n-1}^{(1)}(i\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n-1}\right.$$

$$\left.-2H_n^{(1)}(i\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^n-iH_{n+1}^{(1)}(i\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n+1}\right]$$

$$+\alpha^2(1-\nu)\frac{\eta^2\omega'(\eta)}{\omega'(\eta)}\left[H_{n-2}^{(1)}(i\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n-2}\right.$$

$$-2iH_{n-1}^{(1)}(i\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n}-H_{n}^{(1)}(i\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n}\Bigg]$$

$$-\alpha^{2}(1-\nu)\frac{\bar{\eta}^{2}\overline{\omega'(\eta)}}{\omega'(\eta)}\Bigg[H_{n}^{(1)}(i\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n}$$

$$+2iH_{n+1}^{(1)}(i\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n+1}-H_{n+2}^{(1)}(i\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n+2}\Bigg]\Bigg\}$$

$$\cdot\exp\Big[-\frac{\alpha}{2}(\omega(\eta)+\overline{\omega(\eta)})\Big]i^{n+1}=\mathrm{Re}\,\delta_{6}$$

$$\epsilon_{n}^{34}=-\mathrm{Im}\,\delta_{6}$$

$$\epsilon_{n}^{41}=-\mathrm{Re}\Bigg\{4\alpha^{2}(1-\nu)\frac{\eta^{2}}{[\overline{\omega'(\eta)}]^{2}}\mathrm{Re}\{\omega'(\eta)[\overline{\omega'(\eta)}+\bar{\eta}^{2}\overline{\omega''(\eta)}]\}$$

$$\cdot\Bigg[H_{n-2}^{(1)}(i\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n-2}+2iH_{n-1}^{(1)}(i\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n-1}$$

$$-H_{n}^{(1)}(i\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n}\Bigg]$$

$$-4\alpha^{2}(1-\nu)\frac{\bar{\eta}^{2}}{[\omega'(\eta)]^{2}}\mathrm{Re}\{\omega'(\eta)[\overline{\omega'(\eta)}+\bar{\eta}\overline{\omega''(\eta)}]\}$$

$$\cdot\Bigg[H_{n}^{(1)}(i\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n}-2iH_{n+1}^{(1)}(i\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n+1}$$

$$-H_{n+2}^{(1)}(i\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n+2}\Bigg]$$

$$+\alpha^{3}(1-\nu)\frac{\eta^{3}\omega'(\eta)}{\overline{\omega'(\eta)}}\Bigg[iH_{n-3}^{(1)}(i\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n-3}$$

$$+3H_{n-2}^{(1)}(i\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n-2}$$

$$-3iH_{n-1}^{(1)}(i\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n-1}-H_{n}^{(1)}(i\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n}\Bigg]$$

$$-\alpha^{3}(1-\nu)\frac{\bar{\eta}^{3}\overline{\omega'(\eta)}}{\omega'(\eta)}\Bigg[H_{n}^{(1)}(i\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n}$$

$$-3iH_{n+1}^{(1)}(i\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n+1}$$

$$-3H_{n+2}^{(1)}(i\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n+2}+iH_{n+3}^{(1)}(i\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n+3}\Bigg]$$

$$-\alpha^{3}(5-\nu)\eta\omega'(\eta)\Bigg[H_{n-2}^{(1)}(i\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n-2}$$

$$-3iH_{n-1}^{(1)}(i\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n-1}$$

$$-3H_{n}^{(1)}(i\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n}+iH_{n+1}^{(1)}(i\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n+1}$$

$$+\alpha^3(5-\nu)\bar{\eta}\overline{\omega'(\eta)}\Big[iH^{(1)}_{n-1}(i\alpha|\omega(\eta)|)\Big\{\frac{\omega(\eta)}{|\omega(\eta)|}\Big\}^{n-1}$$

$$+3H^{(1)}_{n}(i\alpha|\omega(\eta)|)\Big\{\frac{\omega(\eta)}{|\omega(\eta)|}\Big\}^{n}$$

$$-3iH^{(1)}_{n+1}(i\alpha|\omega(\eta)|)\Big\{\frac{\omega(\eta)}{|\omega(\eta)|}\Big\}^{n+1}-H^{(1)}_{n+2}(i\alpha|\omega(\eta)|)\Big\{\frac{\omega(\eta)}{|\omega(\eta)|}\Big\}^{n+2}\Big]\Big\}$$

$$\cdot\exp\Big[\frac{\alpha}{2}(\omega(\eta)+\overline{\omega(\eta)})\Big]\frac{i^{n+1}}{|\omega'(\eta)|}=-\mathrm{Re}\,\delta_7$$

$$\epsilon^{42}_n=\mathrm{Im}\,\delta_7$$

$$\epsilon^{43}_n=-\mathrm{Re}\Big\{4\alpha^2(1-\nu)\frac{\eta^2}{[\overline{\omega'(\eta)}]^2}\mathrm{Re}\{\omega'(\eta)[\overline{\omega'(\eta)}+\bar{\eta}\overline{\omega''(\eta)}]\}$$

$$\cdot\Big[H^{(1)}_{n-2}(i\alpha|\omega(\eta)|)\Big\{\frac{\omega(\eta)}{|\omega(\eta)|}\Big\}^{n-2}+2iH^{(1)}_{n-1}(i\alpha|\omega(\eta)|)\Big\{\frac{\omega(\eta)}{|\omega(\eta)|}\Big\}^{n-1}$$

$$-H^{(1)}_{n}(i\alpha|\omega(\eta)|)\Big\{\frac{\omega(\eta)}{|\omega(\eta)|}\Big\}^{n}\Big]$$

$$-4\alpha^2(1-\nu)\frac{\bar{\eta}^2}{[\omega'(\eta)]^2}\mathrm{Re}\{\omega'(\eta)[\overline{\omega'(\eta)}+\bar{\eta}\overline{\omega''(\eta)}]\}$$

$$\cdot\Big[H^{(1)}_{n}(i\alpha|\omega(\eta)|)\Big\{\frac{\omega(\eta)}{\bar{\eta}|\omega(\eta)|}\Big\}^{n}+2iH^{(1)}_{n+1}(i\alpha|\omega(\eta)|)\Big\{\frac{\omega(\eta)}{|\omega(\eta)|}\Big\}^{n+1}$$

$$-H^{(1)}_{n+2}(i\alpha|\omega(\eta)|)\Big\{\frac{\omega(\eta)}{|\omega(\eta)|}\Big\}^{n+2}\Big]$$

$$+\alpha^3(1-\nu)\frac{\eta^3\omega'(\eta)}{\overline{\omega'(\eta)}}\Big[iH^{(1)}_{n-3}(i\alpha|\omega(\eta)|)\Big\{\frac{\omega(\eta)}{|\omega(\eta)|}\Big\}^{n-3}$$

$$-3H^{(1)}_{n-2}(i\alpha|\omega(\eta)|)\Big\{\frac{\omega(\eta)}{|\omega(\eta)|}\Big\}^{n-2}$$

$$-3iH^{(1)}_{n-1}(i\alpha|\omega(\eta)|)\Big\{\frac{\omega(\eta)}{|\omega(\eta)|}\Big\}^{n-1}+H^{(1)}_{n}(i\alpha|\omega(\eta)|)\Big\{\frac{\omega(\eta)}{|\omega(\eta)|}\Big\}^{n}\Big]$$

$$+\alpha^3(1-\nu)\frac{\bar{\eta}^3\overline{\omega'(\eta)}}{\omega'(\eta)}\Big[iH^{(1)}_{n}(i\alpha|\omega(\eta)|)\Big\{\frac{\omega(\eta)}{|\omega(\eta)|}\Big\}^{n}$$

$$+3iH^{(1)}_{n+1}(i\alpha|\omega(\eta)|)\Big\{\frac{\omega(\eta)}{|\omega(\eta)|}\Big\}^{n+1}$$

$$-3H^{(1)}_{n+2}(i\alpha|\omega(\eta)|)\Big\{\frac{\omega(\eta)}{|\omega(\eta)|}\Big\}^{n+2}-iH^{(1)}_{n+3}(i\alpha|\omega(\eta)|)\Big\{\frac{\omega(\eta)}{|\omega(\eta)|}\Big\}^{n+3}$$

$$+\alpha^3(5-\nu)\eta\omega'(\eta)\Big[H^{(1)}_{n-2}(i\alpha|\omega(\eta)|)\Big\{\frac{\omega(\eta)}{|\omega(\eta)|}\Big\}^{n-2}$$

$$+3iH^{(1)}_{n-1}(i\alpha|\omega(\eta)|)\Big\{\frac{\omega(\eta)}{|\omega(\eta)|}\Big\}^{n-1}$$

$$-3H^{(1)}_{n}(i\alpha|\omega(\eta)|)\Big\{\frac{\omega(\eta)}{|\omega(\eta)|}\Big\}^{n}-iH^{(1)}_{n+1}(i\alpha|\omega(\eta)|)\Big\{\frac{\omega(\eta)}{|\omega(\eta)|}\Big\}^{n+1}\Big]$$

$$+\alpha^3(5-\nu)\bar{\eta}\overline{\omega'(\eta)}\Big[iH^{(1)}_{n-1}(i\alpha|\omega(\eta)|)\Big\{\frac{\omega(\eta)}{|\omega(\eta)|}\Big\}^{n-1}$$

$$-3H^{(1)}_n(i\alpha|\omega(\eta)|)\Big\{\frac{\omega(\eta)}{|\omega(\eta)|}\Big\}^n$$

$$-3iH^{(1)}_{n+1}(i\alpha|\omega(\eta)|)\Big\{\frac{\omega(\eta)}{|\omega(\eta)|}\Big\}^{n+1}+H^{(1)}_{n+2}(i\alpha|\omega(\eta)|)\Big\{\frac{\omega(\eta)}{|\omega(\eta)|}\Big\}^{n+2}\Big]\Big\}$$

$$\cdot\exp\Big[-\frac{\alpha}{2}(\omega(\eta)+\overline{\omega(\eta)})\Big]\frac{i^{n+1}}{|\omega'(\eta)|}=-\mathrm{Re}\,\delta_8$$

$$\epsilon_n^{44}=\mathrm{Im}\,\delta_8$$

$$\epsilon_1=-\frac{N_0}{2\sqrt{DEh}}\Big[6-\frac{\eta^2\omega'(\eta)}{\omega'(\eta)}-\frac{\bar{\eta}^2\overline{\omega'(\eta)}}{\omega'(\eta)}\Big]$$

$$\epsilon_2=\frac{N_0 i}{2\sqrt{DEh}}\Big[\frac{\eta^2\omega'(\eta)}{\omega'(\eta)}-\frac{\bar{\eta}^2\overline{\omega'(\eta)}}{\omega'(\eta)}\Big],\quad \epsilon_3=0,\quad \epsilon_4=0$$

另外

$$X_n^1=A_n,\quad X_n^2=B_n$$

$$X_n^3=C_n,\quad X_n^4=D_n$$

将方程式(4.5)两边乘以$\exp[-is\theta]$，并在区间$(-\pi,\ \pi)$上积分正交，则可以得到

$$\sum_{j=1}^{4}\Big(\sum_{-\infty}^{+\infty}\epsilon_{ns}^{ij}X_n^j\Big)=\epsilon_{is}\qquad(i=1,\ 2,\ 3,\ 4;\ s=0,\ \pm1,\ \pm2,\ \cdots)\tag{4.6}$$

其中

$$\epsilon_{ns}^{ij}=\frac{1}{2\pi}\int_{-\pi}^{\pi}\epsilon_n^{ij}\exp[-is\theta]d\theta$$

$$\epsilon_{is}=\frac{1}{2\pi}\int_{-\pi}^{\pi}\epsilon_i\exp[-is\theta]d\theta$$

方程式(4.6)即为决定未知系数A_n，B_n，C_n，D_n的无穷代数方程组。

3 应力集中系数

弹性圆柱壳承受内压作用时，确定开孔周边上的应力集中系数是研究问题的热点。圆柱壳承受内压时，壳体的应力为双向应力状态，且未开孔前的最大膜应力（膜力）$N_y=N_0$。应力集中系数可理解为：开孔周边上任一点的应力与圆柱壳无开孔时环向应力的比值。由于假设开孔周边上给出应力自由的边界条件，则沿开孔周边上只有环向的膜应力及弯曲应力。膜应力（膜力）集中系数应为

$$N_\theta^*=\frac{N_\theta}{N_0}\tag{4.7}$$

式中 N_θ，M_θ为开孔周边上任意一点的环向膜力及环向弯矩。

孔边周向膜力和周向弯矩可分别表示为

$$N_\theta=N_\theta^1+N_\theta^0$$

$$=\sqrt{DEh}\sum_{-\infty}^{+\infty}(A_n\mathrm{Im}+B_n\mathrm{Re})\Big\{\alpha^2\Big[iH^{(1)}_{n-1}(i\alpha|\omega(\eta)|)\Big\{\frac{\omega(\eta)}{|\omega(\eta)|}\Big\}^{n-1}$$

$$+2H_n^{(1)}(i\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^n-iH_{n+1}^{(1)}(i\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n+1}\Big]\Big\}$$

$$\cdot\exp\left[\frac{\alpha}{2}\omega(\eta)+\overline{\omega(\eta)}\ )\right]i^{n+1}$$

$$-\sqrt{DEh}\sum_{-\infty}^{+\infty}(C_n\mathrm{Im}+D_n\mathrm{Re})\left\{\alpha^2\Big[iH_{n-1}^{(1)}(i\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n-1}\right.$$

$$-2H_n^{(1)}(i\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^n-iH_{n+1}^{(1)}(i\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n+1}\Big]\Big\}$$

$$\cdot\exp\left[-\frac{\alpha}{2}(\omega(\eta)+\overline{\omega(\eta)})\right]i^{n+1}+\frac{3}{2}N_0 \tag{4.8}$$

弯曲应力集中系数为

$$M_\theta^*=\frac{6M_\theta}{hN_0} \tag{4.9}$$

式中 M_θ为开孔周边上任一点的环向弯矩，它可以表示为

$$M_\theta=M_\theta^1+M_\theta^0$$

$$=-D\sum_{-\infty}^{+\infty}(A_n\mathrm{Re}-B_n\mathrm{Im})\left\{\alpha^2(1+\nu)\Big[iH_{n-1}^{(1)}(i\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n-1}\right.$$

$$+2H_n^{(1)}(i\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^n-iH_{n+1}^{(1)}(i\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n+1}\Big]\Big\}$$

$$\cdot\exp\left[\frac{\alpha}{2}\omega(\eta)+\overline{\omega(\eta)}\ )\right]i^{n+1}$$

$$+D\sum_{-\infty}^{+\infty}(C_n\mathrm{Re}-D_n\mathrm{Im})\left\{\alpha^2(1+\nu)\Big[iH_{n-1}^{(1)}(i\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n-1}\right.$$

$$-2H_n^{(1)}(i\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^n-iH_{n+1}^{(1)}(i\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n+1}\Big]\Big\}$$

$$\cdot\exp\left[-\frac{\alpha}{2}(\omega(\eta)+\overline{\omega(\eta)}\ )\right]i^{n+1} \tag{4.10}$$

这样，圆柱壳自由开孔边界条件下，膜应力及弯曲应力集中系数的一般表达式分别为

$$N_\theta^*=\frac{\sqrt{DEh}}{N_0}\sum_{-\infty}^{+\infty}(A_n\mathrm{Im}+B_n\mathrm{Re})\left\{\alpha^2\Big[iH_{n-1}^{(1)}(i\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n-1}\right.$$

$$+2H_n^{(1)}(i\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^n-iH_{n+1}^{(1)}(i\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n+1}\Big]\Big\}$$

$$\cdot\exp\left[\frac{\alpha}{2}(\omega(\eta)+\overline{\omega(\eta)}\ )\right]i^{n+1}$$

$$-\frac{\sqrt{DEh}}{N_0}\sum_{-\infty}^{+\infty}(C_n\mathrm{Im}+D_n\mathrm{Re})\left\{\alpha^2\Big[iH_{n-1}^{(1)}(i\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n-1}\right.$$

$$-2H_n^{(1)}(i\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^n-iH_{n+1}^{(1)}(i\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n+1}\Bigg]\Bigg\}$$

$$\cdot\exp\left[-\frac{\alpha}{2}(\omega(\eta)+\overline{\omega(\eta)})\right]i^{n+1}+\frac{3}{2} \tag{4.11}$$

$$M_\theta^*=-\frac{6D(1+\nu)}{hN_0}\sum_{-\infty}^{+\infty}(A_n\mathrm{Re}-B_n\mathrm{Im})\left\{\alpha^2\left[iH_{n-1}^{(1)}(i\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n-1}\right.\right.$$

$$+2H_n^{(1)}(i\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^n-iH_{n+1}^{(1)}(i\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n+1}\Bigg]\Bigg\}$$

$$\cdot\exp\left[\frac{\alpha}{2}(\omega(\eta)+\overline{\omega(\eta)})\right]i^{n+1}$$

$$+\frac{6D(1+\nu)}{hN_0}\sum_{-\infty}^{+\infty}(C_n\mathrm{Re}-D_n\mathrm{Im})\left\{\alpha^2\left[iH_{n-1}^{(1)}(i\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n-1}\right.\right.$$

$$-2H_n^{(1)}(i\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^n-iH_{n+1}^{(1)}(i\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n+1}\Bigg]\Bigg\}$$

$$\cdot\exp\left[-\frac{\alpha}{2}(\omega(\eta)+\overline{\omega(\eta)})\right]i^{n+1} \tag{4.12}$$

五、求解算例

上述分析可以很方便地用于计算圆柱壳自由开孔边界的应力集中系数。下面给定保角映射变换，计算圆柱壳开圆孔及椭圆孔的应力集中系数，以说明本文的复变函数方法适用于求解薄圆柱壳开任意形状小孔的应力集中问题。

1 圆形开孔附近的应力集中

圆柱壳上含有一个半径为a的圆孔，此时映射函数可取为

$$\zeta=\omega(\eta)=r_0\eta \tag{5.1}$$

式中 $r_0=a$。

代式(5.1)至(4.6)，取$n=s=8$，$\nu=0.30$，图1，2的上半部给出了膜应力集中系数，而下半部给出了弯曲应力集中系数沿周边的变化；图3，4分别给出了膜应力集中系数（$\theta=0$），弯曲应力集中系数($\theta=0$)随曲率参数$\frac{r_0}{\sqrt{Rh}}$的变化规律。

2 椭圆形开孔附近的应力集中

圆柱壳上含有一个长短轴半长分别为b和a的椭圆形孔，若长轴和短轴分别置于y和x轴，则将其映射成单位圆的映射函数为

$$\zeta=\omega(\eta)=r_0\left(\eta+\frac{m}{\eta}\right) \tag{5.2}$$

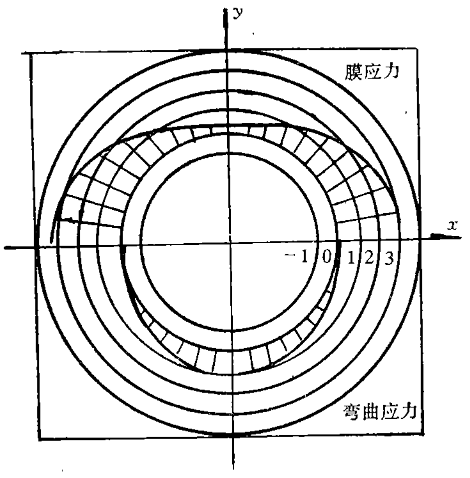

图1 $a/b=1.0$，$r_0/\sqrt{Rh}=0.5$时，圆孔沿孔边膜应力及弯曲应力分布

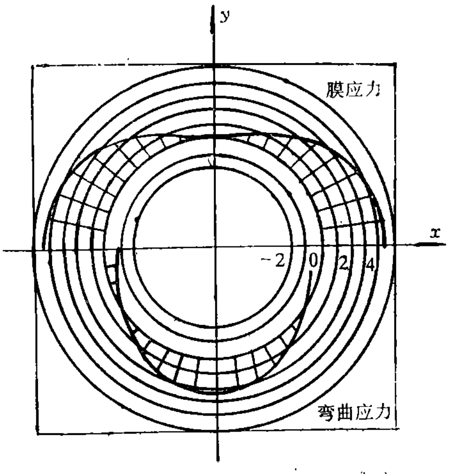

图2 $a/b=1.0$，$r_0/\sqrt{Rh}=1.0$时，圆孔沿孔边膜应力及弯曲应力分布

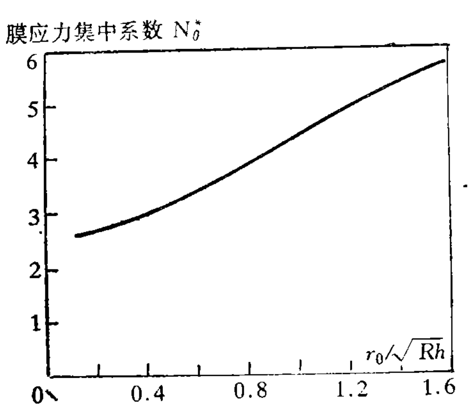

图3 $a/b=1.0$，$\theta=0$时，圆孔膜应力集中系数随壳体曲率参数变化示意图

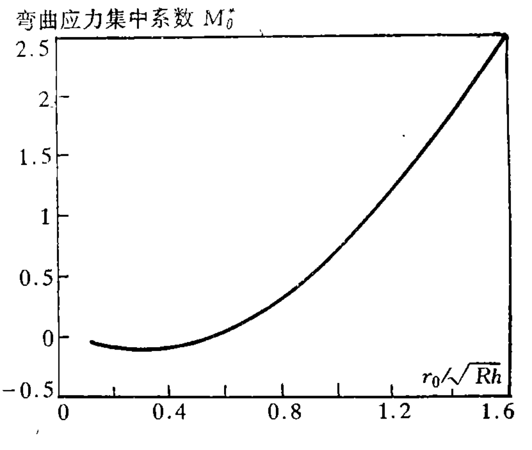

图4 $a/b=1.0$，$\theta=0$时，圆孔弯曲应力集中系数随壳体曲率参数变化示意图

式中 $r_0=\frac{1}{2}(a+b)$， $m=\frac{a-b}{a+b}$.

代式(5.2)至(4.6)，取$n=s=11$，$\nu=0.30$，长短轴之比$\frac{a}{b}=\frac{3}{4}$. 图5，6的上半部及下半部分别给出了膜应力集中系数及弯曲应力集中系数沿开孔周边的变化；图7，8分别给出了膜应力集中系数和弯曲应力集中系数($\theta=0$)随曲率参数$\frac{r_0}{\sqrt{Rh}}$的变化规律。

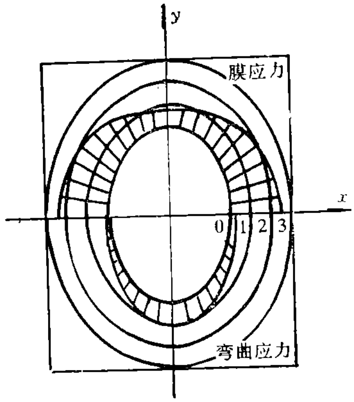

图5 $a/b=3/4$, $r_0/\sqrt{Rh}=0.5$时，椭圆孔沿孔边膜应力及弯曲应力分布

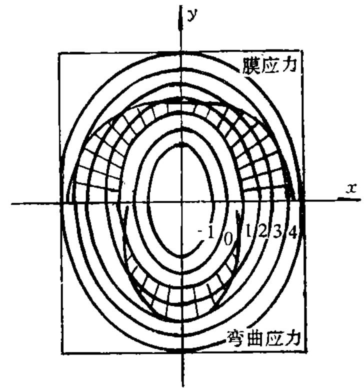

图6 $a/b=3/4$, $r_0/\sqrt{Rh}=1.0$时，椭圆孔沿孔边膜应力及弯曲应力分布

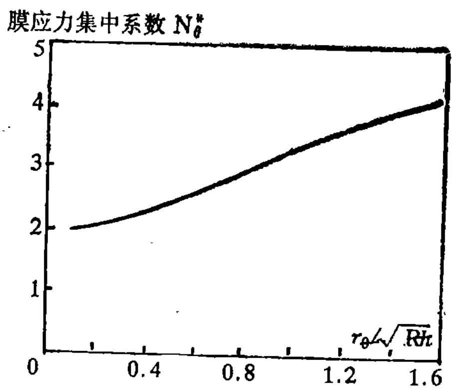

图7 $a/b=3/4$, $\theta=0$时，椭圆孔膜应力集中系数随壳体曲率参数变化示意图

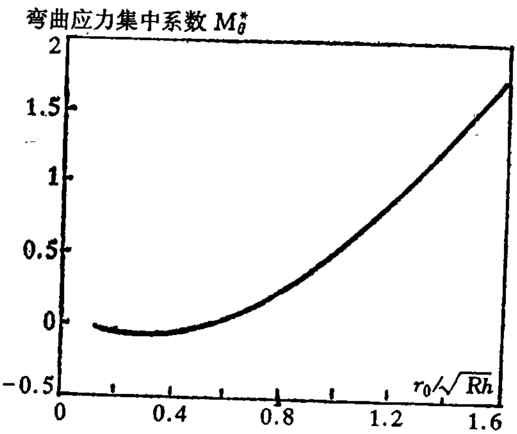

图8 $a/b=3/4$, $\theta=0$时，椭圆孔弯曲应力集中系数随壳体曲率参数变化示意图

六、讨　论

通过计算可知，对于圆柱壳含有圆形开孔，本文分析计算结果完全与文献[10]的结果相同．说明本文分析方法是可靠的，对于求解圆柱壳开非圆孔的应力集中问题具有统一和规范化之优点，是一种可推广的新方法．

通过对圆柱壳开孔应力集中问题的分析，计算可以看到：

（1） 圆柱壳开孔应力集中系数与壳体曲率参数$\frac{r_0}{\sqrt{Rh}}$有和，对圆孔而言r_0为开孔半径，而对非圆孔而言，它是体现在映射函数中的一个特征参数，与非圆孔的特征尺寸有关．

（2） 作者认为，以上结果只适用于圆柱壳开小孔的情况，因为它假定了开孔在柱壳切平面与在展开面上的开孔边界线相差很小．如果不放弃这一限制是很难解决圆柱壳开大孔问

题的，做再多的努力也是无效的。

（3） 本文给出的薄圆柱壳开非圆孔问题解的一般表达式是精确的，对本文中的算例，只取$n=s=8$（圆孔），$n=s=11$即可获得较好的数值结果。为确定项数n和s，可通过回代和验算给定的边界条件来实现。

（4）本文是作为一种新的分析方法而加以研究的，选例也较为简单，要真正了解圆柱壳开非圆孔之一般规律，还要做大量的研究工作。

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The General Solution for the Stress Problem of Circular Cylindrical Shells with an Arbitrary Cutout

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Abstract

In this paper a complex variable analytic method for solving stress concentrations in the circular cylindrical shell is proposed. The problem to be solved can be summerized into the solution of an infinite algebraic equation series. The solution can be normal and effective by means of this method. Numerical results for stress concentrations in the shell with a circular, elliptic cutout are graphically presented.

Key words circular cylindrical shell, non-circular cutout, stress concentration, complex variable method and conformal mapping technique.