应用数学和力学，第18卷 第1期（1997年1月）
Applied Mathematics and Mechanics

应用数学和力学编委会编
重庆出版社出版

弹性圆板在一侧受均载而四周固定的条件下不用Kirchhoff-Love假设的一级近似理论（Ⅰ）

钱伟长[1]

（1995年11月30日收到）

摘　要

根据一般形状的三维弹性板不用 Kirchhoff-Love 假设的近似理论[1],[2]，作者导出了三维弹性圆板的广义变分泛函，从而得到了圆板四周固定和一侧受均布载荷下的一级近似理论的微分方程和有关边界条件，其解析解答留待另文处理。

关键词　弹性圆板　Kirchhoff-Love 假设　广义变分原理

一、引　论

三维圆板的轴对称弹性问题可以作为三维轴对称弹性力学问题处理。板厚 h 为一常数，我们将用中面上的极坐标$(r,\ \theta)$和垂直于中面的横坐标z来研究这一问题（图1），我们将用 σ_r，σ_θ，σ_z，$\sigma_{r\theta}=\sigma_{\theta r}$，$\sigma_{zr}=\sigma_{rz}$，$\sigma_{\theta z}=\sigma_{z\theta}$ 表示应力张量的分量，用 e_r，e_θ，$e_{r\theta}=e_{\theta r}$，$e_{zr}=e_{rz}$，$e_{\theta z}=e_{z\theta}$表示应变张量的分量。对于轴对称问题而言，我们有

$$\left.\begin{aligned}&\sigma_{r\theta}=\sigma_{\theta r}=0, \qquad \sigma_{\theta z}=\sigma_{z\theta}=0\\ &e_{r\theta}=e_{\theta r}=0, \qquad e_{\theta z}=e_{z\theta}=0\end{aligned}\right\} \tag{1.1}$$

我们在轴对称问题中，只有两个位移分量：即轴向位移$W(r,\ z)$，和径向位移$U(r,\ z)$。上述应力分量、应变分量、和位移分量满足下列关系。

(i)　**应变位移关系**

$$\left.\begin{aligned}&e_r=\frac{\partial U}{\partial r},\quad e_\theta=\frac{U}{r},\quad e_z=\frac{\partial W}{dz}\\ &e_{rz}=\frac{1}{2}\left(\frac{\partial U}{\partial z}+\frac{\partial W}{\partial r}\right),\quad e_{\theta z}=e_{\theta r}=0\end{aligned}\right\} \tag{1.2}$$

1　上海大学，上海市应用数学和力学研究所，上海 200072.

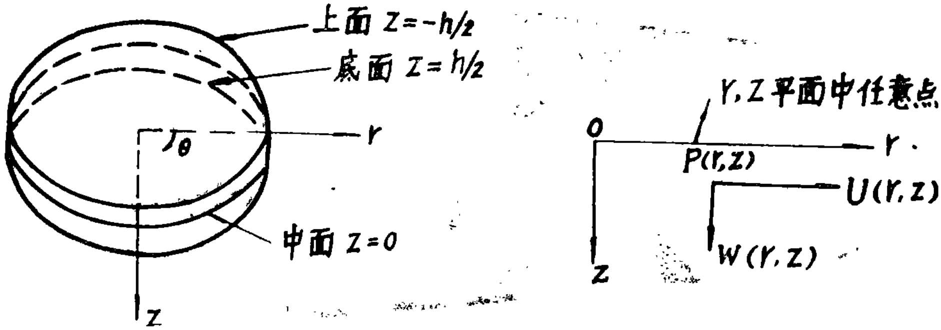

图1 轴对称板 坐标$(r,\ \theta,\ z)$，位移$U(r,\ z)$，$W(r,\ z)$

(ii) **应变应力关系**

$$\left.\begin{aligned} Ee_r&=\sigma_r-\nu(\sigma_\theta+\sigma_z), & Ee_{rz}&=(1+\nu)\sigma_{rz} \\ Ee_\theta&=\sigma_\theta-\nu(\sigma_r+\sigma_z), & Ee_{\theta z}&=(1+\nu)\sigma_{\theta z}=0 \\ Ee_z&=\sigma_z-\nu(\sigma_r+\sigma_\theta), & Ee_{\theta r}&=(1+\nu)\sigma_{\theta r}=0 \end{aligned}\right\} \tag{1.3}$$

或应力应变关系

$$\left.\begin{aligned} \sigma_r&=\frac{E_1}{1-\nu_1^2}[e_r+\nu_1(e_\theta+e_z)], & \sigma_{rz}&=\frac{E_1}{1+\nu_1}e_{rz} \\ \sigma_\theta&=\frac{E_1}{1-\nu_1^2}[e_\theta+\nu_1(e_r+e_z)], & \sigma_{\theta z}&=\frac{E_1}{1+\nu_1}e_{\theta z}=0 \\ \sigma_z&=\frac{E_1}{1-\nu_1^2}[e_z+\nu_1(e_r+e_\theta)], & \sigma_{\theta r}&=\frac{E_1}{1+\nu_1}e_{\theta r}=0 \end{aligned}\right\} \tag{1.4}$$

其中 E，ν 分别为材料的杨氏模量和泊松比，而E_1，ν_1 分别为平面应变问题的折合杨氏模量和折合泊松比。它们满足下列关系

$$E_1=\frac{E}{1-\nu^2},\quad \nu_1=\frac{\nu}{1-\nu},\quad \frac{E}{1+\nu}=\frac{E_1}{1+\nu_1} \tag{1.5}$$

(iii) **应力平衡方程**

$$\frac{1}{r}\ \frac{d}{dr}(r\sigma_r)-\frac{\sigma_\theta}{r}+\frac{\partial\sigma_{rz}}{\partial z}=0 \tag{1.6a}$$

$$\frac{\partial\sigma_z}{\partial z}+\frac{1}{r}\ \frac{\partial}{\partial r}(r\sigma_{zr})=0 \tag{1.6b}$$

其中体力业已略去。

(iv) **作用上表面($z=-h/2$)和底表面($z=h/2$)的外力**

$$\sigma_z=-q,\ \sigma_{rz}=0 \qquad (\text{在上表面}z=-h/2\text{上，}q>0\text{为压力}) \tag{1.7a}$$

$$\sigma_z=0,\ \sigma_{rz}=0 \qquad (\text{在底表面}z=h/2\text{上，未受力}) \tag{1.7b}$$

这里必须指出，在经典薄板理论中，上述受力状态和底面受拉载q上表面未受力的结果相同，但在不用Kirchhoff-Love假定的理论中，其结果是不同的。

(v)　四周边界表面条件（完全固定）

$$U(a,\ z)=0,\qquad W(a,\ z)=0 \tag{1.8}$$

其中　a是圆板的半径。

本文的目的是在(1.7)，(1.8)的条件下，从(1.2)，(1.4)，(1.6)等 14 个偏微分方程中求解应力分量σ_r，σ_θ，σ_z，σ_{rz}，$\sigma_{r\theta}=\sigma_{z\theta}=0$，应变分量$e_r$，$e_\theta$，$e_z$，$e_{rz}$，$e_{r\theta}=e_{z\theta}=0$以及$U(r,\ z)$，$W(r,\ z)$等14个待定量。由于$\sigma_{r\theta}$，$\sigma_{z\theta}$，$e_{r\theta}$，$e_{z\theta}$恒等于零。其实只有 10 个不恒等于零的待定量和不恒等于零的10个偏微分方程。

二、弹性圆板在均布载荷（在z＝－h/2）和四周固定的边界条件下平衡的广义变分原理

弹性板在已给表面载荷和四周支撑条件下平衡时的广义变分原理业已在前文[1]研究过。本文将研究弹性圆板在均布表面载荷和四周固定的边界条件下轴对称平衡时的广义变分原理。

三维弹性体在轴对称应变下的应变能密度ε为

$$\varepsilon=\frac{E_1}{2(1-\nu_1^2)}\left[(e_r+e_\theta+e_z)^2+2(1-\nu_1)(e_{rz}^2-e_re_z-e_re_\theta-e_ze_\theta)\right] \tag{2.1}$$

只要σ_θ，σ_r，σ_z，σ_{rz}和e_r，e_θ，e_z，e_{rz}之间满足应力应变关系（1.4）式，我们很易证明下列变分关系：

$$\delta\varepsilon=\sigma_r\delta e_r+\sigma_\theta\delta e_\theta+\sigma_z\delta e_z+2\sigma_{rz}\delta e_{rz} \tag{2.2}$$

我们在这里将证明，以σ_r，σ_θ，σ_z，σ_{rz}；e_r，e_θ，e_z，e_{rz}；U，W为变量的本问题的广义变分原理可以写成

$$\delta\Pi=0\qquad(\text{驻值}) \tag{2.3}$$

其中　Π为下列广义变分原理泛函：

$$\begin{aligned}\Pi=&\int_0^a\int_{(h)}\varepsilon 2\pi r dr dz-\int_0^a qW_-2\pi r dr-\int_{(h)}(U\sigma_r+W\sigma_{rz})_{r=a}2\pi a dz\\&-\int_0^a\int_{(h)}\left\{\sigma_r\left[e_r-\frac{\partial U}{\partial r}\right]+\sigma_\theta\left[e_\theta-\frac{U}{r}\right]+\sigma_z\left[e_z-\frac{\partial W}{\partial z}\right]\right.\\&\left.+2\sigma_{rz}\left[e_{rz}-\frac{1}{2}\left(\frac{\partial W}{\partial r}+\frac{\partial U}{\partial z}\right)\right]\right\}2\pi r dr dz\end{aligned} \tag{2.4}$$

其中W_-为上表面的垂直位移，$\int_{(h)}(\cdots)dz$为跨板厚的简写积分号

$$\int_{(h)}(\cdots\cdots)dz=\int_{-h/2}^{+h/2}(\cdots\cdots)dz \tag{2.5}$$

现在让我们证明，在σ_r，σ_θ，σ_z，σ_{zr}；e_r，e_θ，e_z，e_{zr}；U，W都是独立变量的条件下，(2.3)式的变分驻值条件给出：(1)应变位移关系(1.2)式，(2)应力应变关系(1.4)式，(3)应力平衡方程(1.6)式，(4)上下表面受力条件(1.7)式，(5)周环边界固定条件（1.8）式等本题应满足的一切条件。

Π变分后，得

$$\delta\Pi=\int_0^a\int_{(h)}\left\{\left[\frac{E_1}{1-\nu_1^2}(e_r+\nu_1e_\theta+\nu_1e_z)-\sigma_r\right]\delta e_r\right.$$
$$+\left[\frac{E_1}{1-\nu_1^2}(e_\theta+\nu_1e_r+\nu_1e_z)-\sigma_\theta\right]\delta e_\theta$$
$$\left.+\left[\frac{E_1}{1-\nu_1^2}(e_z+\nu_1e_r+\nu_1e_\theta)-\sigma_z\right]\delta e_z+2\left[\frac{E_1}{1+\nu}e_{rz}-\sigma_{rz}\right]\delta e_{rz}\right\}2\pi rdrdz$$
$$-\int_0^a\int_{(h)}\left\{\left[e_r-\frac{\partial U}{\partial r}\right]\delta\sigma_r+\left[e_\theta-\frac{U}{r}\right]\delta\sigma_\theta+\left[e_z-\frac{\partial W}{\partial z}\right]\delta\sigma_z\right.$$
$$\left.+\left[e_{rz}-\frac{1}{2}\left(\frac{\partial U}{\partial z}+\frac{\partial W}{\partial r}\right)\right]\delta\sigma_{rz}\right\}2\pi rdrdz+\int_0^a\int_{(h)}\left\{\sigma_r\frac{\partial\delta U}{\partial r}+\sigma_\theta\frac{\delta U}{r}\right.$$
$$\left.+\sigma_z\frac{\partial\delta W}{\partial z}+\sigma_{rz}\left(\frac{\partial\delta U}{\partial z}+\frac{\partial\delta W}{\partial r}\right)\right\}2\pi rdrdz-\int_0^a(q\delta W_-)2\pi rdr$$
$$-\int_{(h)}[\sigma_r\delta U+\sigma_{rz}\delta W+U\delta\sigma_r+W\delta\sigma_{rz}]_{r=a}2\pi adz \tag{2.6}$$

通过分部积分，我们可以证明

$$\int_0^a\int_{(h)}\sigma_r\frac{\partial\delta U}{\partial r}2\pi rdrdz=\int_{(h)}(\sigma_r\delta U)_{r=a}2\pi adz-\int_0^a\int_{(h)}\frac{\partial}{\partial r}(r\sigma_r)\delta U2\pi drdz$$
$$\int_0^a\int_{(h)}\sigma_{rz}\frac{\partial\delta W}{\partial r}2\pi rdrdz=\int_{(h)}(\sigma_{rz}\delta W)_{r=a}2\pi adz-\int_0^a\int_{(h)}\frac{\partial}{\partial r}(r\sigma_{rz})\delta W2\pi drdz$$
$$\int_0^a\int_{(h)}\sigma_z\frac{\partial\delta W}{\partial r}2\pi rdrdz=\int_{(h)}(\sigma_z^+\delta W_+-\sigma_z^-\delta W_-)2\pi rdr-\int_0^a\int_{(h)}\frac{\partial\sigma_z}{\partial z}\delta W2\pi rdrdz$$
$$\int_0^a\int_{(h)}\sigma_{rz}\frac{\partial\delta U}{\partial r}2\pi rdrdz=\int_{(h)}(\sigma_{rz}^+\delta U_+-\sigma_{rz}^-\delta U_-)2\pi rdr-\int_0^a\int_{(h)}\frac{\partial}{\partial r}(\sigma_{rz})\delta U2\pi rdrdz \tag{2.7}$$

于是(2.6)式可以写成

$$\delta\Pi=\int_0^a\int_{(h)}\left\{\left[\frac{E_1}{1-\nu_1^2}(e_r+\nu_1e_\theta+\nu_1e_z)-\sigma_r\right]\delta e_r+\left[\frac{E_1}{1-\nu_1^2}(e_\theta+\nu_1e_r+\nu_1e_z)-\sigma_\theta\right]\delta e_\theta\right.$$
$$\left.+\left[\frac{E_1}{1-\nu_1^2}(e_z+\nu_1e_r+\nu_1e_\theta)-\sigma_z\right]\delta e_z+2\left[\frac{E_1}{1+\nu_1}e_{rz}-\sigma_{rz}\right]\delta e_{rz}\right\}2\pi rdrdz$$
$$-\int_0^a\int_{(h)}\left\{\left[e_r-\frac{\partial U}{\partial r}\right]\delta\sigma_r+\left[e_\theta-\frac{U}{r}\right]\delta\sigma_\theta+\left[e_z-\frac{\partial W}{\partial z}\right]\delta\sigma_z\right.$$
$$\left.+\left[e_{rz}-\frac{1}{2}\left(\frac{\partial U}{\partial z}+\frac{\partial W}{\partial r}\right)\right]\delta\sigma_{rz}\right\}2\pi rdrdz-\int_0^a\int_{(h)}\left\{\frac{1}{r}\frac{\partial}{\partial r}(r\sigma_r)\right.$$
$$\left.-\frac{\sigma_\theta}{r}+\frac{\partial\sigma_{rz}}{\partial z}\right]\delta U+\left[\frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{rz})+\frac{\partial\sigma_z}{\partial z}\right]\delta W\left.\right\}2\pi rdrdz$$
$$+\int_0^a\left\{[\sigma_z^+\delta W_+-(q+\sigma_z^-)\delta W_-+\sigma_{rz}^+\delta U_+-\sigma_{rz}^-\delta U_-\right\}2\pi rdr$$
$$-\int_{(h)}(U\delta\sigma_r+W\delta\sigma_{rz})_{r=a}2\pi adz \tag{2.8}$$

这里必须指出：$\delta\sigma_r$，$\delta\sigma_\theta$，$\delta\sigma_z$，$\delta\sigma_{zr}$在板内各点和边界面内各点上，$\delta\theta_r$，$\delta\theta_r$，$\delta\theta_z$，$\delta\theta_{zr}$在

极内各点上，和δU，δW在板的上下表面各点上和板内各点上都是独立变分，变分$\delta\Pi=0$的驻值条件给出(1.2)，(1.4)，(1.6)，(1.7)，(1.8)式。这就证明了$\delta\Pi=0$的驻值条件是代表本题的广义变分原理，或即是说：这一变分原理，业已解除了所有变分约束条件，是最广义的变分原理。

Π这一泛函虽已消除了一切约束条件，但其计算相当繁重，并不实用。如果我们保留一部分约束条件，必能进一步简化计算。

现设在σ_r，σ_θ，σ_z，σ_{zr}，e_r，e_θ，e_z，e_{zr}，U，W中保留(1.2)，(1.4)式作为变分约束条件，则这些变量所应满足的变分泛函可以写成

$$\Pi^*=\int_0^a\int_{(h)}\varepsilon 2\pi rdrdz-\int_0^a qW_-2\pi rdr-\int_{(h)}(U\sigma_r+W\sigma_{rz})_{r=a}2\pi adz \tag{2.9}$$

而且$\delta\varepsilon$满足(2.2)式。如果σ_r，σ_θ，σ_z，σ_{zr}，e_r，e_θ，e_z，e_{rz}，U,W在约束条件(1.2),(1.4)式下变分，则泛函Π^*的变分驻值条件必须满足(1.7)，(1.8)，(1.9)式；也必为本题的解。其证明和Π在没有任何约束条件的变分驻值问题相同，不再重复。

三、最一般的不用克希霍夫-拉夫假设的弹性圆板理论（e_z，$e_{rz}\neq 0$）和一级近似理论

我们在前文[1]，[2]中业已指出，在不用 Kirchhoff-Love 假设的条件下，e_z，e_{rz}既然不等于零，则可以用z的幂级数来表示，一般可以设

$$e_z=\sum_{k=0}^{\infty}A_kz^k \tag{3.1a}$$

$$e_{rz}=\sum_{k=0}^{\infty}(S_{2k}+zS_{2k+1})\left(\frac{1}{4}h^2-z^2\right)z^{2k} \tag{3.1b}$$

其中A_k，S_{2k}，S_{2k+1}都是待定的r的场函数。这里必须指出，e_{rz}的表达式业已满足上下表面上剪应力为零的条件(1.7)式。用(1.2c)式对z积分，得W的z幂级数表达式。把它代入(1.2d)式，再对z积分，得U的z幂级数表达式，结果为

$$W(r,z)=w(r)+\sum_{k=0}^{\infty}\frac{1}{k+1}A_k(r)z^{k+1} \tag{3.2a}$$

$$U(r,z)=u(r)-\frac{dw}{dr}z-\sum_{k=0}^{\infty}\frac{1}{(k+1)(k+2)}\frac{dA_k}{dr}z^{k+2}+2\sum_{k=0}^{\infty}\left\{\frac{1}{2k+1}\left[\frac{1}{4}h^2-\frac{2k+1}{2k+3}z^2\right]z^{2k+1}S_{2k}+\frac{1}{2k+2}\left[\frac{1}{4}h^2-\frac{2k+2}{2k+4}z^2\right]z^{2k+2}S_{2k+1}\right\} \tag{3.2b}$$

其中 $u(r)$，$w(r)$实际上是中面各点的位移分量，它们和A_k,S_{2k},S_{2k+1}都是r的待定场函数。把(3.2)式代入(1.2)式就可以计算e_r，e_θ分量的表达式，它们所满足的微分方程和边界条件，都可以通过泛函Π^*的变分驻值条件求得。我们必须指出，用上述方法决定的U，W，e_r，e_θ，e_z，e_{zr}，σ_r，σ_θ，σ_z，σ_{zr}等表达式都可满足了(1.2)，(1.4)式。所以我们可以用$\delta\Pi^*=0$的广义变分原理（驻值条件）来求得u，w，A_k，S_{2k}，$S_{2k+1}(k=0,1,2,\cdots)$的常微分

方程和相关的边界条件.

为了求得合理的近似理论，我们可以近似地取幂级数的少数项来进行近似运算. 本文将给出e_z中的两项和e_{zr}中的三项的近似理论，称之为一级近似理论.

取近似表达式

$$e_z=A_0+A_1z \tag{3.3a}$$

$$e_{rz}=\left(\frac{h^2}{4}-z^2\right)[S_0+S_1z+S_2z^2] \tag{3.3b}$$

其中有A_0，A_1，S_0，S_1，S_2五个待定函数. e_z，e_{zr}对z积分，求得$U(r, z)$，$W(r, z)$的z幂级数表达式

$$W(r, z)=w(r)+A_0z+\frac{1}{2}A_1z^2 \tag{3.4a}$$

$$U(r, z)=u(r)-\frac{dw}{dr}z-\frac{1}{2}\frac{dA_0}{dr}z^2-\frac{1}{6}\frac{dA_1}{dr}z^3+2\left(\frac{1}{4}h^2-\frac{1}{3}z^2\right)zS_0$$
$$+\left(\frac{1}{4}h^2-\frac{1}{2}z^2\right)z^2S_1+\frac{2}{3}\left(\frac{1}{4}h^2-\frac{3}{5}z^2\right)z^3S_2 \tag{3.4b}$$

其中 $w(r)$，$u(r)$ 为待定的积分函数，(3.4) 为待定量 u，w，A_0，A_1，S_0，S_1，S_2 表示的$U(r, z)$，$W(r, z)$表达式. 除(3.3)式的e_z，e_{zr} 外，其余应变分量为

$$e_r=\frac{\partial U}{\partial r}=\frac{du}{dr}-\frac{d^2w}{dr^2}z-\frac{1}{2}\frac{d^2A_0}{dr^2}z^2-\frac{1}{6}\frac{d^2A_1}{dr^2}z^3+2\left[\left(\frac{h}{2}\right)^2-\frac{1}{3}z^2\right]z\frac{dS_0}{dr}$$
$$+\left[\left(\frac{h}{2}\right)^2-\frac{1}{2}z^2\right]z^2\frac{dS_1}{dr}+\frac{2}{3}\left[\left(\frac{h}{2}\right)^2-\frac{3}{5}z^2\right]z^3\frac{dS_2}{dr} \tag{3.5a}$$

$$e_\theta=\frac{U}{r}=\frac{u}{r}-\frac{1}{r}\frac{dw}{dr}z-\frac{1}{2r}\frac{dA_0}{dr}z^2-\frac{1}{6r}\frac{dA_1}{dr}z^3+\frac{2}{r}\left[\left(\frac{h}{2}\right)^2-\frac{1}{3}z^2\right]zS_0$$
$$+\frac{1}{r}\left[\left(\frac{h}{2}\right)^2-\frac{1}{2}z^2\right]z^2S_1+\frac{2}{3r}\left[\left(\frac{h}{2}\right)^2-\frac{3}{5}z^2\right]z^2S_2 \tag{3.5b}$$

$$e_{r\theta}=e_{z\theta}=0 \tag{3.5c,d}$$

应力分量(1.4)可以写成

$$\sigma_r=\frac{E_1}{1-\nu_1^2}\left\{\frac{du}{dr}+\nu_1\frac{u}{r}-\left(\frac{d^2w}{dr^2}+\nu_1\frac{1}{r}\frac{dw}{dr}\right)z+\nu_1(A_0+A_1z)\right.$$
$$-\frac{1}{2}\left(\frac{d^2A_0}{dr^2}+\frac{\nu_1}{r}\frac{dA_0}{dr}\right)z^2-\frac{1}{6}\left(\frac{d^2A_1}{dr^2}+\frac{\nu_1}{r}\frac{dA_1}{dr}\right)z^3$$
$$+2\left[\left(\frac{h}{2}\right)^2-\frac{1}{3}z^2\right]z\left(\frac{dS_0}{dr}+\frac{\nu_1S_0}{r}\right)+\left[\left(\frac{h}{2}\right)^2-\frac{1}{2}z^2\right]z^2\left(\frac{dS_1}{dr}+\frac{\nu_1S_1}{r}\right)$$
$$\left.+\frac{2}{3}\left[\left(\frac{h}{2}\right)^2-\frac{3}{5}z^2\right]z^2\left(\frac{dS_2}{dr}+\frac{\nu_1S_2}{r}\right)\right\} \tag{3.6a}$$

$$\sigma_\theta=\frac{E_1}{1-\nu_1^2}\left\{\frac{u}{r}+\nu_1\frac{du}{dr}-\left(\frac{1}{r}\frac{dw}{dr}+\nu_1\frac{dw^2}{dr^2}\right)z+\nu_1(A_0+A_1z)\right.$$
$$-\frac{1}{2}\left(\frac{1}{r}\frac{dA_0}{dr}+\nu_1\frac{d^2A_0}{dr^2}\right)z^2-\frac{1}{6}\left(\frac{1}{r}\frac{dA_1}{dr}+\nu_1\frac{d^2A_1}{dr^2}\right)z^3$$
$$+2\left[\left(\frac{h}{2}\right)^2-\frac{1}{3}z^2\right]z\left(\frac{S_0}{r}+\nu_1\frac{dS_2}{dr}\right)+\left[\left(\frac{h}{2}\right)^2-\frac{1}{2}z^2\right]z^2\left(\frac{S_1}{r}+\nu_1\frac{dS_1}{dr}\right)$$

$$+\frac{2}{3}\left[\left(\frac{h}{2}\right)^2-\frac{3}{5}z^2\right]z^3\left(\frac{S_2}{r}+\nu_1\frac{dS_2}{dr}\right)\Big\} \tag{3.6b}$$

$$\sigma_z=\frac{E_1}{1-\nu_1^2}\Big\{\nu_1\left(\frac{du}{dr}+\frac{u}{r}\right)-\nu_1\left(\frac{d^2w}{dr^2}+\frac{1}{r}\frac{dw}{dr}\right)z+A_0+A_1z$$

$$-\frac{\nu_1}{2}\left(\frac{d^2A_0}{dr^2}+\frac{1}{r}\frac{dA_0}{dr}\right)z^2-\frac{\nu_1}{6}\left(\frac{1}{r}\frac{dA_1}{dr}+\frac{d^2A_1}{dr^2}\right)z^3$$

$$+2\nu_1\left[\left(\frac{h}{2}\right)^2-\frac{1}{3}z^2\right]z\left(\frac{S_0}{r}+\frac{dS_0}{dr}\right)+\nu_1\left[\left(\frac{h}{2}\right)^2-\frac{1}{2}z^2\right]z^2\left(\frac{S_1}{r}+\frac{dS_1}{dr}\right)$$

$$+\frac{2}{3}\nu_1\left[\left(\frac{h}{2}\right)^2-\frac{3}{5}z^2\right]z^3\left(\frac{S_2}{r}+\frac{dS_2}{dr}\right)\Big\} \tag{3.6c}$$

$$\sigma_{rz}=\frac{E_1}{1+\nu_1}\left[\left(\frac{h}{2}\right)^2-z^2\right][S_0+S_1z+S_2z^2] \tag{3.6d}$$

现在让我们在约束条件(1.2)，(1.4)的约束下，简化$\delta\Pi^*$的表达式，根据(2.2)式，从(2.9)式的变分式，得

$$\delta\Pi^*=\int_0^a\int_{(h)}[\sigma_r\delta e_r+\sigma_\theta\delta e_\theta+\sigma_z\delta e_z+2\sigma_{rz}\delta e_{rz}]2\pi rdrdz$$

$$-\int_0^a q\delta W_-2\pi rdr-\int_{(h)}[U\delta\sigma_r+W\delta\sigma_{rz}+\sigma_r\delta U+\sigma_{rz}\delta W]_{r=a}2\pi adz \tag{3.7}$$

从(1.2)式，我们得

$$\delta e_r=\frac{\partial\delta U}{\partial r},\quad \delta e_\theta=\frac{\delta U}{r},\quad \delta e_z=\frac{\partial\delta W}{\partial z},\quad \delta e_{rz}=\frac{1}{2}\left(\frac{\partial\delta U}{\partial z}+\frac{\partial\delta W}{\partial r}\right) \tag{3.8}$$

把(3.8)代入(3.7)，通过分部积分进行简化，得

$$\delta\Pi^*=-\int_0^a\int_{(h)}\Big\{\left[\frac{1}{r}\frac{\partial}{\partial r}(r\sigma_r)+\frac{\partial\sigma_{zr}}{\partial z}-\frac{1}{r}\sigma_\theta\right]\delta U+\left[\frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{rz})\right.$$

$$\left.+\frac{\partial\sigma_z}{\partial z}\right]\delta W\Big\}2\pi rdrdz+\int_0^a\{\sigma_z^+\delta W_+-(\sigma_z^-+q)\delta W_-\}2\pi rdr$$

$$-\int_{(h)}(U\delta\sigma_r+W\delta\sigma_{rz})_{r=a}2\pi adz+\int_0^a(\sigma_{rz}^+\delta U_+-\sigma_{rz}^-\delta U_-)2\pi rdr \tag{3.9}$$

这里必须指出U，W在$r=a$上是已给的[即(1.8)式]，所以$(\sigma_r\delta U+\sigma_{rz}\delta W)_{r=a}$恒等于零．其次，根据(3.6d)，$\sigma_{rz}^+=\sigma_{rz}^-=0$，(3.9)式中最后一个积分必恒等于零．

把δU，δW，$\delta\sigma_r$，$\delta\sigma_{zr}$，从(3.4)，(3.6a，b)中算出，并用δu，δw，δA_0，δA_1，δS_0，δS_1，δS_2表示，即得$\delta\Pi^*$用δu，δw，δA_0，δA_1，δS_0，δS_1，δS_2表示的七个独立部分：$\delta\Pi_u^*$，$\delta\Pi_w^*$，$\delta\Pi_{A_0}^*$，$\delta\Pi_{A_1}^*$，$\delta\Pi_{S_0}^*$，$\delta\Pi_{S_1}^*$，$\delta\Pi_{S_2}^*$即

$$\delta\Pi^*=\delta\Pi_u^*+\delta\Pi_w^*+\delta\Pi_{A_0}^*+\delta\Pi_{A_1}^*+\delta\Pi_{S_0}^*+\delta\Pi_{S_1}^*+\delta\Pi_{S_2}^* \tag{3.10}$$

驻值条件$\delta\Pi^*=0$也必导致$\delta\Pi_u^*$，$\delta\Pi_w^*$，$\delta\Pi_{A_0}^*$，$\delta\Pi_{A_1}^*$，$\delta\Pi_{S_0}^*$，$\delta\Pi_{S_1}^*$，$\delta\Pi_{S_2}^*$的驻值

$$\delta\Pi_u^*=0,\quad \delta\Pi_w^*=0,\quad \delta\Pi_{A_0}^*=0,\quad \delta\Pi_{A_1}^*=0,\quad \delta\Pi_{S_0}^*=0,\quad \delta\Pi_{S_1}^*=0,\quad \delta\Pi_{S_2}^*=0 \tag{3.11}$$

从(3.9)中，可以导出涉及δu的$\delta\Pi_u^*$

$$\delta\Pi_u^*=-\int_0^a\int_{(h)}\left[\frac{1}{r}\frac{\partial}{\partial r}(r\sigma_r)+\frac{\partial\sigma_{rz}}{\partial z}-\frac{1}{r}\sigma_\theta\right]\delta u2\pi rdrdz$$

$$-\int_{(h)}\left\{\frac{E_1}{1-\nu_1^2}U\left(\frac{d\delta u}{dr}+\nu_1\frac{\delta u}{r}\right)\right\}_{r=a}2\pi adz \tag{3.12}$$

对 z 积分，上式可以化成为

$$\delta\Pi_u^*=-\int_0^a\left[\frac{1}{r}\frac{d}{dr}(rN_r)-\frac{N_\theta}{r}\right]\delta u2\pi rdr$$
$$-\frac{E_1}{1-\nu_1^2}\left\{\left(\frac{d\delta_u}{dr}+\nu_1\frac{\delta u}{r}\right)2\pi r\int_{(h)}Udz\right\}_{r=a}\tag{3.13}$$

其中　N_θ，N_r为薄膜拉力分量（内力素）

$$(N_r,\ N_\theta)=\int_{(h)}(\sigma_r,\ \sigma_\theta)dz\tag{3.14}$$

在板内的δu，和在边界面上的$\frac{d\delta u}{dr}+\nu_1\frac{\delta u}{r}$，都是独立变分，所以$\delta\Pi_u^*$的变分驻值条件为

$$\frac{1}{r}\frac{d}{dr}(rN_r)-\frac{1}{r}N_\theta=0\qquad(0\leqslant r\leqslant a)\tag{3.15}$$

$$\int_{(h)}Udz=0\qquad（在边界r=a上）\tag{3.16}$$

现在研究涉及δw的 $\delta\Pi_w^*$，从(3.9)中可以导得

$$\delta\Pi_w^*=\int_0^a\int_{(h)}\left\{\left[\frac{1}{r}\frac{d}{dr}(r\sigma_r)+\frac{\partial\sigma_{rz}}{\partial z}-\frac{\sigma_\theta}{r}\right]\frac{d\delta w}{dr}z-\left[\frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{rz})\right.\right.$$
$$\left.\left.+\frac{\partial\sigma_z}{\partial z}\right]\delta w\right\}2\pi rdrdz+\int_0^a(\sigma_z^+-\sigma_z^--q)\delta w2\pi rdr$$
$$+\frac{E_1}{1-\nu_1^2}\left\{\left(\frac{d^2\delta w}{dr^2}+\frac{\nu_1}{r}\frac{d\delta w}{dr}\right)\int_{(h)}Uzdz2\pi r\right\}_{r=a}\tag{3.17}$$

对 z 积分得

$$\delta\Pi_w^*=\int_0^a\left\{\left[\frac{1}{r}\frac{d}{dr}rM_r)-\frac{M_\theta}{r}-Q\right]\frac{d}{dr}(\delta w)-\left[\frac{1}{r}\frac{d}{dr}(rQ)\right]\delta w\right\}2\pi rdr$$
$$-\int_0^a q\delta w2\pi rdr+\frac{E_1}{1-\nu_1^2}\left\{\left[\frac{d^2}{dr^2}(\delta w)+\frac{\nu_1}{r}\frac{d}{dr}(\delta w)\right]\int_{(h)}Uzdz2\pi r\right\}_{r=a}\tag{3.18}$$

其中　M_r，M_θ为弯矩分量，Q 为横剪

$$(M_r,\ M_\theta)=\int_{(h)}(\sigma_r,\ \sigma_\theta)zdz,\quad Q=\int_{(h)}\sigma_{rz}dz\tag{3.19}$$

还可以对 r 分部积分，得

$$\delta\Pi_w^*=-\int_0^a\left[\frac{1}{r}\frac{d^2}{dr^2}(rM_r)-\frac{1}{r}\frac{dM_\theta}{dr}+q\right]\delta w2\pi rdr+\left\{\left[\frac{d}{dr}(rM_r)-M_\theta\right.\right.$$
$$\left.\left.-Qr\right]\delta w\right\}_{r=0}^{r=a}2\pi+\frac{E_1}{1-\nu_1^2}\left\{\left[\frac{d^2}{dr^2}(\delta w)+\frac{\nu_1}{r}\frac{d}{dr}(\delta w)\right]\int_{(h)}Uzdz2\pi r\right\}_{r=a}\tag{3.20}$$

$\delta\Pi_w^*$的驻值条件为

$$\frac{d^2}{dr^2}(rM_r)-\frac{dM_\theta}{dr}+rq=0\qquad(0\leqslant r\leqslant a)\tag{3.21}$$

$$\frac{d}{dr}(rM_r)-M_\theta-rQ=0\qquad(r=a)\tag{3.22}$$

$$\frac{d}{dr}(rM_r)-M_\theta-rQ=0 \qquad (r=0) \tag{3.23}$$

$$\int_{(h)}Uzdz=0 \qquad (r=a) \tag{3.24}$$

下面研究涉及δA_0的$\delta\Pi^*_{A_0}$，从(3.9)中可以导得

$$\delta\Pi^*_{A_0}=\int_0^a\int_{(h)}\left\{\left[\frac{1}{r}\frac{d}{dr}(r\sigma_r)+\frac{\partial\sigma_{rz}}{\partial z}-\frac{1}{r}\sigma_\theta\right]\frac{1}{2}z^2\frac{d\delta A_0}{dr}-\left[\frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{rz})+\frac{\partial\sigma_z}{\partial z}\right]z\delta A_0\right\}2\pi rdrdz+\int_0^a(\sigma_z^++\sigma_z^-+q)\delta A_0\frac{h}{2}2\pi rdr$$
$$-\frac{E_1}{1-\nu_1^2}\int_{(h)}\left\{U\left[\nu_1\delta A_0-\frac{z^2}{2}\left(\frac{d^2\delta A_0}{dr^2}+\frac{\nu_1}{r}\frac{d\delta A_0}{dr}\right)\right]\right\}_{r=a}2\pi adz \tag{3.25}$$

对z积分，得

$$\delta\Pi^*_{A_0}=\int_0^a\left\{\left(\frac{1}{r}\frac{d}{dr}(rM_r^{(2)})-\frac{1}{r}M_\theta^{(2)}-2Q^{(1)}\right]\frac{1}{2}\frac{d\delta A_0}{dr}-\left[\frac{1}{r}\frac{d}{dr}(rQ^{(1)})-H^{(0)}\right]\delta A_0\right\}2\pi rdr+\int_0^a q\frac{h}{2}\delta A_0 2\pi rdr-\frac{E_1}{1-\nu_1^2}\int_{(h)}\left\{U\left[\nu_1\delta A_0\right.\right.$$
$$\left.\left.-\frac{1}{2}\left(\frac{d^2\delta A_0}{dr^2}+\frac{\nu_1}{r}\frac{d\delta A_0}{dr}\right)z^2\right]\right\}_{r=a}2\pi adz \tag{3.26}$$

其中 $M_r^{(k)}$，$M_\theta^{(k)}$，$Q^{(k)}$，$H^{(k)}$分别为高阶内力素

$$(M_r^{(k)},\ M_\theta^{(k)})=\int_{(h)}(\sigma_r,\ \sigma_\theta)z^kdz,\quad Q^{(k)}=\int_{(h)}\sigma_{rz}z^kdz,\quad H^{(k)}=\int_{(k)}\sigma_z z^kdz \tag{3.27}$$

(3.26)对r分部积分，得

$$\delta\Pi^*_{A_0}=-\int_0^a\left\{\left(\frac{1}{r}\frac{d^2}{dr^2}(rM_r^{(2)})-\frac{1}{r}\frac{dM_\theta^{(2)}}{dr}-2H^{(0)}-qh\right\}\delta A_0\pi rdr+\left\{\left[\frac{d}{dr}(rM_r^{(2)})-M_\theta^{(2)}-2rQ^{(1)}\right]\delta A_0\pi\right\}_{r=0}^{r=a}$$
$$-\frac{E_1}{1-\nu_1^2}\int_{(h)}\{Udz\delta A_0\nu_1\}_{r=a}2\pi a+\frac{E_1}{1-\nu_1^2}\int_{(h)}\left\{Uz^2\frac{1}{2}\left(\frac{d^2\delta A_0}{dr^2}+\frac{\nu_1}{r}\frac{d\delta A_0}{dr}\right)\right\}_{r=a}2\pi a \tag{3.28}$$

$\delta\Pi^*_{A_0}$的变分驻值条件给出

$$\frac{1}{r}\frac{d^2}{dr^2}(rM_r^{(2)})-\frac{1}{r}\frac{dM^{(2)}}{dr}-2H^{(0)}-qh=0 \qquad (0\leqslant r\leqslant a) \tag{3.29}$$

$$\frac{d}{dr}(rM_r^{(2)})-M_\theta^{(2)}-2rQ^{(1)}=0 \qquad (r=a) \tag{3.30}$$

$$\frac{d}{dr}(rM_r^{(2)})-M_\theta^{(2)}-2rQ^{(1)}=0 \qquad (r=0) \tag{3.31}$$

$$\int_{(h)}Udz=0,\quad \int_{(h)}Uz^2dz=0 \qquad (r=a) \tag{3.32a, b}$$

同样，我们有$\delta\Pi^*_{A_1}$

$$\delta\Pi^*_{A_1}=\int_0^a\int_{(h)}\left\{\left[\frac{1}{r}\frac{\partial}{\partial r}(r\sigma_r)-\frac{\sigma_\theta}{r}+\frac{\partial\sigma_{rz}}{\partial z}\right]\frac{1}{6}z^3\frac{d}{dr}(\delta A_1)-\left[\frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{rz})+\frac{\partial\sigma_z}{\partial z}\right]\frac{1}{2}z^2\delta A_1\right\}2\pi rdrdz+\int_0^a[\sigma_z^+-\sigma_z^--q]\frac{h^2}{4}\delta A_1\pi rdr$$
$$-\frac{E_1}{1-\nu_1^2}2\pi a\left\{\nu_1\int_{(h)}Uzdz\delta A_1-\frac{1}{6}\int_{(h)}Uz^3dz\left(\frac{d^2\delta A_1}{dr^2}+\frac{\nu_1}{r}\frac{d\delta A_1}{dr}\right)\right\}_{r=a} \tag{3.33}$$

对z部份积分，得

$$\delta\Pi^*_{A_1}=\int_0^a\left\{\left[\frac{1}{r}\frac{d}{dr}(rM_r^{(3)})-\frac{M_\theta^{(2)}}{r}-3Q^{(2)}\right]\frac{1}{6}\frac{d\delta A_1}{dr}-\left[\frac{1}{r}\frac{d}{dr}(rQ^{(2)})-2H^{(2)}\right]\frac{1}{2}\delta A_1\right\}2\pi rdr-\int_0^a q\frac{h^2}{4}\delta A_1\pi rdr-2\pi a\frac{E_1}{1-\nu_1^2}\left\{\nu_1\int_{(h)}Uzdz\delta A_1\right.$$
$$\left.-\frac{1}{6}\int_{(h)}Uz^3dz\left(\frac{d\delta A_1}{dr^2}+\frac{\nu_1}{r}\frac{d\delta A_1}{dr}\right)\right\}_{r=a} \tag{3.34}$$

其中内力素$M_r^{(3)}$，$M_\theta^{(3)}$，$Q^{(2)}$，$H^{(1)}$分别见(3.27)。

$\delta\Pi^*_{A_1}$ (3.34)式对r分部积分，得

$$\delta\Pi^*_{A_1}=-\int_0^a\left\{\frac{1}{r}\frac{d^2}{dr^2}(rM_r^{(3)})-\frac{1}{r}\frac{dM_\theta^{(3)}}{dr}-6H^{(1)}+\frac{3}{4}qh^2\right\}\frac{1}{3}\delta A_1\pi rdr$$
$$+\left\{\left[\frac{d}{dr}(rM_r^{(3)})-M_\theta^{(3)}-3rQ^{(2)}\right]\frac{\pi}{3}\delta A_1\right\}_{r=0}^{r=a}$$
$$-2\pi a\frac{E_1}{1-\nu_1^2}\left\{\nu_1\int_{(h)}Uzdz\delta A_1-\frac{1}{6}\int Uz^3dz\left(\frac{d^2\delta A_1}{dr^2}+\frac{\nu_1}{r}\frac{d\delta A_1}{dr}\right)\right\}_{r=a} \tag{3.35}$$

$\delta\Pi^*_{A_1}$的驻值条件为

$$\frac{1}{r}\frac{d^2}{dr^2}(rM_r^{(3)})-\frac{1}{r}\frac{dM_\theta^{(3)}}{dr}-6H^{(1)}+\frac{3}{4}qh^2=0\qquad(0\leqslant r\leqslant a) \tag{3.36}$$

$$\frac{d}{dr}(rM_r^{(3)})-M_\theta^{(3)}-3rQ^{(2)}=0\qquad(r=a) \tag{3.37}$$

$$\frac{d}{dr}(rM_r^{(3)})-M_\theta^{(3)}-3rQ^{(2)}=0\qquad(r=0) \tag{3.38}$$

$$\int_{(h)}Uzdz=0,\quad\int_{(h)}Uz^3dz=0\qquad(r=a) \tag{3.39a,b}$$

最后让我们研究$\delta\Pi^*_{S_0}$，$\delta\Pi^*_{S_1}$和$\delta\Pi^*_{S_2}$：

$$\delta\Pi^*_{S_0}=-\int_0^a\int_{(h)}\left\{\frac{1}{r}\frac{\partial}{\partial r}(r\sigma_r)+\frac{\partial\sigma_{rz}}{\partial z}-\frac{1}{r}\sigma_\theta\right\}2\left(\frac{1}{4}h^2-\frac{1}{3}z^2\right)z\delta S_0 2\pi rdrdz$$
$$-\int_{(h)}\left\{2U\left(\frac{1}{4}h^2-\frac{1}{3}z^2\right)z\left(\frac{d\delta S_0}{dr}+\nu_1\frac{\delta S_0}{r}\right)\right.$$
$$\left.-W(1-\nu_1)\left(\frac{1}{4}h^2-z^2\right)\delta S_0\right\}_{r=a}\frac{E_1}{1-\nu_1^2}2\pi adz \tag{3.40}$$

$$\delta\Pi^*_{S_1}=-\int_0^a\int_{(h)}\left\{\frac{1}{r}\frac{\partial}{\partial r}(r\sigma_r)+\frac{\partial\sigma_{rz}}{\partial z}-\frac{1}{r}\sigma_\theta\right\}\left(\frac{1}{4}h^2-\frac{1}{2}z^2\right)z^2\delta S_1 2\pi r dr dz$$

$$-\int_{(h)}\left\{U\left(\frac{1}{4}h^2-\frac{1}{3}z^2\right)z^2\left(\frac{d\delta S_1}{dr}+\nu_1\frac{\delta S_1}{r}\right)\right.$$

$$\left.-W(1-\nu_1)\left(\frac{1}{4}h^2-z^2\right)z\delta S_1\right\}_{r=a}\frac{E_1}{1-\nu_1^2}2\pi a dz \tag{3.41}$$

$$\delta\Pi^*_{S_2}=-\int_0^a\int_{(h)}\left\{\frac{1}{r}\frac{\partial}{\partial r}(r\sigma_r)+\frac{\partial\sigma_{rz}}{\partial z}-\frac{1}{r}\sigma_\theta\right\}\frac{2}{3}\left(\frac{1}{4}h^2-\frac{3}{5}z^2\right)z^3\delta S_2 2\pi r dr dz$$

$$-\int_{(h)}\left\{\frac{2}{3}U\left(\frac{1}{4}h^2-\frac{3}{5}z^2\right)z^3\left(\frac{d\delta S_2}{dr}+\nu_1\frac{\delta S_2}{r}\right)\right.$$

$$\left.-W(1-\nu_1)\left(\frac{1}{4}h^2-z^2\right)z^2\delta S_2\right\}_{r=a}\frac{E_1}{1-\nu_1^2}2\pi a dz \tag{3.42}$$

对 z 进行积分，得

$$\delta\Pi^*_{S_0}=-\int_0^a\left\{\frac{1}{r}\frac{d}{dr}(r\Sigma_r^{(0)})-\frac{1}{r}\Sigma_\theta^{(0)}-\chi^{(0)}\right\}\delta S_0 2\pi r dr$$

$$-\frac{E_1}{1-\nu_1^2}2\pi a\left\{\left[\frac{d\delta S_0}{dr}+\nu_1\frac{\delta S_0}{r}\right]\int_{(h)}Uz\left(\frac{1}{4}h^2-\frac{1}{3}z^2\right)z dz\right.$$

$$\left.+\delta S_0(1-\nu_1)\int_{(h)}W\left(\frac{1}{4}h^2-z^2\right)dz\right\}_{r=a} \tag{3.43}$$

$$\delta\Pi^*_{S_1}=-\int_0^a\left\{\frac{1}{r}\frac{d}{dr}(r\Sigma_r^{(1)})-\frac{1}{r}\Sigma_\theta^{(1)}-\chi^{(1)}\right\}\delta S_1 2\pi r dr$$

$$-\frac{E_1}{1-\nu_1^2}2\pi a\left\{\left[\frac{d\delta S_1}{dr}+\nu_1\frac{\delta S_1}{r}\right]\int_{(h)}U\left(\frac{1}{4}h^2-\frac{1}{2}z^2\right)z^2 dz\right.$$

$$\left.+\delta S_1(1-\nu_1)\int_{(h)}W\left(\frac{1}{4}h^2-z^2\right)z dz\right\}_{r=a} \tag{3.44}$$

$$\delta\Pi^*_{S_2}=-\int_0^a\left\{\frac{1}{r}\frac{d}{dr}(r\Sigma_r^{(2)})-\frac{1}{r}\Sigma_\theta^{(2)}-\chi^{(2)}\right\}\delta S_2 2\pi r dr$$

$$-\frac{E_1}{1-\nu_1^2}2\pi a\left\{\left[\frac{d\delta S_2}{dr}+\nu_1\frac{\delta S_2}{r}\right]\int_{(h)}U\left(\frac{1}{4}h^2-\frac{3}{5}z^2\right)\frac{2}{3}z^3 dz\right.$$

$$\left.+\delta S_2(1-\nu_1)\int_{(h)}W\left(\frac{1}{4}h^2-z^2\right)z^2 dz\right\}_{r=a} \tag{3.45}$$

其中$\Sigma_r^{(k)}$，$\Sigma_\theta^{(k)}$，$\chi^{(k)}$分别为高阶内力素.

$$\left.\begin{aligned}&(\Sigma_r^{(k)},\ \Sigma_\theta^{(k)})=\int_{(h)}(\sigma_r,\ \sigma_\theta)\frac{2}{k+1}\left(\frac{1}{4}h^2-\frac{k+1}{k+3}z^2\right)z^{k+1}dz\\&\chi^{(k)}=\int_{(h)}\sigma_{rz}2\left(\frac{1}{4}h^2-z^2\right)z^k dz\end{aligned}\right\}\quad(k=0,1,2) \tag{3.46}$$

(3.43)，(3.44)，(3.45)的驻值条件给出

$$\frac{1}{r}\frac{d}{dr}(r\Sigma_r^{(0)})-\frac{1}{r}\Sigma_\theta^{(0)}-\chi^{(0)}=0\qquad(0\leqslant r\leqslant a) \tag{3.47}$$

$$\int_{(h)} U\left(\frac{1}{4}h^2-\frac{1}{3}z^2\right)zdz=0,\ \int_{(h)} W\left(\frac{1}{4}h^2-z^2\right)dz=0 \qquad (r=a) \qquad (3.48\text{a,b})$$

$$\frac{1}{r}\frac{d}{dr}(r\Sigma_r^{(1)})-\frac{1}{r}\Sigma_\theta^{(1)}-\chi^{(1)}=0 \qquad (0\leqslant r\leqslant a) \qquad (3.49)$$

$$\int_{(h)} U\left(\frac{1}{4}h^2-\frac{1}{2}z^2\right)z^2dz=0,\ \int_{(h)} W\left(\frac{1}{4}h^2-z^2\right)z^2dz=0 \qquad (r=a) \qquad (3.50\text{a,b})$$

$$\frac{1}{r}\frac{d}{dr}(r\Sigma_r^{(2)})-\frac{1}{r}\Sigma_\theta^{(2)}-\chi^{(2)}=0 \qquad (0\leqslant r\leqslant a) \qquad (3.51)$$

$$\int_{(h)} U\left(\frac{1}{4}h^2-\frac{3}{5}z^2\right)z^3dz=0,\ \int_{(h)} W\left(\frac{1}{4}h^2-z^2\right)z^2dz=0 \qquad (r=a) \qquad (3.52\text{a,b})$$

上面共计有7个平衡方程：(3.15)，(3.21)，(3.29)，(3.36)，(3.47)，(3.49)，(3.51)．还有(3.16)，(3.22)，(3.24)，(3.30)，(3.32a,b)，(3.37)，(3.39a，b) (3.48a，b)，(3.50a,b)，(3.52a,b)等在边界$r=a$上满足的边界条件，以及(3.23) (3.31)，(3.38)等在中心处满足的中心条件．这里必须指出：(3.16)和(3.32a)，(3.24) 和 (3.39a) 是相同的．(3.48a)可由(3.39)导出．还有，在利用了(3.32b) (3.39b) 以后，(3.50a)，(3.52a) 还可以简化．

四、边界条件的简化

可以证明，在(3.16)，(3.24)，(3.32a,b)，(3.39a,b)，(3.48a)，(3.50a)，(3.52a)等9个边界条件中，只有下列6个条件是独立的

$$\int_{(h)} Uz^k dz=0 \qquad (k=0,\ 1,\ 2,\ 3,\ 4,\ 5;\ r=a) \qquad (4.1)$$

由此可以导出$U(a,z)=0$；于是，我们得下列6式

$$u(a)=0,\ A_0'(a)=0,\ S_1(a)=0,\ S_2(a)=0 \qquad (4.2\text{a,b,c,d})$$

$$-w'(a)+\frac{1}{2}h^2S_0(a)=0 \qquad (4.3\text{a})$$

$$A_1'(a)+4S_0(a)=0 \qquad (4.3\text{b})$$

将(3.21)式积分一次，得

$$\frac{d}{dr}(rM_r)-M_\theta=-\frac{1}{2}qr^2+C_1 \qquad (0\leqslant r\leqslant a) \qquad (4.4)$$

其中 C_1为待定积分常数．上式适用于$0<r<a$中，也必适用于中心($r=0$)．在$r=0$中心处，上式可以写成

$$\left[\frac{d}{dr}(rM_r)-M_\theta\right]_{r=0}=C_1 \qquad (4.5)$$

但当$r=0$时，$Q(0)$应该上有限的，(3.23)式给出

$$\left[\frac{d}{dr}(rM_r)-M_\theta\right]_{r=0}=0 \qquad (4.6)$$

于是，(4.5)，(4.6)两式相比，我们求得

$$C_1=0 \qquad (4.7)$$

(4.4)式于是可以写成

$$\frac{d}{dr}(rM_r)-M_\theta=-\frac{1}{2}qr^2 \qquad (0\leqslant r\leqslant a) \tag{4.8}$$

但(3.22)可以写成

$$\left[\frac{d}{dr}(rM_r)-M_\theta\right]_{r=a}=aQ(a) \tag{4.9}$$

(4.8)式也适用于 $r=a$ 处，它是

$$\left[\frac{d}{dr}(rM_r)-M_\theta\right]_{r=a}=-\frac{1}{2}qa^2 \tag{4.10}$$

(4.9)，和(4.10)相比，得

$$Q(a)=-\frac{1}{2}qa \tag{4.11}$$

或根据$Q(a)$的定义，(4.11)可以写成

$$Q(a)=\int_{(h)}\sigma_{rz}(a,z)dz=-\frac{1}{2}qa \tag{4.12}$$

把(3.6d)代入，积分得

$$S_0(a)=-\frac{qa}{4(1-\nu_1)D_1} \tag{4.13}$$

将上式代入(4.3)，求得

$$w'(a)=-\frac{qah^2}{8(1-\nu_1)D_1'} \qquad A_1'(a)=\frac{qa}{(1-\nu_1)D_1} \tag{4.14a,b}$$

(4.2a,b,c,d)，(4.13)，(4.14a,b)给出了$u(r)$，$A_0'(r)$，$A_1'(r)$，$w'(r)$，$S_0(r)$，$S_1(r)$，$S_2(r)$的边界值。

同样，通过积分，从(3.48b)，(3.50b)，(3.52b)，我们可以证明

$$w(a)=0,\quad A_0(a)=0,\quad A_1(a)=0 \tag{4.15a,b,c}$$

五、用u,w,A_0,A_1,S_0,S_1,S_2表示的一级近似微分方程和边界条件

把(3.6a,b,c,d)代入内力素的定义(3.14)，(3.19)，(3.27)，(3.46)，得

$$N_r=B_1\left(\frac{du}{dr}+\nu_1\frac{u}{r}+\nu_1A_0\right)-\frac{1}{2}D_1\left(\frac{d^2A_0}{dr^2}+\nu_1\frac{1}{r}\frac{dA_0}{dr}\right)+\frac{7}{6}D_1^{(5)}\left(\frac{dS_1}{dr}+\nu_1\frac{S_1}{\nu}\right) \tag{5.1a}$$

$$N_\theta=B_1\left(\frac{u}{r}\nu\frac{du}{dr}+\nu_1A_0\right)-\frac{1}{2}D_1\left(\frac{1}{r}\frac{dA_0}{dr}+\nu_1\frac{1}{r}\frac{d^2A_0}{dr^2}\right)+\frac{7}{6}D_1^{(5)}\left(\frac{S_1}{r}+\nu_1\frac{dS_1}{dr}\right) \tag{5.1b}$$

$$M_r=-D_1\left(\frac{d^2w}{dr^2}+\nu_1\frac{1}{r}\frac{dw}{dr}-\nu_1A_1\right)-\frac{1}{6}D_1^{(5)}\left(\frac{d^2A_1}{dr^2}+\nu_1\frac{1}{r}\frac{dA_1}{dr}\right)$$
$$+\frac{8}{3}D_1^{(5)}\left(\frac{dS_0}{dr}+\nu_1\frac{S_0}{r}\right)+\frac{8}{15}D_1^{(7)}\left(\frac{dS_2}{dr}+\nu_1\frac{S_2}{r}\right) \tag{5.2a}$$

$$M_\theta=-D_1\left(\frac{1}{r}\frac{dw}{dr}+\nu_1\frac{d^2w}{dr^2}-\nu_1A_1\right)-\frac{1}{6}D_1^{(5)}\left(\frac{1}{r}\frac{dA_1}{dr}+\nu_1\frac{d^2A_1}{dr}\right)$$
$$+\frac{8}{3}D_1^{(5)}\left(\frac{S_0}{r}+\nu_1\frac{dS_0}{dr}\right)+\frac{8}{15}D_1^{(7)}\left(\frac{S_2}{r}+\nu_1\frac{dS_2}{dr}\right) \tag{5.2b}$$

$$Q=2(1-\nu_1)\left\{D_1S_0+\frac{1}{3}D_1^{(5)}S_2\right\} \tag{5.2c}$$

$$M_r^{(2)}=D_1\left(\frac{du}{dr}+\nu_1\frac{u}{r}+\nu_1A_0\right)-\frac{1}{2}D_1^{(5)}\left(\frac{d^2A_0}{dr^2}+\nu_1\frac{1}{r}\frac{dA_0}{dr}\right)$$
$$+\frac{9}{10}D_1^{(7)}\left(\frac{dS_1}{dr}+\nu_1\frac{S_1}{r}\right) \tag{5.3a}$$

$$M_\theta^{(2)}=D_1\left(\frac{u}{r}+\nu_1\frac{du}{dr}+\nu_1A_0\right)-\frac{1}{2}D_1^{(5)}\left(\frac{1}{r}\frac{dA_0}{dr}+\nu_1\frac{d^2A_0}{dr^2}\right)$$
$$+\frac{9}{10}D_1^{(7)}\left(\frac{S_1}{r}+\nu_1\frac{dS_1}{dr}\right) \tag{5.3b}$$

$$Q^{(1)}=\frac{2}{3}D_1^{(5)}(1-\nu_1)S_1 \tag{5.3c}$$

$$H^{(0)}=B_1\left[\nu_1\frac{1}{r}\frac{d}{dr}(ru)+A_0\right]-\frac{1}{2}D_1\nu_1\frac{1}{r}\frac{d}{dr}r\frac{dA_0}{dr}+\frac{7}{6}\nu_1D_1^{(5)}\frac{1}{r}\frac{d}{dr}(rS_1) \tag{5.3d}$$

$$M_r^{(3)}=-D_1^{(5)}\left(\frac{d^2w}{dr^2}+\nu_1\frac{1}{r}\frac{dw}{dr}-\nu_1A_1\right)-\frac{1}{6}D_1^{(7)}\left(\frac{d^2A_1}{dr^2}+\nu_1\frac{1}{r}\frac{dA_1}{dr}\right)$$
$$+\frac{32}{15}D_1^{(7)}\left(\frac{dS_0}{dr}+\nu_1\frac{S_0}{r}\right)+\frac{16}{35}D_1^{(9)}\left(\frac{dS_2}{dr}+\nu_1\frac{S_2}{r}\right) \tag{5.4a}$$

$$M_\theta^{(3)}=-D_1^{(5)}\left(\frac{1}{r}\frac{dw}{dr}+\nu_1\frac{d^2w}{dr^2}-\nu_1A_1\right)-\frac{1}{6}D_1^{(7)}\left(\frac{1}{r}\frac{dA_1}{dr}+\nu_1\frac{d^2A_1}{dr^2}\right)$$
$$+\frac{32}{15}D_1^{(7)}\left(\frac{S_0}{r}+\nu_1\frac{dS_0}{dr}\right)+\frac{16}{35}D_1^{(9)}\left(\frac{S_2}{r}+\nu_1\frac{dS_2}{dr}\right) \tag{5.4b}$$

$$Q^{(2)}=\frac{2}{3}D_1^{(5)}(1-\nu_1)S_0+\frac{2}{5}D_1^{(7)}(1-\nu_1)S_2 \tag{5.4c}$$

$$H^{(1)}=D_1\left\{-\nu_1\frac{1}{r}\frac{d}{dr}r\frac{dw}{dr}+A_1\right\}-\frac{1}{6}D_1^{(5)}\nu_1\frac{1}{r}\frac{d}{dr}r\frac{dA_1}{dr}+\frac{8}{3}\nu_1D_1^{(5)}\frac{1}{r}\frac{d}{dr}(rS_0)$$
$$+\frac{8}{15}\nu_1D_1^{(7)}\frac{1}{r}\frac{d}{dr}(rS_2) \tag{5.4d}$$

$$\Sigma_r^{(0)}=-\frac{8}{3}D_1^{(5)}\left\{\frac{d^2w}{dr^2}+\nu_1\frac{1}{r}\frac{dw}{dr}-\nu_1A_1\right\}-\frac{16}{45}D_1^{(7)}\left(\frac{d^2A_1}{dr^2}+\nu_1\frac{1}{r}\frac{dA_1}{dr}\right)$$
$$+\frac{272}{45}D_1^{(7)}\left(\frac{dS_0}{dr}+\nu_1\frac{S_0}{r}\right)+\frac{16}{15}D_1^{(9)}\left(\frac{dS_2}{dr}+\nu_1\frac{S_2}{r}\right) \tag{5.5a}$$

$$\Sigma_\theta^{(0)}=-\frac{8}{3}D_1^{(5)}\left\{\frac{1}{r}\frac{dw}{dr}+\nu_1\frac{d^2w}{dr^2}-\nu_1A_1\right\}-\frac{16}{45}D_1^{(7)}\left(\frac{1}{r}\frac{dA_1}{dr}+\nu_1\frac{d^2A_1}{dr^2}\right)$$
$$+\frac{272}{45}D_1^{(7)}\left(\frac{S_0}{dr}+\nu_1\frac{dS_0}{dr}\right)+\frac{16}{15}D_1^{(9)}\left(\frac{S_2}{r}+\nu_1\frac{dS_2}{dr}\right) \tag{5.5b}$$

$$\chi^{(0)}=\frac{16}{3}(1-\nu_1)D_1^{(5)}S_0+\frac{16}{15}(1-\nu_1)D_1^{(7)}S_2 \tag{5.5c}$$

$$\Sigma_r^{(1)}=\frac{7}{6}D_1^{(5)}\left(\frac{du}{dr}+\nu_1\frac{u}{r}+\nu_1A_0\right)-\frac{9}{20}D_1^{(7)}\left(\frac{d^2A_0}{dr^2}+\nu_1\frac{1}{r}\frac{dA_0}{dr}\right)$$
$$+\frac{107}{140}D_1^{(9)}\left(\frac{dS_1}{dr}+\nu_1\frac{S_1}{r}\right)\tag{5.6a}$$

$$\Sigma_\theta^{(1)}=\frac{7}{6}D_1^{(5)}\left(\frac{u}{r}+\nu_1\frac{du}{dr}+\nu_1A_0\right)-\frac{9}{20}D_1^{(7)}\left(\frac{1}{r}\frac{dA_0}{dr}+\nu_1\frac{d^2A_0}{dr^2}\right)$$
$$+\frac{107}{140}D_1^{(9)}\left(\frac{S_1}{r}+\nu_1\frac{dS_1}{dr}\right)\tag{5.6b}$$

$$\chi^{(1)}=\frac{16}{15}D_1^{(7)}(1-\nu_1)S_1\tag{5.6c}$$

$$\Sigma_r^{(2)}=-\frac{8}{15}D_1^{(7)}\left(\frac{d^2w}{dr^2}+\nu_1\frac{1}{r}\frac{dw}{dr}-\nu_1A_1\right)-\frac{8}{105}D_1^{(9)}\left(\frac{d^2A_1}{dr^2}+\nu_1\frac{1}{r}\frac{dA_1}{dr}\right)$$
$$+\frac{16}{15}D_1^{(9)}\left(\frac{dS_0}{dr}+\nu_1\frac{S_0}{r}\right)+\frac{976}{4725}D_1^{(11)}\left(\frac{dS_2}{dr}+\nu_1\frac{S_2}{r}\right)\tag{5.7a}$$

$$\Sigma_\theta^{(2)}=-\frac{8}{15}D_1^{(7)}\left(\frac{1}{r}\frac{dw}{dr}+\nu_1\frac{d^2w}{dr^2}-\nu_1A_1\right)-\frac{8}{105}D_1^{(9)}\left(\frac{1}{r}\frac{dA_1}{dr}\right.$$
$$\left.+\nu_1\frac{d^2A_1}{dr^2}\right)+\frac{16}{15}D_1^{(9)}\left(\frac{S_0}{r}+\nu_1\frac{dS_0}{dr}\right)+\frac{976}{4725}D_1^{(11)}\left(\frac{S_2}{r}+\nu_1\frac{dS_2}{dr}\right)\tag{5.7b}$$

$$\chi^{(2)}=\frac{16}{15}D_1^{(7)}(1-\nu_1)S_0+\frac{16}{35}D_1^{(9)}(1-\nu_1)S_2\tag{5.7c}$$

其中 B_1, D_1, $D_1^{(3)}$, $D_1^{(5)}$, $D_1^{(7)}$, $D_1^{(9)}$, $D_1^{(11)}$为各级刚度系数:

$$\left.\begin{array}{l}D_1^{(1)}=B_1=\dfrac{E_1h}{1-\nu_1^2},\quad D_1^{(3)}=D_1=-\dfrac{E_1h^3}{12(1-\nu_1^2)},\quad D_{(1)}^{(3)}=\dfrac{E_1h^5}{80(1-\nu_1^2)}\\[2ex] D_1^{(7)}=\dfrac{E_1h^7}{448(1-\nu_1^2)},\quad D_1^{(9)}=\dfrac{E_1h^9}{2304(1-\nu_1^2)},\quad D_1^{(11)}=\dfrac{E_1h^{11}}{11264(1-\nu_1^2)}\end{array}\right\}\tag{5.8}$$

或

$$D_1^{(2k+1)}=\frac{E_1h^{2k+1}}{(2k+1)4^k(1-\nu_1^2)}\qquad(k=0,\ 1,\ 2,\ \cdots)\tag{5.9}$$

把(4.1)～(4.7) 各式代入 (3.21), (3.47), (3.51), (3.36), 得用 $w(r)$, $A_1(r)$, $S_0(r)$, $S_2(r)$ 表示的4个联立微分方程组:

$$\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}\frac{1}{r}\frac{d}{dr}r\frac{dw}{dr}-\nu_1\frac{1}{r}\frac{d}{dr}r\frac{dA_1}{dr}+\frac{1}{40}h^2\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}\frac{1}{r}\frac{d}{dr}r\frac{dA_1}{dr}$$
$$-\frac{2}{5}h^2\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}\frac{1}{r}\frac{d}{dr}(rS_0)-\frac{1}{70}h^4\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}\frac{1}{r}\frac{d}{dr}(rS_2)=\frac{q}{D_1}\tag{5.10}$$

$$\frac{d}{dr}\frac{1}{r}\frac{d}{dr}r\frac{dw}{dr}-\nu_1\frac{dA_1}{dr}+\frac{1}{42}h_2\frac{d}{dr}\frac{1}{r}\frac{d}{dr}r\frac{dA_1}{dr}-\frac{17}{42}h^2\frac{d}{dr}\frac{1}{r}\frac{d}{dr}(rS_0)$$
$$-\frac{1}{72}h^4\frac{d}{dr}\frac{1}{r}\frac{d}{dr}(rS_2)+2(1-\nu_1)S_0+\frac{1}{14}h^2(1-\nu_1)S_2=0\tag{5.11}$$

$$\frac{d}{dr}\frac{1}{r}\frac{d}{dr}r\frac{dw}{dr}-\nu_1\frac{dA_1}{dr}+\frac{1}{36}h^2\frac{d}{dr}\frac{1}{r}\frac{d}{dr}r\frac{dA_1}{dr}-\frac{7}{18}h^2\frac{d}{dr}\frac{1}{r}\frac{d}{dr}(rS_0)$$

$$-\frac{61}{3960}h^4\frac{d}{dr}\ \frac{1}{r}\ \frac{d}{dr}(rS_2)+2(1-\nu_1)S_0+\frac{1}{6}h^2(1-\nu_1)S_2=0 \tag{5.12}$$

$$\nu_1\frac{1}{r}\ \frac{d}{dr}r\frac{dw}{dr}-A_1-\frac{11}{40}h^2\left[\frac{1}{r}\ \frac{d}{dr}r\frac{d}{dr}\frac{1}{r}\ \frac{d}{dr}r\frac{dw}{dr}-2\nu_1\frac{1}{r}\ \frac{d}{dr}r\frac{dA_1}{dr}\right]$$

$$-\frac{1}{1344}h^4\frac{1}{r}\ \frac{d}{dr}r\frac{d}{dr}\ \frac{1}{r}\ \frac{d}{dr}r\frac{dA_1}{dr}-\frac{2}{5}h^2\nu_1\frac{1}{r}\ \frac{d}{dr}(rS_0)$$

$$-\frac{1}{70}h^4\nu_1\frac{1}{r}\ \frac{d}{dr}(rS_2)+\frac{1}{105}h^4\frac{1}{r}\ \frac{d}{dr}r\frac{d}{dr}\ \frac{1}{r}\ \frac{d}{dr}(rS_0)$$

$$+\frac{1}{2520}h^6\frac{1}{r}\ \frac{d}{dr}r\frac{d}{dr}\ \frac{1}{r}\ \frac{d}{dr}(rS_2)+\frac{qh^2}{8D_1}=0 \tag{5.13}$$

把(4.1)～(4.7)式代入(3.15)，(3.29)，(3.49)，得用$u(r)$，$A_0(r)$，$S_1(r)$表示的3个联立微分方程相：

$$\frac{d}{dr}\ \frac{1}{r}\ \frac{d}{dr}(ru)+\nu_1\frac{dA_0}{dr}-\frac{1}{24}h^2\frac{d}{dr}\ \frac{1}{r}\ \frac{d}{dr}r\frac{dA_0}{dr}+\frac{7}{480}h^4\frac{d}{dr}\ \frac{1}{r}\ \frac{d}{dr}(rS_1)=0 \tag{5.14}$$

$$-\left[\nu_1\frac{1}{r}\ \frac{d}{dr}(ru)+A_0\right]+\frac{h^2}{24}\left[\frac{1}{r}\ \frac{d}{dr}r\frac{d}{dr}\ \frac{1}{r}\ \frac{d}{dr}(ru)+2\nu_1\frac{1}{r}\ \frac{d}{dr}r\frac{dA_0}{dr}\right]$$

$$-\frac{1}{320}h^4\frac{1}{r}\ \frac{d}{dr}r\frac{d}{dr}\ \frac{1}{r}\ \frac{d}{dr}r\frac{dA_0}{dr}-\frac{7}{480}\nu_1h^4\frac{1}{r}\ \frac{d}{dr}(rS_1)$$

$$+\frac{9}{8960}h^6\frac{1}{r}\ \frac{d}{dr}r\frac{d}{dr}\ \frac{1}{r}\ \frac{d}{dr}(rS_1)-\frac{qh}{2B_1}=0 \tag{5.15}$$

$$\frac{d}{dr}\ \frac{1}{r}\ \frac{d}{dr}(ru)+\nu_1\frac{dA_0}{dr}-\frac{27}{392}h^2\frac{d}{dr}\ \frac{1}{r}\ \frac{d}{dr}\left(r\frac{dA_0}{dr}\right)-\frac{8}{49}h^2(1-\nu_1)S_1$$

$$+\frac{107}{4704}h^4\frac{d}{dr}\ \frac{1}{r}\ \frac{d}{dr}(rS_1)=0 \tag{5.16}$$

边界条件除(4.2a，b，c)，(4.14a，b，c，d)，(4.15a，b，c)外，还有(3.22)，(3.30)，(3.37)。把(4.1)～(4.7)代入这3个边界条件，得

$$r\left\{\frac{d}{dr}\ \frac{1}{r}\ \frac{d}{dr}r\frac{dw}{dr}-\nu_1\frac{dA_1}{dr}+\frac{1}{40}h^2\frac{d}{dr}\ \frac{1}{r}\ \frac{d}{dr}r\frac{dA_1}{dr}-\frac{2}{5}h^2\frac{d}{dr}\ \frac{1}{r}\ \frac{d}{dr}(rS_0)\right.$$

$$\left.-\frac{1}{70}h^4\frac{d}{dr}\ \frac{1}{r}\ \frac{d}{dr}(rS_2)+2(1-\nu_1)\left[S_0+\frac{1}{20}h^2S_2\right]\right\}=0,\qquad (r=a) \tag{5.17a}$$

$$r\left\{\frac{d}{dr}\ \frac{1}{r}\ \frac{d}{dr}(ru)+\nu_1\frac{dA_0}{dr}-\frac{3}{40}h^2\frac{d}{dr}\ \frac{1}{r}\ \frac{d}{dr}r\frac{dA_0}{dr}+\frac{27}{1120}h^4\frac{d}{dr}\ \frac{1}{r}\ \frac{d}{dr}(rS_1)\right.$$

$$\left.-\frac{1}{5}h^2(1-\nu_1)S_1\right\}=0\qquad (r=a) \tag{5.17b}$$

$$r\left\{\frac{d}{dr}\ \frac{1}{r}\ \frac{d}{dr}\left(r\frac{dw}{dr}\right)-\nu_1\frac{dA_1}{dr}+\frac{5}{168}h^2\frac{d}{dr}\ \frac{1}{r}\ \frac{d}{dr}r\frac{dA_1}{dr}-\frac{8}{21}h^2\frac{d}{dr}\ \frac{1}{r}\ \frac{d}{dr}(rS_0)\right.$$

$$\left.-\frac{1}{63}h^4\frac{d}{dr}\ \frac{1}{r}\ \frac{d}{dr}(rS_2)+2(1-\nu_1)S_0+\frac{3}{14}h^2(1-\nu_1)S_2\right\}=0\qquad (r=a) \tag{5.17c}$$

上面3式也适用于$r=0$，即中心处的条件。显然这3个条件是自然满足的。

参 考 文 献

[1] 钱伟长，不用Kirchhoff-Love假定的三维弹性板近似理论及其边界条件，应用数学和力学，**16**(3) (1995), 189—206.

[2] 钱伟长，不用Kirchhoff-Love假定的三维弹性板二级近似理论及其边界条件，应用数学和力学，**16**(5) (1995), 38—394.

[3] Chien, Weizang, Further study of generalized variational principles in elasticity, *Advances of Applied Mathematics and Mechanics in China*, Edited by W. Z. Chien. and Z. Z. Fu., **1** (1987), 1—10.

[4] Chien, Weizang, Incompatible elements and generalised variational principles, *Advances in Applied Mechanics, U.S.A*, **14** (1984), 93—153.

The First Order Approximation of Non-Kirchhoff-Love Theory for Elastic Circular Plate with Fixed Boundary under Uniform Surface Loading(Ⅰ)

Chien Weizang

(*Shanghai University; Shanghai Institute of Applied Mathematics and Mechanics, Shanghai* 20072, *P. R. China*)

Abstract

Based on the approximation theory adopting non-Kirchhoff-Love assumption for three dimensional elastic plates with arbitrary shapes, the author derives a functional of generalized variation for three dimensional elastic circular plates, thereby obtains a set of differential equations and the related boundary conditions to establish a first order approxmiation theory for elastic circular plate with fixed boundary and under uniform loading on one of its turface. The analytical solution of this problem will be presented in another paper.

Key words elastic circular plate, Kirchhoff-Love assumption, generalized variational principle