应用数学和力学编委会编
重庆出版社出版

非均匀切向荷载下弹性半空间二阶效应的一般解答*

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（樊大钧推荐，1995年9月11日收到，1996年9月20日收到修改稿）

摘　要

继文[1]，本文运用积分变换方法，获得了可压缩各向同性弹性半空间在非均匀分布的切向荷载下，二阶弹性效应问题的封闭形式一般解答.

关键词　弹性半空间　切向荷载　二阶弹性效应　积分变换

一、引　言

在有限弹性理论中，控制可压缩各向同性弹性体变形的数学方程是强非线性的，其边值问题的精确解，仅在某些限制条件下才能获得，并常常求助于近似方法得到结果．成功的近似方法成为一种受到高度重视的技术．获得包含位移梯度二次项，并取得特殊椭圆积分形式的二阶解，是一项十分困难的工作．Rivlin[2]，Green 和 Spratt[3] 几乎同时首次导出二阶理论，并且Truesdell和Noll[4]，以及Green和 Adkins[5] 提出了一个综合计算方法．Goodman与Naghdi[6] 用位移势解答了可压缩的二阶弹性问题，并将其应用于平面应变问题；对于不可压缩材料的二阶理论问题的各种求解方法已由Chan和 Carlson[7] 给出；Selvadurai与Spencer[8]，Carroll 与Mooney[9]、Lindsay[10]、以及Choi与Shield[11]分别用反变形方法研究了二阶弹性材料的某些轴对称问题.

用上述成功的近似方法，将位移、应力等量展开成用一些适当参数表示的具有非零收敛半径的幂级数形式．Signorini[12] 和 Stoppeli[13,14] 讨论了在适当可微条件下级数解答结果的存在性和唯一性．特别是 Stoppeli 还证明了位移能够按某些参数展开为具有非零收敛半径的绝对收敛的幂级数．若参数足够小，经典线弹性方程的充分光滑解答是存在的.

在文[1]中，我们已经研究了弹性半空间在非均匀分布法向荷载下的二阶效应．在本文中我们继Rivlin的文[2]，研究可压缩弹性半空间在非均匀分布切向荷载作用下的二阶效应问题．Rivlin 的研究将二阶问题简化为考虑体力的线弹性问题的解答．回顾在经典弹性理论中，弹性半空间在一个切向力作用下的应力场分布问题首次由 Cerruti[16]求得．这个解

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* 湖南省自然科学基金资助课题

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答除了数学上的意义，在土力学中具有实际的用途．非均匀的法向荷载分布在一个圆内的相应问题解答由Sneddon[15]获得．Sneddon 证明了这个法向荷载问题相当于求半空间在一个刚性平底的圆柱相压下的应力分布问题．注意到，当荷载是任意分布时，我们需要分别考虑法向和切向荷载．我们参照Sneddon[15]关于一般线性和二阶问题的解答，采用积分变换方法求解．详细研究了一种特殊情形的一般解答，最后对 z 方向的应力与位移进行了数值计算．

二、基 本 方 程

设弹性体在单位体积质量受体力X_i以及在变形前单位面积上所受面力$X_{v\,i}$作用下发生变形，则平衡方程与边界条件为

$$\frac{\partial\tau}{\partial H_{kj}}\frac{\partial t_{ik}}{\partial x_j}+\rho_0X_i=0 \tag{2.1}$$

$$X_{vi}=(\partial\tau/\partial H_{ks})l_st_{ik} \tag{2.2}$$

式中t_{ik}为柯西应力，u_i为位移分量，x_k为变形前弹性体内某点的坐标分量，l_s是变形前弹性体表面法向的方向余弦，且

$$H_{ik}=\partial u_i/\partial x_k,\quad \tau=\det(\delta_{ik}+\partial u_i/\partial x_k)\qquad (i,\ k=1,\ 2,\ 3)$$

对于可压缩的各向同性材料，应变能函数W的形式为

$$W=a_1J_2+a_2J_1^2+a_3J_1J_2+a_4J_1^3+a_5J_3 \tag{2.3}$$

式中a_1，…，a_5为材料常数，J_1，J_2，J_3分别是关于应变量的1，2，3次应变不变量．Rivlin[2]证明了由方程(2.1)和(2.2)，略去高阶的位移u_i的空间导数，分别得到

$$\left[(1+\Delta)\delta_{sk}-\frac{\partial u_s}{\partial x_k}\right]\frac{\partial t'_{ik}}{\partial x_s}+\frac{\partial t''_{ik}}{\partial x_k}+\rho_0X_i=0 \tag{2.4}$$

且

$$X_{vi}=[(1+\Delta)\delta_{sk}-\partial u_s/\partial x_k]l_st'_{ik}+l_kt''_{ik} \tag{2.5}$$

式中

$$t_{ik}=t'_{ik}+t''_{ik} \tag{2.6}$$

且

$$t'_{ik}=2[-a_1e_{ik}+2(a_1+2a_2)\Delta\delta_{ik}]$$

$$t''_{ik}=2[\{(4a_2-2a_3+a_1)\Delta e_{ik}-a_1\alpha_{ik}-(a_1-a_5)E_{ik}\}$$
$$+\{(a_1+2a_2)\alpha+(a_1+a_3)E+2(6a_4+3a_3-a_1-2a_2)\Delta^2\}\delta_{ik}]$$

式(2.3)中，应变不变量J_i与通常的右柯西—格林应变张量**C**的不变量I_i的关系为

$$J_1=I_1-3,\quad J_2=I_2-2I_1+3,\quad J_3=I_3-I_2+I_1-1 \tag{2.7}$$

且常数a_1，a_2由经典弹性理论确定为

$$\lambda=4(a_1+2a_2),\quad \mu=-2a_1$$

式中λ和μ为Lame常数，另外三个常数a_3，a_4，a_5称为二阶弹性常数．

令

$$u_i=v_i+w_i \tag{2.8}$$

并定义

$$\left.\begin{aligned}\tau_{ik}&=2[-a_1e'_{ik}+2(a_1+2a_2)\Delta'\delta_{ik}]\\ e'_{ik}&=\frac{\partial v_i}{\partial x_k}+\frac{\partial v_k}{\partial x_i},\quad \Delta'=\frac{1}{2}e'_{ss}\end{aligned}\right\}\tag{2.9}$$

且

$$\left.\begin{aligned}\tau''_{ik}&=2[-a_1e''_{ik}+2(a_1+2a_2)\Delta''\delta_{ik}]\\ e''_{ik}&=\frac{\partial w_i}{\partial x_k}+\frac{\partial w_k}{\partial x_i},\quad \Delta''=\frac{1}{2}e''_{ik}\end{aligned}\right\}\tag{2.10}$$

综上所述，求解一个二阶边值问题，必须

（Ⅰ）求解下式表达的线弹性问题：

$$\partial\tau_{ik}/\partial x_k+\rho_0X_i=0\tag{2.11}$$

受面力为

$$X_{vi}=l_k\tau_{ik}\tag{2.12}$$

（Ⅱ）获得由下式给出的二阶弹性解：

$$\partial\tau''_{ik}/\partial x_k+\rho_0X'_i=0\tag{2.13}$$

受面力

$$X'_{vi}=l_k\tau''_{ik}\tag{2.14}$$

式中

$$X'_{vi}=-[\Delta'\delta_{ik}-\partial v_s/\partial x_k]l_s\tau_{ik}-l_k\tau'_{ik}\tag{2.15}$$

$$\rho_0X'_i=\left(\Delta'\delta_{ik}-\frac{\partial v_s}{\partial x_k}\right)\frac{\partial\tau_{ik}}{\partial x_s}+\frac{\partial\tau'_{ik}}{\partial x_k}\tag{2.16}$$

且式中

$$\begin{aligned}\tau'_{ik}=2[&\{(4a_2-2a_3+a_1)\Delta'e'_{ik}-a_1\alpha'_{ik}-(a_1-a_5)E'_{ik}\}\\&+\{(a_1+2a_2)\alpha'+(a_1+a_3)E'+2(6a_4+2a_3-a_1-2a_2)\Delta'^2\}\delta_{ik}]\end{aligned}\tag{2.17}$$

其中 $\alpha'_{ik}=(\partial v_k/\partial x_s)(\partial v_k/\partial x_s)$，$\alpha'=\alpha'_{ss}$，$E'=E'_{ss}$

且E'_{ik}为$\det e'_{ik}$中e'_{ik}的余因子.

三、圆形分布的非均匀荷载

我们考察一个可压缩的弹性半空间，在其表面半径为a的圆内，承受着总量为P的非均匀切向荷载．在经典弹性理论中，这个由表面作用均匀切向力的弹性半空间的应力分布问题首次由Cerruti[16]解决．Muki[17]运用Hakel变换方法，获得了这一问题的另一种解答．本文考虑作用非均匀分布切向荷载的二阶问题．我们选取圆柱坐标$(r,\ \theta,\ z)$，弹性材料占据$z\geqslant0$的半空间，荷载作用于$z=0$的平面上，并沿x轴方向．其边界条件是

$$\left.\begin{aligned}t_{zz}&=0\\ t_{rz}&=[(1+\delta)P/\pi a^{2(1+\delta)}](a^2-r^2)^\delta H(a-r)\cos\theta\\ t_{\theta z}&=-[(1+\delta)P/\pi a^{2(1+\delta)}](a^2-r^2)^\delta H(a-r)\sin\theta\end{aligned}\right\}\tag{3.1}$$

式中$\delta>-1$是一常数，且H是Heaviside单位函数，设不计体力，由Rivlin 方法，该问题能分解为如下两个子问题

（Ⅰ）线性解

$$\left.\begin{aligned}&\frac{\partial\tau_{rr}}{\partial r}+\frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta}+\frac{\partial\tau_{rz}}{\partial z}+\frac{\tau_{rr}-\tau_{\theta\theta}}{r}=0\\&\frac{\partial\tau_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial\tau_{\theta\theta}}{\partial\theta}+\frac{\partial\tau_{\theta z}}{\partial z}+\frac{2}{r}\tau_{r\theta}=0\\&\frac{\partial\tau_{rz}}{\partial r}+\frac{1}{r}\frac{\partial\tau_{\theta z}}{\partial\theta}+\frac{\partial\tau_{zz}}{\partial z}+\frac{1}{r}\tau_{rz}=0\end{aligned}\right\}\tag{3.2}$$

受面力

$$\left.\begin{aligned}&\tau_{zz}|_{z=0}=0\\&\tau_{rz}|_{z=0}=[(1+\delta)P/\pi a^{2(1+\delta)}](a^2-r^2)^\delta H(a-r)\cos\theta\\&\tau_{\theta z}|_{z=0}=-[(1+\delta)P/\pi a^{2(1+\delta)}](a^2-r^2)^\delta H(a-r)\sin\theta\end{aligned}\right\}\tag{3.3}$$

（Ⅱ）二阶解

$$\left.\begin{aligned}&\frac{\partial\tau''_{rr}}{\partial r}+\frac{1}{r}\frac{\partial\tau''_{r\theta}}{\partial\theta}+\frac{\partial\tau''_{rz}}{\partial z}+\frac{\tau_{rr}-\tau''_{\theta\theta}}{r}+\rho_0X'_r=0\\&\frac{\partial\tau''_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial\tau''_{\theta\theta}}{\partial\theta}+\frac{\partial\tau''_{\theta z}}{\partial z}+\frac{2}{r}\tau''_{r\theta}+\rho_0x'_\theta=0\\&\frac{\partial\tau''_{rz}}{\partial r}+\frac{1}{r}\frac{\partial\tau''_{\theta z}}{\partial\theta}+\frac{\partial\tau''_{zz}}{\partial z}+\frac{1}{r}\tau''_{rz}+\rho_0X'_z=0\end{aligned}\right\}\tag{3.4}$$

受面力

$$\tau''_{zz}|_{z=0}=-\bar{X}''_z,\quad \tau''_{rz}|_{z=0}=-\bar{X}''_r,\quad \tau''_{\theta z}|_{z=0}=-\bar{X}''_\theta\tag{3.5}$$

式中面力和体力列于附录Ⅰ中.

3.1 线性解答

我们采用Muki的位移解来求解子问题（Ⅰ），

$$\boldsymbol{v}=(1/2\mu)\{2(1-\eta)\nabla^2\boldsymbol{G}-\nabla(\nabla\cdot\boldsymbol{G})+\nabla\times\boldsymbol{A}\}\tag{3.6}$$

式中$\boldsymbol{G}$是一个双调矢量，$\boldsymbol{A}$是一个调和矢量．Sneddon[18]在其近期的一篇论文中证明了不使用调和与双调和函数也能够获得Muki型式的解答．Muki对于$\boldsymbol{G}$和$\boldsymbol{A}$均提出了单一的z分量

$$\boldsymbol{G}=(0,\ 0,\ G_z(r,\ \theta,\ z)),\quad \boldsymbol{A}=(0,\ 0,\ A_z(r,\ \theta,\ z))\tag{3.7}$$

我们选$G_z(r,\ \theta,\ z)=\phi(r,\ z)\cos\theta$和$A_z(r,\ \theta,\ z)=\psi(r,\ z)\sin\theta$，则位移分量$(v_r,\ v_\theta,\ v_z)$变为

$$\left.\begin{aligned}&v_r=\frac{1}{2\mu}\left[-\frac{\partial^2\phi}{\partial r\partial z}+\frac{2\psi}{r}\right]\cos\theta,\quad v_\theta=\frac{1}{2\mu}\left[\frac{1}{r}\frac{\partial\phi}{\partial z}-2\frac{\partial\psi}{\partial r}\right]\sin\theta\\&v_z=\frac{1}{2\mu}\left[2(1-\eta)\nabla_1^2\phi-\frac{\partial^2\psi}{\partial z^2}\right]\cos\theta\end{aligned}\right\}\tag{3.8}$$

并满足式(3.2)，其中$\phi(r,\ z)$与$\psi(r,\ z)$分别满足：

$$\nabla_1^4\phi=\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}+\frac{\partial^2}{\partial z^2}\right)^2\phi=0\tag{3.9}$$

$$\nabla_1^2\psi=\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}+\frac{\partial^2}{\partial z^2}\right)\psi=0\tag{3.10}$$

相应于位移场(3.8)的应力场可写为

$$\left.\begin{aligned}
\tau_{rr}&=\left[\frac{\partial(\eta\nabla_1^2\phi-\partial^2\phi/\partial r^2)}{\partial z}+\left(\frac{2}{r}\frac{\partial\phi}{\partial r}-\frac{2\psi}{r^2}\right)\right]\cos\theta\\
\tau_{\theta\theta}&=\left[\frac{\partial(\eta\nabla_1^2\phi-\partial\phi/r\partial r+\phi/r^2)}{\partial z}-\left(\frac{2}{r}\frac{\partial\psi}{\partial r}-\frac{2\psi}{r^2}\right)\right]\cos\theta\\
\tau_{zz}&=\left[\frac{\partial((2-\eta)\nabla_1^2\phi-\partial^2\phi/\partial z^2)}{\partial z}\right]\cos\theta\\
\tau_{\theta z}&=\left[-\frac{1}{r}\left((1-\eta)\nabla_1^2\phi-\frac{\partial^2\phi}{\partial z^2}\right)-\frac{\partial^2\psi}{\partial z\partial r}\right]\sin\theta\\
\tau_{rz}&=\left[\frac{\partial((1-\eta)\nabla_1^2\phi-\partial^2\phi/\partial z^2)}{\partial r}+\frac{1}{r}\frac{\partial\psi}{\partial z}\right]\cos\theta\\
\tau_{r\theta}&=\left[\frac{\partial^2(\phi/r)}{\partial z\partial r}-\left(2\frac{\partial^2\psi}{\partial r^2}+\frac{\partial^2\psi}{\partial z^2}\right)\right]\sin\theta
\end{aligned}\right\}\tag{3.11}$$

由(3.3)式导出

$$\left.\frac{\partial((2-\eta)\nabla_1^2\phi-\partial^2\phi/\partial z^2)}{\partial z}\right]\Bigg|_{z=0}=0\tag{3.12a}$$

$$\left[-\frac{1}{r}\left((1-\eta)\nabla_1^2\phi-\frac{\partial^2\phi}{\partial z^2}\right)-\frac{\partial^2\psi}{\partial z\partial r}\right]\Bigg|_{z=0}=-\frac{(1+\delta)P}{\pi a^{2(1+\delta)}}(a^2-r^2)^{\delta}H(a-r)\tag{3.12b}$$

$$\left[\frac{\partial((1-\eta)\nabla_1^2\phi-\partial^2\phi/\partial z^2)}{\partial r}+\frac{1}{r}\frac{\partial\psi}{\partial z}\right]\Bigg|_{z=0}=\frac{(1+\delta)P}{\pi a^{2(1+\delta)}}(a^2-r^2)^{\delta}H(a-r)\tag{3.12c}$$

设 $\bar{\phi}=\int_0^\infty rJ_1(\xi r)\phi(r,\ z)dr,\ \bar{\psi}=\int_0^\infty rJ_1(\xi r)\psi(r,\ z)dr$

为了得到关于$\bar{\phi}(\xi,\ z)$与$\bar{\psi}(\xi,\ z)$的常微分方程，分别对式(3.9)和(3.10)进行一阶Hankel变换. 这些微分方程对于我们有用的解答是

$$\bar{\phi}(\xi,\ z)=(A_2+A_3\xi z)e^{-\xi z},\ \bar{\psi}(\xi,\ z)=A_1e^{-\xi z}\tag{3.13}$$

式中A_1, A_2, A_3是ξ的任意函数. 作(3.12a)式的一阶Hankel变换，作(3.12b)+(3.12c)式的二阶Hankel变换以及(3.12b)-(3.12c)式的零阶Hankel变换，并解关于A_1, A_2, A_3所得到的方程，我们得到

$$A_1=-\frac{T}{2}\frac{J_{1+\delta}(a\xi)}{\xi^{3+\delta}},\quad A_2=-\frac{T}{2}\frac{(1-2\eta)J_{1+\delta}(a\xi)}{\xi^{4+\delta}},\quad A_3=\frac{T}{2}\frac{J_{1+\delta}(a\xi)}{\xi^{4+\delta}}\tag{3.14}$$

式中 $T=2^{1+\delta}(1+\delta)\Gamma(1+\delta)P/(\pi a^{1+\delta})$

现在作应力与位移函数的如下Hankel变换

$H_1[\tau_{zz}/\cos\theta]$, $H_2[\tau_{rz}/\cos\theta+\tau_{\theta z}/\sin\theta]$, $H_0[\tau_{rz}/\cos\theta-\tau_{\theta z}/\sin\theta]$

$H_2[v_r/\cos\theta+v_\theta/\sin\theta]$, $H_0[v_r/\cos\theta-v_\theta/\sin\theta]$, $H_1[\tau_{rr}/\cos\theta+\tau_{\theta\theta}/\sin\theta]$

$H_1[\tau_{rr}/\cos\theta+2\mu v_r/(r\cos\theta)+2\mu v_\theta/(r\sin\theta)]$

并反演所得到的方程，我们得到

$$
\left.\begin{aligned}
v_r &= (T/4\mu)[-(2-\eta)L(0, -(1+\delta), z) - \eta L(2, -(1+\delta), z) \\
&\quad + (z/2)L(0, -\delta, z) - (z/2)L(2, -\delta, z)]\cos\theta \\
v_\theta &= (T/4\mu)[(2-\eta)L(0, -(1+\delta), z) - \eta L(2, -(1+\delta), z) \\
&\quad - (z/2)L(0, -\delta, z) - (z/2)L(2, -\delta, z)]\sin\theta \\
v_z &= -(T/4\mu)[(1-2\eta)L(1, -(1+\delta), z) + zL(1, -\delta, z)]\cos\theta \\
\tau_{rr} &= T[(\eta/r)L(2, -(1+\delta), z) + (z/2r)L(2, -\delta, z) + L(1, -\delta, z) \\
&\quad - (z/2)L(0, -\delta, z) - (z/2)L(1, 1-\delta, z)]\cos\theta \\
\tau_{\theta\theta} &= T[(-\eta/r)L(2, -(1+\delta), z) - (z/2r)L(2, -\delta, z) - 3\eta L(1, -\delta, z)]\cos\theta \\
\tau_{zz} &= T[(z/2)L(1, 1-\delta, z)]\cos\theta \\
\tau_{rz} &= T[(1/2)L(0, -\delta, z) + (z/4)L(0, 1, -\delta, z) - (z/4)L(2, 1-\delta, z)]\cos\theta \\
\tau_{\theta z} &= T[(1/2)L(0, -\delta, z) - (z/4)L(0, 1, -\delta, z) - (z/4)L(2, 1-\delta, z)]\sin\theta \\
\tau_{r\theta} &= T[-(\eta/r)L(2, -(1+\delta), z) - (z/2r)L(2, -\delta, z) - (1/2)L(1, -\delta, z)]\sin\theta
\end{aligned}\right\} \tag{3.15}
$$

式中$L(n, s, z)$定义为

$$
L(n, s, z) = \int_0^\infty \xi^s J_n(\xi r) J_{1+\delta}(\xi a) e^{-\xi z} d\xi \tag{3.16}
$$

式(3.15)和(3.16)给出了线弹性问题的位移和应力分量．下面给出它们在半空间表面上的值．将$L(n, s, 0)$表示为$L(n, s)$，线性位移与应力分量可写为

$$
\left.\begin{aligned}
v_r &= -[T/4\mu][(2-\eta)L(0, -(1+\delta)) + \eta L(2, -(1+\delta))]\cos\theta \\
v_\theta &= [T/4\mu][(2-\eta)L(0, -(1+\delta)) - \eta L(2, -(1+\delta))]\sin\theta \\
v_z &= -[T/4\mu](1-2\eta)L(1, -(1+\delta))\cos\theta \\
\tau_{rr} &= T[(\eta/r)L(2, -(1+\delta)) + L(1, -\delta)]\cos\theta \\
\tau_{\theta\theta} &= -T[(\eta/r)rL(2, -(1+\delta)) + 3\eta L(1, -\delta)]\cos\theta \\
\tau_{zz} &= 0 \\
\tau_{r\theta} &= T[(\eta/r)L(2, -(1+\delta)) - (1/2)L(1, -\delta)]\sin\theta \\
\tau_{rz} &= (T/2)L(0, -\delta)\cos\theta \\
\tau_{\theta z} &= -(T/2)L(0, -\delta)\sin\theta
\end{aligned}\right\} \tag{3.17}
$$

3.2 二阶解答

为了解二阶问题，我们需要求解子问题(Ⅱ)，目前，附加力与面力可写为

$$
\left.\begin{aligned}
&\rho_0 X_r' = f_r^1(r, z) + f_r^2(r, z)\cos 2\theta, \quad \rho_0 X_\theta' = f_\theta(r, z)\cos\theta\sin\theta \\
&\rho_0 X_z' = f_z^1(r, z) + f_z^2(r, z)\cos 2\theta
\end{aligned}\right\} \tag{3.18}
$$

且

$$
\left.\begin{aligned}
&\bar{X}_r'' = -X_{vr}^1(r, z) - X_{vr}^2(r, z)\cos 2\theta, \quad \bar{X}_\theta'' = -X_{v\theta}(r, z)\cos\theta\sin\theta \\
&\bar{X}_z'' = -X_{vz}^1(r, z) - X_{vz}^2(r, z)\cos 2\theta
\end{aligned}\right\} \tag{3.19}
$$

我们选位移矢量为Garlerkin的解答再加一个无旋项

$$
\boldsymbol{w} = (1/2\mu)\{2(1-\eta)\nabla^2 \boldsymbol{G} - \nabla(\nabla\cdot\boldsymbol{G}) + \nabla\Psi\}
$$

式中

$$
\left.\begin{aligned}
&\boldsymbol{G} = \{G_1(r, z)\cos\theta, \ G_2(r, z)\sin\theta, \ G_3(r, z)\cos 2\theta + G_4(r, z)\} \\
&\Psi = (1-2\eta)\mu\Phi(r, z)\cos 2\theta
\end{aligned}\right\} \tag{3.20}
$$

由此，位移分量变为

$$\left.\begin{aligned}
2\mu w_r=&2(1-\eta)[\nabla_1^2G_1\cos^2\theta+\nabla_1^2G_2\sin^2\theta]-\frac{\partial G_0}{\partial r}-\frac{\partial^2G_4}{\partial r\partial z}\\
&-\frac{\partial^2G_3}{\partial r\partial z}\cos2\theta+(1-2\eta)\mu\frac{\partial\Phi}{\partial r}\cos2\theta\\
2\mu w_\theta=&2(1-\eta)[\nabla_1^2G_2-\nabla_1^2G_1]\cos\theta\sin\theta-\frac{1}{r}\frac{\partial G_0}{\partial\theta}\\
&+\frac{2}{r}\frac{\partial G_3}{\partial z}\sin2\theta-(1-2\eta)\mu\frac{2\Phi}{r}\sin2\theta\\
2\mu w_z=&-\frac{\partial G_0}{\partial z}+\left[2(1-\eta)\nabla_0^2G_4-\frac{\partial^2G_4}{\partial z^2}\right]+\left[2(1-\eta)\nabla_2^2G_3\right.\\
&\left.-\frac{\partial^2G_3}{\partial z^2}\right]\cos2\theta+(1-2\eta)\mu\frac{\partial\Phi}{\partial z}\cos2\theta
\end{aligned}\right\}\tag{3.21}$$

式中

$$\left.\begin{aligned}
G_0=&\frac{1}{2}\left[\frac{\partial G_1}{\partial r}+\frac{G_1}{r}+\frac{\partial G_2}{\partial r}+\frac{G_2}{r}\right]+\frac{1}{2}\left[\frac{\partial G_1}{\partial r}-\frac{G_1}{r}-\frac{\partial G_2}{\partial r}+\frac{G_2}{r}\right]\cos2\theta\\
\nabla_n^2=&\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{n^2}{r^2}+\frac{\partial^2}{\partial z^2}\right)
\end{aligned}\right\}\tag{3.22}$$

应力分量为

$$\left.\begin{aligned}
\tau''_{rr}=&\frac{\eta}{2}g_0+(1+\eta)\left[\frac{\partial[\nabla_1^2G_1+\nabla_2^2G_2]}{\partial r}+\frac{\partial[\nabla_1^2-\nabla_1^2G_2]}{\partial r}\cos2\theta\right]-\frac{\partial^2G_0}{\partial r^2}\\
&+\frac{\partial[\eta\nabla_0^2G_4-\partial^2G_4/\partial r^2]}{\partial z}+\frac{\partial[\eta\nabla_2^2G_3-\partial^2G_3/\partial r^2]}{\partial z}\cos2\theta\\
&+\left[\mu\eta\nabla_2^2\Phi+(1-2\eta)\mu\frac{\partial^2\Phi}{\partial r^2}\right]\cos2\theta\\
\tau''_{\theta\theta}=&\frac{\eta}{2}g_0+(1+\eta)\left[\frac{\nabla_1^2G_1+\nabla_2^2G_2}{r}+\frac{\nabla_1^2G_2-\nabla_1^2G_1}{r}\cos2\theta\right]-\left[\frac{1}{r}\frac{\partial G_0}{\partial r}+\frac{1}{r^2}\frac{\partial^2G_0}{\partial\theta^2}\right]\\
&+\frac{\partial[\eta\nabla_0^2G_4-\partial G_4/r\partial r]}{\partial z}+\frac{\partial[\eta\nabla_2^2G_3-\partial G_3/r\partial r+4G_3/r^2]}{\partial z}\cos2\theta\\
&+\left[\mu\eta\nabla_2^2\Phi+\frac{(1-2\eta)\mu}{r}\left(\frac{\partial\Phi}{\partial r}-\frac{4\Phi}{r}\right)\right]\cos2\theta\\
\tau''_{zz}=&\frac{\eta}{2}g_0-\frac{\partial^2G_0}{\partial z^2}+\frac{\partial[(2-\eta)\nabla_0^2G_4-\partial^2G_4/\partial z^2]}{\partial z}+\frac{\partial[(2-\eta)\nabla_2^2G_3-\partial^2G_3/\partial z^2]}{\partial z}\cos2\theta\\
&+\left[\mu\eta\nabla_2^2\Phi+(1-2\eta)\mu\frac{\partial^2\Phi}{\partial z^2}\right]\cos2\theta\\
\tau''_{rz}=&\frac{1-\eta}{2}\left[\frac{\partial[\nabla_1^2G_1+\nabla_1^2G_2]}{\partial z}+\frac{\partial[\nabla_1^2G_1-\nabla_1^2G_2]}{\partial z}\cos2\theta\right]-\frac{\partial^2G_0}{\partial r\partial z}\\
&+\frac{\partial[(1-\eta)\nabla_0^2G_4-\partial^2G_4/\partial z^2]}{\partial r}+\frac{\partial[(1-\eta)\nabla_2^2G_3-\partial^2G_3/\partial z^2]}{\partial r}\cos2\theta
\end{aligned}\right\}\tag{3.23}$$

$$+(1-2\eta)\mu\frac{\partial^2\Phi}{\partial r\partial z}\cos2\theta$$

$$\tau''_{\theta z}=(1-\eta)\frac{\partial[\nabla_1^2G_2+\nabla_1^2G_1]}{\partial z}\cos\theta\sin\theta+\frac{2}{r}\frac{\partial[\partial G_1/\partial r-G_1-\partial G_2/\partial r+G_2]}{\partial z}$$

$$\cdot\cos\theta\sin\theta-\frac{4}{r}\left[(1-\eta)\nabla_0^2G_3-\frac{\partial^2G_3}{\partial z^2}\right]\cos\theta\sin\theta-\frac{4(1-2\eta)\mu}{r}\frac{\partial\Phi}{\partial r}\cos\theta\sin\theta$$

$$\tau''_{r\theta}=(1-\eta)\left[\frac{\nabla_1^2G_2-\nabla_1^2G_1}{r}+\frac{\partial[\nabla_1^2G_1-\nabla_1^2G_2]}{\partial r}\right]\cos\theta\sin\theta-\frac{1}{r}\frac{\partial^2G_0}{\partial r\partial\theta}$$

$$+\frac{1}{2r^2}\frac{\partial G_0}{\partial\theta}-\frac{4}{r}\frac{\partial[G_3/r-\partial G_3/\partial r]}{\partial z}\cos\theta\sin\theta+\frac{4(1-2\eta)\mu}{r}\left[\frac{\Phi}{r}-\frac{\partial\Phi}{\partial r}\right]$$

$$\cdot\cos\theta\sin\theta$$

式中

$$g_0=\frac{\partial[\nabla_1^2G_1+\nabla_1^2G_2]}{\partial r}+\frac{\nabla_1^2G_1-\nabla_1^2G_2}{r}$$

$$+\left\{\frac{\partial[\nabla_1^2G_1-\nabla_1^2G_2]}{\partial r}+\frac{\nabla_1^2G_2-\nabla_1^2G_1}{r}\right\}\cos2\theta \tag{3.24}$$

将(3.18)与(3.23)式代入(3.4)式，并重新整理，得到

$$\nabla_1^4(G_1+G_2)=-2f_r^1 \tag{3.25}$$

$$\frac{\partial\nabla_2^2\Phi}{\partial r}-\frac{2}{r}\nabla_2^2\Phi=-\frac{2f_r^2+f_\theta}{2} \tag{3.26}$$

$$\nabla_1^4G_1=[f_\theta-2f_r^1-(4/r)\nabla_2^2\Phi]/2 \tag{3.27}$$

$$\nabla_0^4G_4=-f_z^1 \tag{3.28}$$

$$\nabla_2^4G_3+\frac{\partial\nabla_2^2\Phi}{\partial z}=-f_z^2 \tag{3.29}$$

现在对式(3.26)的两边分别作三阶Hankel变换得到

$$\left(\frac{d^2}{dz^2}-\xi^2\right)\bar{\Phi}=\frac{1}{\xi}\int_0^\infty rJ_3(\xi r)\frac{2f_r^2+f_\theta}{2}dr\triangleq\phi(\xi,z) \tag{3.30}$$

式中 $\bar{\Phi}=\int_0^\infty rJ_2(\xi r)\Phi(\xi,\ r)dr$

由(3.30)式，我们找到适合(3.26)式的解答是

$$\Phi=H_2[(A+\phi^*)e^{-\xi z};\ \xi\to r] \tag{3.31}$$

式中

$$\phi^*=\int_0^z e^{2\xi z}\int_0^{z_2}\phi(\xi,z_1)\exp[-\xi z_1]dz_1dz_2$$

并且A是ξ的一个任意函数．由式(3.27)得

$$\left(\frac{d^2}{dz^2}-\xi^2\right)^2H_1[G_1]=\int_0^\infty rJ_1(\xi r)\frac{f_\theta-2f_r^1-(4/r)\nabla_2^2\Phi}{2}dr\triangleq g_1(\xi,\ z) \tag{3.32}$$

适合式(3.27)的解答为

$$G_1=H_1[(A_1\xi z+g_1^*)e^{-\xi z};\ \xi\to r] \tag{3.33}$$

式中

$$g_1^*(\xi,z)=\frac{1}{2\xi}\int_0^z\exp[2\xi z_2]\int_0^{z_2}(2z_2-z-z_1)g_1(\xi,z_1)\exp[-\xi z_1]dz_1dz_2 \tag{3.34}$$

且A_1是ξ的一个任意函数.

用类似方法可得

$$\left.\begin{aligned}G_2&=H_1[(A_2\xi z+g_2^*)e^{-\xi z};\ \xi\to r]\\G_3&=H_2[(A_3\xi z+g_3^*)e^{-\xi z};\ \xi\to r]\\G_4&=H_0[(A_4\xi z+g_4^*)e^{-\xi z};\ \xi\to r]\end{aligned}\right\}\tag{3.35}$$

式中A_2，A_3，A_4是ξ的任意函数，且

$$g_i^*(\xi,z)=\frac{1}{2\xi}\int_0^z\exp[2\xi z_2]\int_0^{z_2}(2z_2-z-z_1)g_i(\xi,z_1)\exp[-\xi z_1]dz_1dz_2,\qquad i=2,3,4\tag{3.36}$$

$$\left.\begin{aligned}g_2(\xi,z)&=\int_0^\infty rJ_1(\xi r)\,\frac{-2f_r^1-f_\theta+(4/r)\nabla_2^2\Phi}{2}dr\\g_3(\xi,z)&=-\int_0^\infty rJ_2(\xi r)\left[f_z^2+\frac{\partial(\nabla_2^2\Phi)}{\partial z}\right]dr\\g_4(\xi,z)&=-\int_0^\infty rJ_0(\xi r)f_z^1dr\end{aligned}\right\}\tag{3.37}$$

求得Φ，G_1至G_4的解答后，需要确定任意函数A，A_1至A_4. 这可由位移分量代入应力分量，并使用边界条件(3.5)达到. 经大量代数变换后，我们得到

$$\left.\begin{aligned}A&=\frac{1}{(1-2\eta)\mu\xi^2}\left[\frac{(9\eta-8\eta^2)h_5}{4-5\eta+2\eta^2}+\frac{(3-4\eta)h_2+(1+2\eta-4\eta^2)h_3}{2(4-5\eta+2\eta^2)}\right]\\A_1&=\frac{1}{\xi^3}\left[\frac{(1-2\eta)h_1+2\eta h_4}{3-4\eta}+\frac{(1-\eta)h_2+h_3+2\eta h_5}{2(4-5\eta+2\eta^2)}\right]\\A_2&=\frac{1}{\xi^3}\left[\frac{(1-2\eta)h_1+2\eta h_4}{3-4\eta}-\frac{(1-\eta)h_2+h_3+2\eta h_5}{2(4-5\eta+2\eta^2)}\right]\\A_3&=\frac{1}{\xi^3}\left[\frac{4(1-\eta)h_5}{4-5\eta+2\eta^2}+\frac{(1-2\eta)h_3-(3-2\eta)h_2}{2(4-5\eta+2\eta^2)}\right]\\A_4&=\frac{(3-2\eta)h_4-2(1-\eta)h_1}{(3-4\eta)\xi^3}\end{aligned}\right\}\tag{3.38}$$

式中

$$\left.\begin{aligned}&h_1=\int_0^\infty rJ_1(\xi r)X_{vr}^1dr,\quad h_2=\int_0^\infty rJ_3(\xi r)(2X_{vr}^2+X_{v\theta})dr\\&h_3=\int_0^\infty rJ_1(\xi r)(2X_{vr}^2-X_{v\theta})dr,\quad h_4=\int_0^\infty rJ_0(\xi r)X_{vz}^1dr\\&h_5=\int_0^\infty rJ_2(\xi r)X_{vz}^2dr-\mu(1-\eta)\phi(\xi,0)\end{aligned}\right\}\tag{3.39}$$

由于$\boldsymbol{G}$和Ψ已知，我们可从方程(3.21)至(3.24)写出完整的二阶解答. 在半空间表面的二阶位移和应力分量可写为

$$
\left.
\begin{aligned}
2\mu w_r = & -2(1-\eta)\int_0^\infty xX_{vr}^1K_{11}(0,x)dx+(1-2\eta)\int_0^\infty xX_{vz}^1K_{10}(0,x)dx \\
& +\frac{\cos 2\theta}{4-5\eta+2\eta^2}\left\{\int_0^\infty x(2X_{vr}^2+X_{v\theta})\left[(1+\eta-2\eta^2)K_{13}(0,x)\right.\right. \\
& \left.+\frac{2\eta}{r}K_{33}(-1,x)\right]dx-\int_0^\infty x(2X_{vr}^2-X_{v\theta})\left[2(1-\eta)^2K_{11}(0,x)\right. \\
& \left.+\frac{2(1-2\eta)^2}{r}K_{21}(-1,x)\right]dx-\int_0^\infty xX_{vz}^2\left[(4+9\eta+4\eta^2)K_{12}(0,x)\right. \\
& \left.+\frac{2(4+5\eta-8\eta^2)}{r}K_{22}(-1,x)\right]dx+\mu(1-\eta)\int_0^\infty x\frac{2f_r^2+f_\theta}{2} \\
& \left.\cdot\left[(4+9\eta+\eta^2)K_{13}(-1,x)+\frac{2(4+5\eta-8\eta^2)}{r}K_{23}(-2,x)\right]dx\right\} \\
2\mu w_\theta = & \frac{2\sin\theta}{4-5\eta+2\eta^2}\left\{\int_0^\infty x(2X_{vr}^2-X_{v\theta})\left[(1-\eta)^2K_{13}(0,x)-\frac{3(1-\eta)}{r}\right.\right. \\
& \left.\cdot K_{23}(-1,x)\right]dx+\int_0^\infty x(2X_{vr}^2-X_{v\theta})[(1-\eta)K_{11}(0,x)_{2\eta(1-\eta)/r} \\
& \cdot K_{21}(-1,x)]dx+\int_0^\infty xX_{vz}^2\left[2\eta(1-\eta)K_{12}(0,x)\right. \\
& \left.+\frac{4-13\eta+8\eta^2}{r}K_{22}(-1,x)\right]dx-\mu(1-\eta)\int_0^\infty x\frac{f_r^2-f_\theta}{2} \\
& \left.\cdot\left[2\eta(1-\eta)K_{13}(-1,x)+\frac{4-13\eta+8\eta^2}{r}K_{23}(-2,x)\right]dx\right\} \\
2\mu w_z = & (1-2\eta)\int_0^\infty xX_{vr}^1K_{01}(0,x)dx-2(1-\eta)\int_0^\infty xX_{vz}^1K_{00}(0,x)dx \\
& +\frac{\cos 2\theta}{4-5\eta+2\eta^2}\left\{\frac{10-21\eta+8\eta^2}{2}\int_0^\infty x(X_{vr}^2+X_{v\theta})K_{23}(0,x)dx\right\} \\
& +(5\eta-6\eta^2)\int_0^\infty x(2X_{vr}^2-X_{v\theta})K_{12}(0,x)dx \\
& -(8-34\eta+24\eta^2)\left[\int_0^\infty xX_{vz}^2K_{22}(0,x)dx\right. \\
& \left.-\mu(1-\eta)\int_0^\infty x\frac{f_r^2+f_\theta}{2}K_{23}(-1,x)dx\right]
\end{aligned}
\right\}
\quad (3.40)
$$

$$
\tau''_{zz}=X_{vz}^1+X_{vz}^2\cos 2\theta,\quad \tau''_{rz}=X_{vr}^1+X_{vr}^2\cos 2\theta,\quad \tau''_{\theta z}=X_{v\theta}\cos\theta\sin\theta \quad (3.41)
$$

式中

$$
\begin{aligned}
X_{vr}^1= & \frac{T^2}{8\mu}\left[\frac{2\eta(1-2\eta)}{r^2}L(1,-(1+\delta))L(2,-(1+\delta))\right. \\
& -\frac{\eta(1-2\eta)}{r}L(0,-\delta)L(2,-(1+\delta))+\frac{1-2\eta}{2r}L(1,-\delta) \\
& \left.\cdot L(1,-(1+\delta))-\eta L(1,-\delta)L(0,-\delta)\right]-\tau_{rz}^1 \\
X_{vr}^2= & \frac{T^2}{8\mu}\left[\frac{3(1-2\eta)}{2r}L(1,-\delta)L(1,-(1+\delta))\right.
\end{aligned}
$$

$$-\frac{\eta(1-2\eta)}{r}L(0,-\delta)L(2,-(1+\delta))-\eta L(1,-\delta)L(0,-\delta)\Big]-\tau^{2}_{rz}$$

$$X^{1}_{vz}=\frac{(1-2\eta)T^{2}}{16\mu}I^{2}(1,-\delta)-\tau^{1}_{zz}$$

$$X^{2}_{vz}=\frac{T^{2}}{8\mu}\Big[\frac{1-2\eta}{2}I^{2}(0,-\delta)-\frac{1-2\eta}{2}L(0,-\delta)L(1,-(1+\delta))\Big]-\tau^{2}_{zz}$$

$$X_{v\theta}=\frac{T^{2}}{4\mu}\Big[\frac{3-4\eta}{2}L(0,-\delta)L(1,-\delta)-\frac{\eta(1-\eta)}{r}L(0,-\delta)L(2,-(1+\delta))$$

$$-\frac{(1-2\eta)(1+6\eta)}{2r}L(1,-\delta)L(1,-(1+\delta))\Big]-\tau^{1}_{\theta z} \qquad (3.42)$$

$$\tau^{1}_{rz}=\frac{T^{2}}{16\mu^{2}}\Big[b_{1}L(0,-\delta)L(1,-\delta)+b_{2}L(1,-\delta)L(2,-\delta)$$

$$+\frac{b_{3}}{r}L(1,-\delta)L(1,-(1+\delta))+\frac{b_{4}}{r^{2}}L(1,-(1+\delta))L(2,-(1+\delta))$$

$$+\frac{b_{5}}{r}L(0,-\delta)L(2,-(1+\delta))\frac{b_{6}}{r}L(2,-\delta)L(2,-(1+\delta))\Big]$$

$$\tau^{2}_{rz}=\frac{T^{2}}{16\mu^{2}}\Big[b_{7}L(1,-\delta)L(0,-\delta)+b_{8}L(1,-\delta)L(2,-\delta)$$

$$+\frac{b_{9}}{r}L(1,-\delta)L(1,-(1+\delta))+\frac{b_{10}}{r}L(0,-\delta)L(2,-(1+\delta))\Big]$$

$$\tau^{1}_{zz}=\frac{T^{2}}{16\mu^{2}}\Big[b_{11}I^{2}(1,-\delta)b_{12}I^{2}(0,-\delta)+\frac{b_{13}}{r^{2}}I^{2}(1,-(1+\delta))$$

$$+\frac{b_{14}}{r^{2}}I^{2}(2,-(1+\delta))+\frac{b_{15}}{r}L(0,-\delta)L(1,-(1+\delta))$$

$$+\frac{b_{16}}{r}L(1,-\delta)L(2,-(1+\delta))+b_{17}I^{2}(2,-\delta)$$

$$+\frac{b_{18}}{r}L(2,-\delta)L(1,-(1+\delta))b_{19}L(0,-\delta)L(2,-\delta)\Big]$$

$$\tau^{2}_{zz}=\frac{T^{2}}{16\mu^{2}}\Big[b_{20}I^{2}(1,-\delta)+b_{21}I^{2}(0,-\delta)+\frac{b_{22}}{r}L(0,-\delta)L(1,-(1+\delta))$$

$$+b_{23}L(1,-\delta)L(2,-(1+\delta))+b_{24}L(0,-\delta)L(2,-\delta)\Big]$$

$$\tau^{1}_{\theta z}=\frac{T^{2}}{8\mu^{2}}\Big[\frac{b_{25}}{r}L(1,-\delta)L(1,-(1+\delta))+b_{26}L(1,-\delta)L(0,-\delta)$$

$$+b_{27}L(1,-\delta)L(2,-\delta)+\frac{b_{28}}{r}L(0,-\delta)L(2,-(1+\delta))$$

$$+\frac{b_{29}}{r^{2}}L(1,-(1+\delta))L(2,-(1+\delta))\Big]$$

$$f^{2}_{r}=\frac{T^{2}}{16\mu^{2}}\Big[b_{30}L(0,-\delta)L(0,-\delta)+\frac{b_{32}}{r^{2}}L(1,-\delta)L(2,-(1+\delta)) \qquad (3.43)$$

$$
\begin{aligned}
&+\frac{b_{33}}{r}L(0,1-\delta)L(2,-(1+\delta))+b_{34}L(0,-\delta)L(1,1-\delta)\\
&+b_{35}L(2,-\delta)L(1,1-\delta)+\frac{b_{36}}{r}L(1,-(1+\delta))L(1,1-\delta)\\
&+\frac{b_{37}}{r}L(0,-\delta)L(2,-\delta)+\frac{b_{38}}{r^2}L(0,-\delta)L(1,-(1+\delta))\\
&+b_{39}L(1,-\delta)L(2,1-\delta)+\frac{b_{40}}{r^2}L(2,-\delta)L(1,-(1+\delta))\\
&+\frac{b_{41}}{r}I^2(0,-\delta)\Big]
\end{aligned}
$$

$$
\begin{aligned}
f_\theta=\frac{T^2}{8\mu^2}\Big[&b_{42}L(1,-\delta)L(0,1-\delta)+\frac{b_{43}}{r}I^2(0,-\delta)\\
&+b_{44}L(0,-\delta)L(1,1-\delta)\\
&+b_{45}L(2,-\delta)L(1,1-\delta)+\frac{b_{46}}{r}I^2(2,-\delta)\\
&+\frac{b_{47}}{r}L(0,1-\delta)L(2,-(1+\delta))\\
&+\frac{b_{48}}{r^2}L(1,-\delta)L(2,-(1+\delta))+\frac{b_{49}}{r}L(1,1-\delta)L(1,-(1+\delta))\\
&+b_{50}L(1,-\delta)L(2,-\delta)+\frac{b_{51}}{r}L(0,-\delta)L(2,-\delta)\\
&+\frac{b_{52}}{r^2}L(2,-\delta)L(1,-(1+\delta))+\frac{b_{53}}{r}I^2(0,-\delta)\\
&+\frac{b_{54}}{r^2}L(0,-\delta)L(1,-(1+\delta))\Big]
\end{aligned}
$$

且式中有如下关系

$$\tau'_{rz}=\tau^1_{rz}+\tau^2_{rz}\cos2\theta,\quad \tau'_{\theta z}=\tau^1_{\theta z}+\tau^2_{\theta z}\cos\theta\sin\theta,\quad \tau'_{zz}=\tau^1_{zz}+\tau^2_{zz}\cos2\theta \tag{3.44}$$

τ''_{rr}，$\tau''_{\theta\theta}$，$\tau''_{r\theta}$具有相类似的表达式（参见Guo[19]），注意到量$K_{ij}(s,x)$列于附录Ⅱ中，而由a_1至a_5及η表达的b_{ij}可在Guo[19]中找到。

附 录 Ⅰ

设边界是$z=0$，且弹性体占据半空间，$z\geqslant0$，在边界上有$(l_1,l_2,l_3)=(0,0,-1)$，边界$z=0$上作用力取如下形式

$$
\begin{aligned}
X'_{\nu r}&=-\frac{\partial v_z}{\partial r}\tau_{rr}-\frac{\tau_{r\theta}}{r}\frac{\partial v_z}{\partial\theta}+\left(\Delta'-\frac{\partial v_z}{\partial z}\right)\tau_{rz}+\tau'_{rz}\\
X'_{\nu\theta}&=-\frac{\partial v_z}{\partial r}\tau_{r\theta}-\frac{\tau_{\theta\theta}}{r}\frac{\partial v_z}{\partial\theta}+\left(\Delta'-\frac{\partial v_z}{\partial z}\right)\tau_{\theta z}+\tau'_{\theta z}\\
X'_{\nu z}&=-\frac{\partial v_z}{\partial r}\tau_{rz}-\frac{\tau_{\theta z}}{r}\frac{\partial v_z}{\partial\theta}+\left(\Delta'-\frac{\partial v_z}{\partial z}\right)\tau_{zz}+\tau'_{zz}
\end{aligned}
$$

此处

$$\Delta'=\frac{\partial v_r}{\partial r}+\frac{v_r}{r}+\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{\partial v_z}{\partial z}$$

体力项表示为

$$\rho_0 X'_r=-\frac{\partial v_r}{\partial r}\frac{\partial \tau_{rr}}{\partial r}+\frac{2\tau_{r\theta}}{r}\frac{\partial v_\theta}{\partial r}-\frac{\partial v_z}{\partial r}\frac{\partial \tau_{rr}}{\partial z}+\frac{v_\theta}{r}\frac{\partial \tau_{r\theta}}{\partial r}-\frac{v_r}{r^2}(\tau_{rr}-\tau_{\theta\theta})$$

$$-\frac{\partial v_r}{\partial z}\frac{\partial \tau_{rz}}{\partial r}+\frac{\tau_{\theta z}}{r}\frac{\partial v_\theta}{\partial z}-\frac{\partial v_z}{\partial z}\frac{\partial \tau_{rz}}{\partial z}-\frac{1}{r}\left[\frac{\partial v_r}{\partial \theta}\frac{\partial \tau_{r\theta}}{\partial r}+\frac{\partial \tau_{rr}}{\partial \theta}\frac{\partial v_\theta}{\partial r}+\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}\frac{\partial \tau_{r\theta}}{\partial \theta}\right]$$

$$+\left[\frac{v_r}{r}\frac{\partial \tau_{r\theta}}{\partial \theta}+\frac{\partial v_\theta}{\partial \theta}\frac{\tau_{rr}-\tau_{\theta\theta}}{r}+\frac{\partial \tau_{rz}}{\partial \theta}\frac{\partial v_\theta}{\partial z}+\frac{\partial v_z}{\partial \theta}\frac{\partial \tau_{r\theta}}{\partial z}\right]$$

$$+\frac{\partial \tau'_{rr}}{\partial r}+\frac{1}{r}\frac{\partial \tau'_{rr}}{\partial \theta}+\frac{\partial \tau'_{rz}}{\partial z}+\frac{\tau'_{rr}-\tau'_{\theta\theta}}{r}$$

附　录　Ⅱ

$$K_{00}(0,x)=\begin{cases}(2/\pi r)F(x/r), & x<r\\(2/\pi x)F(r/x), & x>r\end{cases}$$

$$K_{11}(0,x)=\begin{cases}(2/\pi r)[F(x/r)-E(x/r)], & x<r\\(2/\pi x)[F(r/x)-E(r/x)], & x>r\end{cases}$$

$$K_{10}(0,x)=\begin{cases}1/r, & x<r\\0, & x>r\end{cases}$$

$$K_{01}(0,x)=\begin{cases}0, & x<r\\1/x, & x>r\end{cases}$$

$$K_{02}(0,x)=\begin{cases}\frac{2}{\pi r}F\left(\frac{x}{r}\right)+\frac{4r}{\pi x^2}\left[E\left(\frac{x}{r}\right)-F\left(\frac{x}{r}\right)\right], & x<r\\\frac{4}{\pi x}E\left(\frac{r}{x}\right)-\frac{2}{\pi x}F\left(\frac{r}{x}\right), & x>r\end{cases}$$

$$K_{03}(0,x)=\begin{cases}0, & x<r\\(1/x)(1-2r^2/x^2), & x>r\end{cases}$$

$$K_{12}(0,x)=\begin{cases}0, & x<r\\r/x^2, & x>r\end{cases}$$

$$K_{21}(0,x)=\begin{cases}x/r^2, & x<r\\0, & x>r\end{cases}$$

$$K_{13}(0,x)=\begin{cases}\frac{13F(x/r)-5E(x/r)}{3\pi x}+\frac{16r^2}{3\pi x^3}\left[E\left(\frac{x}{r}\right)-F\left(\frac{x}{r}\right)\right], & x<r\\\frac{11}{3\pi r}\left[F\left(\frac{r}{x}\right)-E\left(\frac{r}{x}\right)\right]+\frac{8r}{3\pi x^2}\left[2E\left(\frac{r}{x}\right)-F\left(\frac{r}{x}\right)\right], & x>r\end{cases}$$

$$K_{22}(0,x)=\begin{cases}\frac{2}{3\pi r}\left[F\left(\frac{x}{r}\right)-2E\left(\frac{x}{r}\right)\right]-\frac{4r}{3\pi x^2}\left[E\left(\frac{x}{r}\right)-F\left(\frac{x}{r}\right)\right], & x<r\\\frac{2}{3\pi x}\left[F\left(\frac{r}{x}\right)-2E\left(\frac{r}{x}\right)\right]-\frac{4x}{3\pi r^2}\left[E\left(\frac{r}{x}\right)-F\left(\frac{r}{x}\right)\right], & x>r\end{cases}$$

$$K_{23}(0,x)=\begin{cases}0, & x<r\\r^2/x^3, & x>r\end{cases}$$

$$K_{13}(-1,x)=\begin{cases}0, & x<r\\(r/2x)(1-r^2/x^2), & x>r\end{cases}$$

$$K_{21}(-1,x)=\begin{cases}\frac{2x}{3\pi r}\left[F\left(\frac{x}{r}\right)+2E\left(\frac{x}{r}\right)\right]+\frac{2r}{3\pi x}\left[E\left(\frac{x}{r}\right)-F\left(\frac{x}{r}\right)\right], & x<r\\ \frac{2}{3\pi}\left[2F\left(\frac{r}{x}\right)-E\left(\frac{r}{x}\right)\right]+\frac{4x^2}{3\pi r^2}\left[E\left(\frac{r}{x}\right)-F\left(\frac{r}{x}\right)\right], & x>r\end{cases}$$

$$K_{22}(-1,x)=\begin{cases}x^2/4r^2, & x<r\\ r^2/4x^2, & x>r\end{cases}$$

$$K_{23}(-2,x)=\begin{cases}x^3/24r^2, & x<r\\ (r^2/8x)(1-2r^2/3x^3), & x>r\end{cases}$$

$$K_{23}(-1,x)=-\frac{2x}{15\pi r}\left[9F\left(\frac{x}{r}\right)+2E\left(\frac{x}{r}\right)\right]+\frac{8r}{15\pi x}F\left(\frac{x}{r}\right)$$
$$+\left(\frac{16r^3}{15\pi x^3}-\frac{24r}{15\pi x}\right)\left[E\left(\frac{x}{r}\right)-F\left(\frac{x}{r}\right)\right], \quad x<r$$

$$K_{23}(-1,x)=\frac{8r^2}{15\pi x^2}\left[2F\left(\frac{r}{x}\right)-E\left(\frac{r}{x}\right)\right]+\frac{2}{15\pi}\left[2F\left(\frac{r}{x}\right)-3E\left(\frac{r}{x}\right)\right]$$
$$-\frac{4x^2}{15\pi r^2}\left[E\left(\frac{r}{x}\right)-F\left(\frac{r}{x}\right)\right], \quad x>r$$

式中$E(r)$与$F(r)$分别是第一类和第二类完全椭圆积分．

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The General Solution of Second Order Effects in Elastic Half-Space Acted upon by a Non-Uniform Shear Load

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Abstract

This paper is a continuation of [1]. A closed form solution to the second order elasticity problem, when an isotropic compressible elastic half-space undergoes a deformation owing to a non-uniformly distributed shear load, is presented. The method of integral transform is employed to determine the solutions.

Key words elastic half-space, shear load, second order elasticity effects, integra l transform