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两相材料空间问题基本解的显式张量表示*

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摘　　要

本文应用张量运算将文献中的三维两相无限体的集中力基本解表示为张量形式，从而使其能够直接用于边界积分方程和边界元方法，以分析两相材料空间弹性力学问题。本文结果包括了Mindlin问题、Lorentz问题和均质体空间问题的基本解。

关键词　三维两相材料　集中力基本解　张量表达式

一、引　　言

随着材料科学的发展，有关复合材料弹性力学问题的研究正受到愈来愈多的、从事固体力学的研究人员的广泛重视。在力学模型上，很多多相材料的弹性力学问题都是以两相材料的弹性力学问题为基础的。

尽管关于两相材料空间问题的基本解研究可以追溯到Rongved[1]的工作，但到目前为止，就作者所知，除Rongved[1]外，只有Dundurs和Hetenyi[2]、黄和王[3]以及乐[4]等先后对基本解也作了一些研究，但他们的解都未表示为方便的张量形式，这给我们在使用边界积分方程和边界元法研究有关断裂、夹杂和界面等弹性力学问题时带来了很多困难。本文作者正是基于上述理由，从实用和便利的角度出发，进一步完善[3]的工作。通过张量运算，求出与各向同性、均质弹性体全空间的Kelvin解相对应的各种显式张量表达式，此时，Mindlin问题、Lorentz问题以及均质体空间问题的基本解的张量表示，都可由本文的结果退化得到。

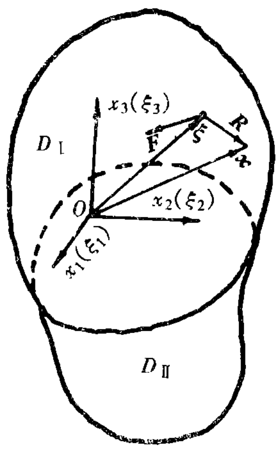

图1　两相材料空间构成

二、黄与王[3]的基本解

图1是三维两相材料无限弹性空间 D 的构图。上半空间和下半空间的量分用别上Ⅰ标和Ⅱ表示，μ_1，ν_1和μ_2,ν_2是

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对应于空间Ⅰ和Ⅱ的弹性常数. 下文要用到的张量指标约定如下：拉丁指标$(i,j,k,\cdots)$取值1，2，3，而希腊指标$(\alpha,\beta,\gamma,\cdots)$取值1,2. 此约定除以后特别申明外，均不再说明.

在图1所示的坐标系中，上半空间$D_{\rm I}$中的$P(\xi_1,\xi_2,\xi_3)$点有一集中力$\boldsymbol{F}(F_1,F_2,F_3)$作用，黄和王使用 Papkovich-Neuber 的位移势函数导出了两相材料空间中的位移基本解(u_1,u_2,u_3)[3]，即：

$$\left.\begin{aligned}u_\alpha^{\rm I}&=U_\alpha+A\varkappa_1\bar{U}_\alpha-2Ax_3\bar{U}_{3,\alpha}-Ax_3^2\nabla^2\bar{U}_\alpha-\beta[(1-2\nu_1)A\bar{K}_\alpha+\bar{G}_{,\alpha}]\\u_3^{\rm I}&=U_3+A_1\varkappa\bar{U}_3-2Ax_3\bar{U}_{\beta,\beta}+Ax_3^2\nabla^2\bar{U}_3-B_1\beta(\bar{K}_3+\bar{g})\end{aligned}\right\}\tag{2.1}$$

$$\left.\begin{aligned}u_\alpha^{\rm II}&=(1+A\varkappa_1)U_\alpha+B_2\beta x_3(K_3+g)_{,\alpha}-\beta[(1-2\nu_1)AK_\alpha+G_{,\alpha}]\\u_3^{\rm II}&=(1+A\varkappa_1)U_3+B_2\beta x_3(K_3+g)_{,3}-B_1\beta(K_3+g)\end{aligned}\right\}\tag{2.2}$$

式中A，B_1，B_2和β，$\varkappa_\alpha$均为材料常数，

$$\left.\begin{aligned}&\Gamma=\frac{\mu_2}{\mu_1},\quad A=\frac{1-\Gamma}{1+\varkappa_1\Gamma},\quad \beta=\frac{2}{1+\Gamma},\quad \varkappa_\alpha=3-4\nu_\alpha\\&B_1=\frac{1}{2}\left[\frac{1-2\nu_1}{1+\varkappa_1\Gamma}-\frac{(1-2\nu_2)\Gamma}{\varkappa_2+\Gamma}\right],\quad B_2=\left[\frac{1-2\nu_2}{\varkappa_2+\Gamma}-\frac{(1-2\nu_1)\Gamma}{1+\varkappa_1\Gamma}\right]\end{aligned}\right\}\tag{2.3}$$

函数U_i由下面均质体空间Kelvin解给出：

$$U_i=K_i-\frac{1}{\varkappa_1+1}(K_0+x_k\cdot K_k)_{,i}\tag{2.4}$$

其中

$$K_0=-\frac{\xi_kF_k}{4\mu_1\pi R},\quad K_l=\frac{F_l}{4\mu_1\pi R},\quad R^2=(x_k-\xi_k)(x_k-\xi_k)\tag{2.5}$$

函数g和G均为如下的调和函数：

$$\left.\begin{aligned}g&=-\frac{1}{4\mu_1\pi}\{[\ln(R-x_3+\xi_3)]_{,\beta}\cdot F_\beta\}\\G&=\frac{1}{4\mu_1\pi}\{(1-2\nu_1)A[-(x_3-\xi_3)\ln(R-x_3+\xi_3)-R]_{,\beta}\cdot F_\beta\\&\quad-B_1[-(x_3-\xi_3)\ln(R-x_3+\xi_3)-R]_{,i}\cdot F_i\}\end{aligned}\right\}\tag{2.6}$$

公式(2.1)和(2.2)中的共轭函数定义如下：

$$\left.\begin{aligned}&\bar{f}(x_1,x_2,x_3)=f(x_1,x_2,-x_3)\\&\overline{(\bar{f})}=f,\quad \bar{f}|_{x_3=0}=f|_{x_3=0}\\&\overline{f_{,\alpha}}=(\bar{f})_{,\alpha},\quad \overline{f_{,3}}=-(\bar{f})_{,3},\quad \overline{\nabla^2f}=\nabla^2(\bar{f})\end{aligned}\right\}\tag{2.7}$$

这里$(\ )_{,i}=\partial(\ )/\partial x_i$，$\nabla^2$为Laplace算子.

从实用的观点出发，以上公式不是最方便的显式张量表示，因此，不能像均质体的Kelvin基本解那样，很简便地用于边界积分方程和边界元方法. 下面使用张量运算，将其转化为显式张量表示，并用此进一步导出文献[3]尚未给出的面力基本解.

三、位移基本解的显式张量表示

运用笛卡尔张量的运算规则，则式(2.4)中的U_i和式(2.6)中的g和G均可用简便的张量形式显式地表示为

$$U_i=\frac{1}{4\mu_1\pi(\varkappa_1+1)R}\{\varkappa_1\delta_{ki}+R_{,i}R_{,k}\}\cdot F_k \tag{3.1}$$

$$\left.\begin{aligned}g&=-\frac{1}{4\mu_1\pi}\left\{\frac{\delta_{k\beta}R_{,\beta}}{R-x_3+\xi_3}\right\}\cdot F_k\\ G&=\frac{1}{4\mu_1\pi}\left\{-[(1-2\nu_1)A-B_1]\frac{\delta_{k\beta}RR_{,\beta}}{R-x_3+\xi_3}+B_1\delta_{k3}\ln(R-x_3+\xi_3)\right\}\cdot F_k\end{aligned}\right\} \tag{3.2}$$

式中$R_{,\beta}=(x_\beta-\xi_\beta)/R$。

把方程(3.1)和(3.2)代入(2.1)式和(2.2)式，并利用关系式（2.7)，则黄和王得到的在上半空间$D_{\rm I}$中位移基本解用张量表示为，

$$\begin{aligned}u^{\rm I}_{k\alpha}=\frac{1}{4\mu_1\pi(\varkappa_1+1)}&\Big\{(\varkappa_1\delta_{k\alpha}+R_{,\alpha}R_{,k})\frac{1}{R}A\varkappa_1(\varkappa_1\delta_{k\alpha}+\bar{R}_{,\alpha}\bar{R}_{,k})\frac{1}{\bar{R}}\\ &+2Ax_3(\varkappa_1\delta_{k3}\bar{R}_{,\alpha}-\delta_{k\alpha}\bar{R}_{,3}+3\bar{R}_{,3}\bar{R}_{,\alpha}\bar{R}_{,k})\frac{1}{\bar{R}^2}\\ &-2Ax_3^2(\delta_{k\alpha}-3\bar{R}_{,\alpha}\bar{R}_{,k})\frac{1}{\bar{R}^3}-A\beta(\varkappa_1+1)(1-2\nu_1)\frac{\delta_{k\alpha}}{\bar{R}}\\ &-B_1\beta(\varkappa_1+1)\frac{\delta_{k3}\bar{R}_{,\alpha}}{\bar{R}+x_3+\xi_3}\\ &+\beta(\varkappa_1+1)[(1-2\nu_1)A-B_1]\left[\frac{\delta_{k\alpha}}{\bar{R}+x_3+\xi_3}-\frac{\delta_{k\beta}\bar{R}\bar{R}_{,\alpha}\bar{R}_{,\beta}}{(\bar{R}+x_3+\xi_3)^2}\right]\Big\}\end{aligned} \tag{3.3}$$

$$\begin{aligned}u^{\rm I}_{k3}=\frac{1}{4\mu_1\pi(\varkappa_1+1)}&\Big\{(\varkappa_1\delta_{k3}+R_{,3}R_{,k})\frac{1}{R}+A\varkappa_1(\varkappa_1\delta_{k3}+\bar{R}_{,3}\bar{R}_{,k})\frac{1}{\bar{R}}\\ &+2Ax_3[(\varkappa_1-1)\delta_{k\beta}\bar{R}_{,\beta}-2\bar{R}_{,k}+3\bar{R}_{,k}\bar{R}_{,\beta}\bar{R}_{,\beta}]\frac{1}{\bar{R}_2}\\ &+2Ax_3^2(\delta_{k3}-3\bar{R}_{,3}\bar{R}_{,k})\frac{1}{\bar{R}^3}-B_1\beta(\varkappa_1+1)\left[\frac{\delta_{k3}}{\bar{R}}-\frac{\delta_{k\beta}\bar{R}_{,\beta}}{\bar{R}+x_3+\xi_3}\right]\Big\}\end{aligned} \tag{3.4}$$

使用以上结果(3.3)和（3.4)，便可很快地获得文献[4]的全部结果，因此本文的张量表示要比文[4]的结果优越，而且要简洁、紧揍得多[1)]。

在下半空间$D_{\rm II}$中位移基本解的张量表示为：

$$\begin{aligned}u^{\rm II}_{k\alpha}=\frac{1}{4\mu_1\pi(\varkappa_1+1)}&\Big\{(1+A\varkappa_1)(\varkappa_1\delta_{k\alpha}+R_{,\alpha}R_{,k})\frac{1}{R}\\ &-\beta(\varkappa_1+1)\left[(1-2\nu_1)A\frac{\delta_{k\alpha}}{R}+B_1\frac{\delta_{k3}R_{,\alpha}}{R-x_3+\xi_3}\right]\\ &-B_2\beta(\varkappa_1+1)x_3\left[\frac{\delta_{k3}R_{,\alpha}}{R^2}-\frac{\delta_{k\alpha}-\delta_{k\beta}R_{,\alpha}R_{,\beta}}{R(R-x_3+\xi_3)}-\frac{\delta_{k\beta}R_{,\alpha}R_{,\beta}}{(R-x_3+\xi_3)^2}\right]\\ &+B(\varkappa_1+1)[(1-2\nu_1)A-B_1]\left[\frac{\delta_{k\alpha}}{R-x_3+\xi_3}-\frac{\delta_{k\beta}RR_{,\alpha}R_{,\beta}}{(R-x_3+\xi_3)^2}\right]\Big\}\end{aligned} \tag{3.5}$$

1）文献[4]只给出了上半空间的解，没有给出下半空间的解，且其中的u_{13}和u_{31}有符号错误。

$$u^{\mathrm{I}}_{k3}=\frac{1}{4\mu_1\pi(\varkappa_1+1)}\left\{(1+A\varkappa_1)(\varkappa_1\delta_{k3}+R_{,3}R_{,k})\frac{1}{R}\right.$$

$$\left.-B_2\beta(\varkappa_1+1)x_3\frac{R_{,k}}{R^2}-B_1\beta(\varkappa_1+1)\left[\frac{\delta_{k3}}{R}-\frac{\delta_{k\beta}R_{,\beta}}{R-x_3+\xi_3}\right]\right\} \tag{3.6}$$

在上述u_{ij}的表达式中，第一个下标i代表作用在P点单位力的方向（见图1），而第二个下标j则表示在Q点所产生的位移方向.在传统的边界元法中，位移通常用U_{ij}来表示，但是由于在黄和王的论文[3]里，字符U已作他用.故这里仍遵循该文的用法.

四、面力基本解的显式张量表示

通过上面的位移场，便可求出域中任一点的应力σ_{ij}.之后，再使用Cauchy[5]应力公式，就可获得作用在任一单位外法线为$\boldsymbol{n}$的表面上应力矢量或面力（见图2）为，

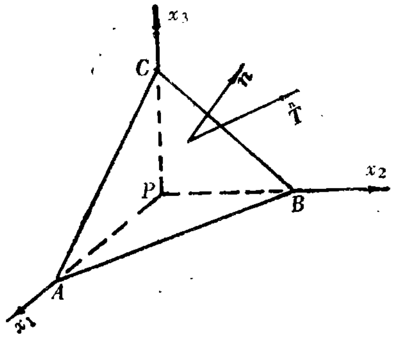

图2 任意面上面力示意图

$$\overset{n}{T}_i=\sigma_{ij}n_j \tag{4.1}$$

根据(4.1)式和广义Hooke定律，面力基本解用位移表示的张量形式可表为，

$$\left.\begin{aligned}\overset{n}{T}{}^{\mathrm{I}}_{k\alpha}&=\mu_1\left\{[u^{\mathrm{I}}_{k\alpha,\gamma}+u^{\mathrm{I}}_{k\gamma,\alpha}]\cdot n_\gamma+[u^{\mathrm{I}}_{k\alpha,3}+u^{\mathrm{I}}_{k3,\alpha}]\cdot n_3+\frac{2\nu_1}{1-2\nu_1}[u^{\mathrm{I}}_{k\gamma,\gamma}+u^{\mathrm{I}}_{k3,3}]\cdot n_\alpha\right\}\\ \overset{n}{T}{}^{\mathrm{I}}_{k3}&=\mu_1\left\{[u^{\mathrm{I}}_{k3,\gamma}+u^{\mathrm{I}}_{k\gamma,3}]\cdot n_\gamma+2[u^{\mathrm{I}}_{k3,3}]\cdot n_3+\frac{2\nu_1}{1-2\nu_1}[u^{\mathrm{I}}_{k\gamma,\gamma}+u^{\mathrm{I}}_{k3,3}]\cdot u_3\right\}\end{aligned}\right\} \tag{4.2}$$

$$\left.\begin{aligned}\overset{n}{T}{}^{\mathrm{II}}_{k\alpha}&=\mu_2\left\{[u^{\mathrm{II}}_{k\alpha,\gamma}+u^{\mathrm{II}}_{k\gamma,\alpha}]\cdot n_\gamma+[u^{\mathrm{II}}_{k\alpha,3}+u^{\mathrm{II}}_{k3,\alpha}]\cdot n_3+\frac{2\nu_2}{1-2\nu_2}[u^{\mathrm{II}}_{k\gamma,\gamma}+u^{\mathrm{II}}_{k3,3}]\cdot n_\alpha\right\}\\ \overset{n}{T}{}^{\mathrm{II}}_{k3}&=\mu_2\left\{[u^{\mathrm{II}}_{k3,\gamma}+u^{\mathrm{II}}_{k\gamma,3}]\cdot n_\gamma+2[u^{\mathrm{II}}_{k3,3}]\cdot n_3+\frac{2\nu_2}{1-2\nu_2}[u^{\mathrm{II}}_{k\gamma,\gamma}+u^{\mathrm{II}}_{k3,3}]\cdot n_3\right\}\end{aligned}\right\} \tag{4.3}$$

在经过繁琐的张量运算和整理之后，就可获得面力基本解的显式张量表达式，即：空间D_{I}中用张量表示的面力显式基本解为，

$$\overset{n}{T}{}^{\mathrm{I}}_{k\alpha}=\overset{n}{T}{}^{\mathrm{I}①}_{k\alpha}+\overset{n}{T}{}^{\mathrm{I}②}_{k\alpha}+\overset{n}{T}{}^{\mathrm{I}③}_{k\alpha}+\overset{n}{T}{}^{\mathrm{I}④}_{k\alpha}+\overset{n}{T}{}^{\mathrm{I}⑤}_{k\alpha} \tag{4.4}$$

这里，

$$\overset{n}{T}{}^{\mathrm{I}①}_{k\alpha}=\frac{1}{2\pi(\varkappa_1+1)R^2}\left\{-\frac{\partial R}{\partial n}[(1-2\nu_1)\delta_{k\alpha}+3R_{,\alpha}R_{,k}]\right.$$

$$\left.+(1-2\nu_1)[R_{,k}n_\alpha-R_{,\alpha}n_k]\right\}$$

$$\overset{n}{T}{}^{\mathrm{I}②}_{k\alpha}=\frac{A\varkappa_1}{2\pi(\varkappa_1+1)\bar{R}^2}\left\{[-(1-2\nu_1)\delta_{k\alpha}\bar{R}_{,\beta}-(1-2\nu_1)\delta_{k\beta}\bar{R}_{,\alpha}+\delta_{\alpha\beta}\bar{R}_{,k}\right.$$

$$-3\bar{R}_{,\alpha}\bar{R}_{,\beta}\bar{R}_{,k}]\cdot n_\beta+\frac{1}{2}(\varkappa_1+1)[\delta_{k\alpha}\bar{R}_{,3}-\delta_{k3}\bar{R}_{,\alpha}]\cdot n_3$$

$$\left.+\frac{\nu_1}{1-2\nu_1}[2(1-2\nu_1)(\delta_{k3}\bar{R}_{,3}-\delta_{k\beta}\bar{R}_{,\beta})+\bar{R}_{,k}+3\bar{R}_{,k}(\bar{R}^2_{,3}-\bar{R}_{,\beta}\bar{R}_{,\beta})]\cdot n_\alpha\right\}$$

$$+\frac{A}{2\pi(\varkappa_1+1)\bar{R}^2}\Big\{[\varkappa_1\delta_{k3}\bar{R}_{,\alpha}-\delta_{k\alpha}\bar{R}_{,3}-3\bar{R}_{,3}\bar{R}_{,\alpha}\bar{R}_{,k}]\cdot n_3$$

$$+\frac{2\nu_1}{1-2\nu_1}[2(1-2\nu_1)\delta_{k\beta}\bar{R}_{,\beta}-2\bar{R}_{,k}+3\bar{R}_{,k}\bar{R}_{,\beta}\bar{R}_{,\beta}]\cdot n_\alpha\Big\}$$

$$\overset{n}{T}{}^{\mathrm{I}\textcircled{3}}_{k\alpha}=\frac{Ax_3}{2\pi(\varkappa_1+1)\bar{R}^3}\Big\{[2\varkappa_1\delta_{k3}(\delta_{\alpha\beta}-3\bar{R}_{,\alpha}\bar{R}_{,\beta})+6(\delta_{k\alpha}\bar{R}_{,3}\bar{R}_{,\beta}+\delta_{k\beta}\bar{R}_{,3}\bar{R}_{,\alpha}$$

$$+\delta_{\alpha\beta}\bar{R}_{,3}\bar{R}_{,k})-30\bar{R}_{,3}\bar{R}_{,\alpha}\bar{R}_{,\beta}\bar{R}_{,k}]\cdot n_\beta+[(\varkappa_1-4)\delta_{k\alpha}+3(\varkappa_1-1)$$

$$\cdot(\delta_{k3}\bar{R}_{,3}-\delta_{k\beta}\bar{R}_{,\beta})\bar{R}_{,\alpha}+3(3\bar{R}_{,\alpha}+2\delta_{\alpha\beta}\bar{R}_{,\beta})\bar{R}_{,k}+3(\delta_{k\alpha}-5\bar{R}_{,\alpha}\bar{R}_{,k})$$

$$\cdot(\bar{R}^2_{,\beta}-\bar{R}^2_{,3})]\cdot n_3+\frac{2\nu_1}{1-2\nu_1}[2(\varkappa_1+2)\delta_{k3}+3(\varkappa_1+1)(\delta_{k\beta}\bar{R}_{,3}\bar{R}_{,\beta}$$

$$-\delta_{k3}\bar{R}_{,\beta}\bar{R}_{,\beta})-6\bar{R}_{,3}\bar{R}_{,k}]\cdot n_\alpha\Big\}$$

$$\overset{n}{T}{}^{\mathrm{I}\textcircled{4}}_{k\alpha}=\frac{Ax_3^2}{2\pi(\varkappa_1+1)\bar{R}^4}\Big\{(6\delta_{\alpha\beta}\bar{R}_{,k}+6\delta_{k\alpha}\bar{R}_{,\beta}+6\delta_{k\beta}\bar{R}_{,\alpha}-30\bar{R}_{,\alpha}\bar{R}_{,\beta}\bar{R}_{,k})\cdot n_\beta$$

$$-(6\delta_{k\alpha}\bar{R}_{,3}+6\delta_{k3}\bar{R}_{,\alpha}-30\bar{R}_{,3}\bar{R}_{,\alpha}\bar{R}_{,k})\cdot n_3\Big\}$$

$$\overset{n}{T}{}^{\mathrm{I}\textcircled{5}}_{k\alpha}=\frac{\beta}{4\pi}\Big\{(1-2\nu_1)A(\delta_{k\alpha}\bar{R}_{,\gamma}+\delta_{k\gamma}\bar{R}_{,\alpha})\frac{1}{\bar{R}^2}-2[(1-2\nu_1)A-B_1]\delta_{k\beta}$$

$$\cdot\left[\frac{\delta_{\alpha\beta}\bar{R}_{,\gamma}+\delta_{\alpha\gamma}\bar{R}_{,\beta}+\delta_{\beta\gamma}\bar{R}_{,\alpha}-\bar{R}_{,\alpha}\bar{R}_{,\beta}\bar{R}_{,\gamma}}{(\bar{R}+x_3+\xi_3)^2}-\frac{2\bar{R}\bar{R}_{,\alpha}\bar{R}_{,\beta}\bar{R}_{,\gamma}}{(\bar{R}+x_3+\xi_3)^3}\right]$$

$$-2B_1\delta_{k3}\left[\frac{\delta_{\alpha\gamma}-\bar{R}_{,\alpha}\bar{R}_{,\gamma}}{\bar{R}(\bar{R}+x_3+\xi_3)}-\frac{\bar{R}_{,\alpha}\bar{R}_{,\gamma}}{(\bar{R}+x_3+\xi_3)^2}\right]\Big\}\cdot n_\gamma+\frac{\beta}{4\pi}\Big\{[(1-2\nu_1)A-B_1]$$

$$\cdot\left[\frac{\delta_{k\beta}\bar{R}_{,\alpha}\bar{R}_{,\beta}-\delta_{k\alpha}}{\bar{R}(\bar{R}+x_3+\xi_3)}+\frac{\delta_{k\beta}\bar{R}_{,\alpha}\bar{R}_{,\beta}}{(\bar{R}+x_3+\xi_3)^2}\right]-(1-2\nu_1)A\frac{\delta_{k\alpha}\bar{R}_{,3}}{\bar{R}^2}$$

$$+B_1\left[\frac{\delta_{k3}\bar{R}_{,\alpha}}{\bar{R}^2}+\frac{\delta_{k\alpha}-\bar{R}_{,k}\bar{R}_{,\alpha}}{\bar{R}(\bar{R}+x_3+\xi_3)}-\frac{\bar{R}_{,k}\bar{R}_{,\alpha}-\delta_{k3}\bar{R}_{,\alpha}}{(\bar{R}+x_3+\xi_3)^2}\right]\Big\}\cdot n_3$$

$$\overset{n}{T}{}^{\mathrm{I}}_{k3}=\overset{n}{T}{}^{\mathrm{I}\textcircled{1}}_{k3}+\overset{n}{T}{}^{\mathrm{I}\textcircled{2}}_{k3}+\overset{n}{T}{}^{\mathrm{I}\textcircled{3}}_{k3}+\overset{n}{T}{}^{\mathrm{I}\textcircled{4}}_{k3}+\overset{n}{T}{}^{\mathrm{I}\textcircled{5}}_{k3} \tag{4.5}$$

式中，

$$\overset{n}{T}{}^{\mathrm{I}\textcircled{1}}_{k3}=\frac{1}{2\pi(\varkappa_1+1)R^2}\Big\{-\frac{\partial R}{\partial n}[(1-2\nu_1)\delta_{k3}+3R_{,3}R_{,k}]$$

$$+(1-2\nu_1)[R_{,k}n_3-R_{,3}n_k]\Big\}$$

$$\overset{n}{T}{}^{\mathrm{I}\textcircled{2}}_{k3}=\frac{A\varkappa_1}{2\pi(\varkappa_1+1)\bar{R}^2}\Big\{\left[\frac{1}{2}(\varkappa_1+1)(\delta_{k\beta}\bar{R}_{,3}-\delta_{k3}\bar{R}_{,\beta})\right]\cdot n_\beta$$

$$+[(\varkappa_1-1)\delta_{k3}\bar{R}_{,3}-\bar{R}_{,k}+3\bar{R}_{,k}\bar{R}^2_{,3}]\cdot n_3+\frac{\nu_1}{1-2\nu_1}[(\varkappa_1-1)(\delta_{k3}\bar{R}_{,3}$$

$$-\delta_{k\beta}\bar{R}_{,\beta})+3\bar{R}_{,k}(\bar{R}^2_{,3}-\bar{R}_{,\beta}\bar{R}_{,\beta})+\bar{R}_{,k}]\cdot n_3\Big\}$$

$$+\frac{A}{2\pi(\varkappa_1+1)\bar{R}^2}\Big\{(\varkappa_1\delta_{k3}\bar{R}_{,\beta}-\delta_{k\beta}\bar{R}_{,3}+3\bar{R}_{,3}\bar{R}_{,\beta}\bar{R}_{,k})\cdot n_\beta$$

$$+[2(\varkappa_1-1)\delta_{k\beta}\bar{R}_{,\beta}-4\bar{R}_{,k}+6\bar{R}_{,k}\bar{R}_{,\beta}\bar{R}_{,\beta}]\cdot n_3$$

$$+\frac{2\nu_1}{1-2\nu_1}[(\varkappa_1-1)\delta_{k\beta}\bar{R}_{,\beta}-2\bar{R}_{,k}+3\bar{R}_{,k}\bar{R}_{,\beta}\bar{R}_{,\beta}]\cdot n_3\Big\}$$

$$\overset{n}{T}{}^{\text{I}③}_{k3}=\frac{Ax_3}{2\pi(\varkappa_1+1)\bar{R}^3}\Big\{[(\varkappa_1-4)\delta_{k\alpha}+3(\varkappa_1-1)(\delta_{k3}\bar{R}_{,3}-\delta_{k\beta}\bar{R}_{,\beta})\cdot\bar{R}_{,\alpha}$$
$$+3(3\bar{R}_{,\alpha}+2\delta_{\alpha\beta}\bar{R}_{,\beta})\bar{R}_{,k}+3(\delta_{k\alpha}-5\bar{R}_{,\alpha}\bar{R}_{,k})(\bar{R}_{,\beta}\bar{R}_{,\beta}-\bar{R}^2_{,3})]\cdot n_\alpha$$
$$+[6(\varkappa_1-1)\delta_{k\beta}\bar{R}_{,3}\bar{R}_{,\beta}+8(\delta_{k3}-3\bar{R}_{,3}\bar{R}_{,k})-6(\delta_{k3}-5\bar{R}_{,3}\bar{R}_{,k})\bar{R}_{,\beta}\bar{R}_{,\beta}]\cdot n_3$$
$$+\frac{2\nu_1}{1-2\nu_1}[2(\varkappa_1+2)\delta_{k3}+3(\varkappa_1+1)(\delta_{k\beta}\bar{R}_{,3}-\delta_{k3}\bar{R}_{,\beta})\bar{R}_{,\beta}-6\bar{R}_{,3}\bar{R}_{,k}]\cdot n_3\Big\}$$

$$\overset{n}{T}{}^{\text{I}④}_{k3}=\frac{Ax_3^2}{2\pi(\varkappa_1+1)\bar{R}^4}\{(-6\delta_{k\beta}\bar{R}_{,3}-6\delta_{k3}\bar{R}_{,\beta}+30\bar{R}_{,3}\bar{R}_{,\beta}\bar{R}_{,k})\cdot n_\beta$$
$$+6(\bar{R}_{,k}+2\delta_{k3}\bar{R}_{,3}-5\bar{R}_{,k}\bar{R}^2_{,3})\cdot n_3\}$$

$$\overset{n}{T}{}^{\text{I}⑤}_{k3}=\frac{\beta}{4\pi}\Big\{[(1-2\nu_1)A-B_1]\cdot\left[\frac{\delta_{k\beta}\bar{R}_{,\alpha}\bar{R}_{,\beta}-\delta_{k\alpha}}{\bar{R}(\bar{R}+x_3+\xi_3)}+\frac{\delta_{k\beta}\bar{R}_{,\alpha}\bar{R}_{,\beta}}{(\bar{R}+x_3+\xi_3)^2}\right]$$
$$-(1-2\nu_1)A\frac{\delta_{k\alpha}\bar{R}_{,3}}{\bar{R}^2}+B_1\Big[\frac{\delta_{k3}\bar{R}_{,\alpha}}{\bar{R}^2}+\frac{\delta_{k\alpha}-\bar{R}_{,k}\bar{R}_{,\alpha}}{\bar{R}(\bar{R}+x_3+\xi_3)}$$
$$-\frac{\bar{R}_{,k}\bar{R}_{,\alpha}-\delta_{k3}\bar{R}_{,\alpha}}{(\bar{R}+x_3+\xi_3)^2}\Big]\Big\}\cdot n_\alpha-\frac{2B_1\beta'}{4\pi}\cdot\left\{\frac{\bar{R}_{,k}}{\bar{R}^2}\right\}\cdot n_3$$

空间D_{I}中用张量表示的面力显式基本解为，

$$\overset{n}{T}{}^{\text{I}}_{k\alpha}=\overset{n}{T}{}^{\text{I}①}_{k\alpha}+\overset{n}{T}{}^{\text{I}②}_{k\alpha}+\overset{n}{T}{}^{\text{I}③}_{k\alpha}+\overset{n}{T}{}^{\text{I}④}_{k\alpha}\tag{4.6}$$

这里，

$$\overset{n}{T}{}^{\text{I}①}_{k\alpha}=\frac{\Gamma}{2\pi(\varkappa_1+1)R^2}\Big\{-\frac{\partial R}{\partial n}[(1-2\nu_1)\delta_{k\alpha}+3R_{,\alpha}R_{,k}]-(1-2\nu_1)R_{,\alpha}n_k$$
$$+\left[1-\frac{2\nu_2}{1-2\nu_2}\cdot(1-2\nu_1)\right]\cdot R_{,k}\cdot n_\alpha\Big\}$$

$$\overset{n}{T}{}^{\text{I}②}_{k\alpha}=A\varkappa_1\overset{n}{T}{}^{\text{I}①}_{k\alpha}$$

$$\overset{n}{T}{}^{\text{I}③}_{k\alpha}=\frac{B_2\beta\Gamma x_3}{4\pi}\Big\{\left[\frac{-2\delta_{k3}\delta_{\alpha\gamma}+6\delta_{k3}R_{,\alpha}R_{,\gamma}}{R^3}\right]\cdot n_\gamma+\left[-\frac{2\delta_{k\alpha}}{R^3}+\frac{6R_{,k}R_{,\alpha}}{R^3}\right]\cdot n_3$$
$$+\delta_{k\beta}\Big[\frac{(2\delta_{\beta\gamma}R_{,\alpha}+2\delta_{\beta\alpha}R_{,\gamma}+2\delta_{\alpha\gamma}R_{,\beta}-6R_{,\alpha}R_{,\beta}R_{,\gamma})(2R-x_3+\xi_3)}{[R(R-x_3+\xi_3)]^2}$$
$$-\frac{4R_{,\alpha}R_{,\beta}R_{,\gamma}}{(R-x_3+\xi_3)^3}\Big]\cdot n_\gamma\Big\}$$

$$\overset{n}{T}{}^{\text{I}④}_{k\alpha}=\frac{\beta\Gamma}{4\pi}\Big\{(1-2\nu_1)A\left[\frac{\delta_{k\alpha}R_{,\gamma}+\delta_{k\gamma}R_{,\alpha}}{R^2}\right]\cdot n_\gamma-2[(1-2\nu_1)A-B_1]\cdot\delta_{k\beta}$$
$$\cdot\left[\frac{\delta_{\alpha\beta}R_{,\gamma}+\delta_{\alpha\gamma}R_{,\beta}+\delta_{\beta\gamma}R_{,\alpha}-R_{,\alpha}R_{,\beta}R_{,\gamma}}{(R-x_3+\xi_3)^2}-\frac{2RR_{,\alpha}R_{,\beta}R_{,\gamma}}{(R-x_3+\xi_3)^3}\right]\cdot n_\gamma$$
$$-2B_1\delta_{k3}\left[\frac{\delta_{\alpha\gamma}-R_{,\alpha}R_{,\gamma}}{R(R-x_3+\xi_3)}-\frac{R_{,\alpha}R_{,\gamma}}{(R-x_3+\xi_3)^2}\right]\cdot n_\gamma$$
$$+\left[(1-2\nu_1)A\frac{\delta_{k\alpha}R_{,3}}{R^2}-B_2\frac{\delta_{k3}R_{,\alpha}}{R^2}\right]\cdot n_3+\frac{2\nu_2}{1-2\nu_2}\left[(2\beta_1-B_2)\frac{R_{,k}}{R^2}\right]\cdot n_\alpha$$

$$-[(1-2\nu_1)A-B_2]\delta_{k\beta}\left[\frac{R_{,\alpha}R_{,\beta}-\delta_{\alpha\beta}}{R(R-x_3+\xi_3)}+\frac{R_{,\alpha}R_{,\beta}}{(R-x_3+\xi_3)^2}\right]\cdot n_3\Big\}$$

$$\overset{n}{T}{}^{\text{I}}_{k3}=\overset{n}{T}{}^{\text{I}①}_{k3}+\overset{n}{T}{}^{\text{I}②}_{k3}+\overset{n}{T}{}^{\text{I}③}_{k3}+\overset{n}{T}{}^{\text{I}④}_{k3} \tag{4.7}$$

式中，

$$\overset{n}{T}{}^{\text{I}①}_{k3}=\frac{\Gamma}{2\pi(\varkappa_1+1)R^2}\Big\{-\frac{\partial R}{\partial n}[(1-2\nu_1)\delta_{k3}+3R_{,3}R_{,k}]$$

$$-(1-2\nu_1)R_{,3}n_k+\left[1-\frac{2\nu_2}{1-2\nu_2}(1-2\nu_1)\right]\cdot R_{,k}\cdot n_3\Big\}$$

$$\overset{n}{T}{}^{\text{I}②}_{k3}=A\varkappa_1\overset{n}{T}{}^{\text{I}①}_{k3}$$

$$\overset{n}{T}{}^{\text{I}③}_{k3}=\frac{B_2\beta\Gamma x_3}{4\pi R^3}\Big\{6R_{,k}\cdot\frac{\partial R}{\partial n}-2n_k\Big\}$$

$$\overset{n}{T}{}^{\text{I}④}_{k3}=\frac{\beta\Gamma}{4\pi}\Big\{\Big[(1-2\nu_1)A\frac{\delta_{k\alpha}R_{,3}}{R^2}-B_2\frac{\delta_{k3}R_{,\alpha}}{R^2}\Big]\cdot n_\alpha-2(B_2-B_1)\Big(\frac{R_{,k}}{R^2}\Big)\cdot n_3$$

$$+\frac{2\nu_2}{1-2\nu_2}\Big[(2B_1-B_2)\frac{R_{,k}}{R^2}\Big]\cdot n_3-[(1-2\nu_1)A-B_2]\cdot\delta_{k\beta}$$

$$\cdot\left[\frac{R_{,\alpha}R_{,\beta}-\delta_{\alpha\beta}}{R(R-x_3+\xi_3)}+\frac{R_{,\alpha}R_{,\beta}}{(R-x_3+\xi_3)^2}\right]\cdot n_\alpha\Big\}$$

以上所获得的基本解是单位力作用点P在空间域 I 时的解．致于当 P 点在空间域Ⅱ时的解，则只需把上面解中的材料常数$\varkappa_1$和$\varkappa_2$互换，并把Γ变为 $1/\Gamma$即可．尽管上面结果看起来似乎冗长，但是从中还是有规律可循的．另外，若使用计算机，那么在设计程序时采用这张量表示的指标运算会大大地减少编程工作量，给计算带来特别的方便．因此，本文的结果具有很大的实用价值．

五、一些特殊情形

很早以前，Mindlin和 Lorentz 就分别给出了不同边界条件的弹性半空间受集中力作用的基本解．然而，他们均未将其表为显式的张量表示．本文将以上获得的结果进行退化，便可将Mindlin和Lorentz的结果用显式张量表示，下面对其分别讨论：

情形1　半空间Mindlin解

Mindlin半空间问题的边界条件是自由的．让$\mu_2=0$，则得 $\Gamma=0$，$\beta=2$，$A=1$及$B_1=(1-2\nu_1)/2$，再令$\varkappa_1=\varkappa$，$\xi_1=\xi_2=0$，$x_3=z$及$\xi_3=c$，则由方程(3.3)和(3.4)退化得Mindlin问题的张量表达式：

$$u_{k\alpha}=\frac{1}{4\mu\pi(\varkappa+1)}\Big\{(\varkappa\delta_{k\alpha}+R_{,\alpha}R_{,k})\frac{1}{R}+\varkappa(\varkappa\delta_{k\alpha}+\bar{R}_{,\alpha}\bar{R}_{,k})\frac{1}{\bar{R}}$$

$$+2z(\varkappa\delta_{k3}\bar{R}_{,\alpha}-\delta_{k\alpha}\bar{R}_{,3}+3\bar{R}_{,3}\bar{R}_{,\alpha}\bar{R}_{,k})\frac{1}{\bar{R}^2}-2z^2(\delta_{k\alpha}-3\bar{R}_{,\alpha}\bar{R}_{,k})\frac{1}{\bar{R}^3}$$

$$-\frac{1}{2}(\varkappa^2-1)\left[\frac{2\delta_{k\alpha}}{\bar{R}}+\frac{\delta_{k3}\bar{R}_{,\alpha}-\delta_{k\alpha}}{\bar{R}+z+c}+\frac{\delta_{k\beta}\bar{R}\bar{R}_{,\alpha}\bar{R}_{,\beta}}{(\bar{R}+z+c)^2}\right]\Bigg\} \tag{5.1}$$

$$u_{k3}=\frac{1}{4\mu\pi(\varkappa+1)}\Bigg\{(\varkappa\delta_{k3}+R_{,3}R_{,k})\frac{1}{R}+\varkappa(\varkappa\delta_{k3}+\bar{R}_{,3}\bar{R}_{,k})\frac{1}{\bar{R}}$$

$$+2z[(\varkappa-1)\delta_{k\beta}\bar{R}_{,\beta}-2\bar{R}_{,k}+3\bar{R}_{,k}\bar{R}_{,\beta}\bar{R}_{,\beta}]\frac{1}{\bar{R}^2}+2z^2(\delta_{k3}-3\bar{R}_{,3}\bar{R}_{,k})\frac{1}{\bar{R}^3}$$

$$-\frac{1}{2}(\varkappa^2-1)\left[\frac{\delta_{k3}}{\bar{R}}-\frac{\delta_{k\beta}\bar{R}_{,\beta}}{\bar{R}+z+c}\right]\Bigg\} \tag{5.2}$$

对(5.1)和(5.2)方程进行变化，便得到与文献[6]给出的相同结果。

情形2 半空间Lorentz解

令 $\mu_2=\infty$，即 $\Gamma=\infty$，$\beta=0$，$A=-1/\varkappa_1$，设 $\varkappa_1=\varkappa$，则此弹性力学问题变为半空间Lorentz问题，其接触为刚性基．从(3.3)和(3.4)式，则得，

$$u_{k\alpha}=\frac{1}{4\mu\pi(\varkappa+1)}\Bigg\{(\varkappa\delta_{k\alpha}+R_{,\alpha}R_{,k})\frac{1}{R}-(\varkappa\delta_{k\alpha}+\bar{R}_{,\alpha}\bar{R}_{,k})\frac{1}{\bar{R}}$$

$$-\frac{2x_3}{\varkappa}[\varkappa\delta_{k3}\bar{R}_{,\alpha}-\delta_{k\alpha}\bar{R}_{,3}+3\bar{R}_{,3}\bar{R}_{,\alpha}\bar{R}_{,k}]\frac{1}{\bar{R}^2}$$

$$+\frac{2x_3^2}{\varkappa}[\delta_{k\alpha}-3\bar{R}_{,\alpha}\bar{R}_{,k}]\frac{1}{\bar{R}^3}\Bigg\} \tag{5.3}$$

$$u_{k3}=\frac{1}{4\mu\pi(\varkappa+1)}\Bigg\{(\varkappa\delta_{k3}+R_{,3}R_{,k})\frac{1}{R}-(\varkappa\delta_{k3}+\bar{R}_{,3}\bar{R}_{,k})\frac{1}{\bar{R}}$$

$$-\frac{2x_3}{\varkappa}[(\varkappa-1)\delta_{k\beta}\bar{R}_{,\beta}-2\bar{R}_{,k}+3\bar{R}_{,k}\bar{R}_{,\beta}\bar{R}_{,\beta}]\frac{1}{\bar{R}^2}$$

$$-\frac{2x_3^2}{\varkappa}[\delta_{k3}-3\bar{R}_{,3}\bar{R}_{,k}]\frac{1}{\bar{R}^3}\Bigg\} \tag{5.4}$$

这与从文献[7]导出的结果完全一致。

情形3 各向同性、均质、无限大空间解

让$\mu_1=\mu_2$，即$\Gamma=1$，$\beta=1$ 和 $A=B_1=B_2=0$，设$\varkappa_1=\varkappa_2=\varkappa$，则两相材料空间问题变为一个由两部分构成的均质体空间问题．由式(3.3)和(3.4)，基本解变为，

$$u^{\mathrm{I}}_{ki}=u^{\mathrm{II}}_{ki}=\frac{1}{4\mu\pi(\varkappa+1)R}\{\varkappa\delta_{ki}+R_{,i}R_{,k}\} \tag{5.5}$$

$$T^{\mathrm{I}}_{ki}=T^{\mathrm{II}}_{ki}=\frac{1}{2\pi(\varkappa+1)R^2}\Bigg\{-\frac{\partial R}{\partial n}[(1-2\nu_1)\delta_{ki}+3R_{,i}R_{,k}]$$

$$+(1-2\nu_1)[R_{,k}n_i-R_{,i}n_k]\Bigg\} \tag{5.6}$$

式(5.5)和(5.6)正是各向同性、均质、无限大体空间的Kelvin解。

六、结 束 语

本文全面地和系统地导出了由两相材料构成的无限大体空间基本解的显式张量表达式。这些解将为使用边界积分方程和边界元方法求解两相材料空间弹性力学问题带来方便.比如，三维两相材料中的断裂问题、夹杂问题和界面问题等等，便可应用以上获得的基本解建立它们的边界积分方程，随后，运用近代的边界元方法求解，使问题获得解决。因而，本文以上给出的结果具有很大的实用价值。另外，本文解包括了 Mindlin问题、Lorentz 问题以及均质体弹性空间问题等一些特殊情况下的解，这些为考察上述问题中的一些疑难点，如Mindlin问题中，当裂纹接近自由边界时，裂纹尖端的奇异性分析，将带来可能。

参 考 文 献

[1] L. Rongved, Force interior to one of two joined semi-infinite solids, *Proc. 2nd Midwestern Conf. Solid Mech.* (1955), 1—13.

[2] J. Dundurs and M. Hetenyi, Transmission of force between two semi-infinite solids, *J. Appl. Mech.*, 32 (1965), 671—674.

[3] 黄克服、王敏中，两相材料构成空间的弹性力学基本解，中国科学，A (1991), 40—46.

[4] 乐金朝，三维复合材料断裂力学问题的超奇异积分方程解法，上海交通大学博士学位论文 (1994).

[5] Y. C. Fung, *Foundations of Solid Mechanics*, Prentice-Hall, NJ (1965).

[6] 木须·结城·浦，三次元境界要素解析におけるMindlinの解の有用性，機論，A (1987), 1864—1869.

[7] N. Phan-Thien, On the image system for the Kelvin-state, *J. Elasticity*, 13 (1983), 231—235.

An Explicit Tensor Expression for the Fundamental Solutions of a Bimaterial Space Problem

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Abstract

In this paper, by using the method of tensor operation, the fundamental solutions, given in the references listed, for a concentrated force in a three-dimensional biphase-infinite solid were expressed in the tensor form, which enables them to be directly applied to the boundary integral equation and the boundary element method for solving elastic mechanics problems of the bimaterial space. The fundamental solutions for Mindlin's problem, Lorentz's problem and homogeneous space problem are involved in the present results.

Key words three-dimensional bimaterial, fundamental solution of a concentrated force, tensor expression