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一阶系统非线性边值问题的奇摄动*

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摘 要

本文应用比较定理研究了一类非线性边界条件的向量非线性奇摄动问题

$$\varepsilon x'=f(t,x,y,\varepsilon)$$
$$\varepsilon y'=g(t,x,y,\varepsilon)$$
$$x(0)=A(\xi_1,\xi_2,x(1)-x(0),y(1)-y(0),\varepsilon)$$
$$y(0)=B(\xi_1,\xi_2,x(1)-x(0),y(1)-y(0),\varepsilon)$$

这里 ξ_1,ξ_2为ε的函数，$0<\varepsilon\ll 1$，在适当的条件下，作出了任意次精度的渐近展式，并得出余项估计。

关键词 非线性边值 奇摄动 比较定理 渐近展式

一、引 言

常微分方程理论告诉我们，高阶标量微分方程的定解问题与一阶系统的定解问题有着密切的联系。Howes[1]曾运用此思想研究了一类高阶问题的奇摄动，至于一阶系统的奇摄动问题，亦有许多学者[2,3]进行了讨论，本文在一定的条件下，对一类非线性边界条件的向量非线性奇摄动边值问题

$$\begin{cases}\varepsilon x'=f(t,x,y,\varepsilon)\\ \varepsilon y'=g(t,x,y,\varepsilon)\end{cases}\quad (0<t<1) \tag{1.1}$$
$$x(0)=A(\xi_1,\xi_2,x(1)-x(0),y(1)-y(0),\varepsilon)=A(\xi_1,\xi_2,v_1,v_2,\varepsilon) \tag{1.2}$$
$$y(0)=B(\xi_1,\xi_2,x(1)-x(0),y(1)-y(0),\varepsilon)=B(\xi_1,\xi_2,v_1,v_2,\varepsilon) \tag{1.3}$$

应用比较定理[4,5]，讨论其渐近展式的一致有效性。这里ξ_1,ξ_2为ε的函数。

二、形式渐近解的构造

对上面给出的奇摄动系统，我们假设：

[H1] f,g,A,B,ξ_1,ξ_2关于其变元无限次连续可微；

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[H2] 退化方程组

$$\left.\begin{array}{l}f(t,x,y,0)=0\\ g(t,x,y,0)=0\end{array}\quad (0\leqslant t\leqslant 1)\right\}\tag{2.1}$$

在[0,1]上存在$C^{(1)}[0,1]$类光滑解$x=x_0(t)$, $y=y_0(t)$;

[H3] 存在正常数l_1,l_2，使得

$$\left.\begin{array}{ll}\dfrac{\partial f}{\partial x}\leqslant -l_1<0, & \left|\dfrac{\partial f}{\partial y}\right|\leqslant\dfrac{l_1}{2}\\ \dfrac{\partial g}{\partial y}\leqslant -l_2<0, & \left|\dfrac{\partial g}{\partial x}\right|\leqslant\dfrac{l_2}{2}\\ \dfrac{\partial A}{\partial v_i}\leqslant 0,\ \dfrac{\partial B}{\partial v_i}\leqslant 0 & (i=1,2)\end{array}\right\}\tag{2.2}$$

[H4] 方程组

$$\left.\begin{array}{l}\lambda_1=A(\xi_1(0),\xi_2(0),x_0(1)-x_0(0)-\lambda_1,y_0(1)-y_0(0)-\lambda_2,0)-x_0(0)\\ \lambda_2=B(\xi_1(0),\xi_2(0),x_0(1)-x_0(0)-\lambda_1,y_0(1)-y_0(0)-\lambda_2,0)-y_0(0)\end{array}\right\}\tag{2.3}$$

有解λ_1,λ_2，且还满足

$$\begin{vmatrix}1+A_{v_1}(\xi_1(0),\xi_2(0),x_0(1)-x_0(0)-\lambda_1,y_0(1)-y_0(0)-\lambda_2,0) & A_{v_2}(\text{——})\\ B_{v_1}(\text{——}) & 1+B_{v_2}(\text{——})\end{vmatrix}\neq 0\tag{2.4}$$

这里 $A_{v_i},B_{v_i}(i=1,2)$分别表示A、B对变元v_i的偏导数.

现构造外部解，假设外部解有如下展式:

$$\left.\begin{array}{l}X(\varepsilon,t)=\sum\limits_{i=0}^{\infty}x_i(t)\varepsilon^i\\ Y(\varepsilon,t)=\sum\limits_{i=0}^{\infty}y_i(t)\varepsilon^i\end{array}\right\}\tag{2.5}$$

代入(1.1)并令系统两端ε^i之系数相等即得

$$\left.\begin{array}{l}f_x(t,x_0,y_0,0)x_n+f_y(t,x_0,y_0,0)y_n+r_{n1}=x_{n-1}\\ g_x(t,x_0,y_0,0)x_n+g_y(t,x_0,y_0,0)y_n+r_{n2}=y_{n-1}\end{array}\right\}\tag{2.6}$$

其中 r_{n1},r_{n2}为关于$x_i,y_i(i=0,1,2,\cdots,n-1)$及其导数的已知函数，由假设[H3]知，(2.6)可解出x_n,y_n，代入(2.5)即得外部解的展开式.

不难觉察到，(2.5)未必满足问题(1.1)～(1.3)的定解条件(1.2)，(1.3)，因此还须构造校正项，令

$$\left.\begin{array}{l}x=X(\varepsilon,t)+U(\varepsilon,\tau)\\ y=Y(\varepsilon,t)+W(\varepsilon,\tau)\end{array}\right\}\tag{2.7}$$

其中 $\tau=\dfrac{t}{\varepsilon}$为伸长变量，且设校正项$U$，$W$有如下展式

$$\left.\begin{aligned}U(\varepsilon,\tau)&=\sum_{i=0}^{\infty}u_i(\tau)\varepsilon^i\\W(\varepsilon,\tau)&=\sum_{i=0}^{\infty}w_i(\tau)\varepsilon^i\end{aligned}\right\}\tag{2.8}$$

将(2.7)，(2.8)代入(1.1)，(1.2)，(1.3)得

$$\left.\begin{aligned}&\frac{dU}{d\tau}=f(\varepsilon\tau,X+U,Y+W,\varepsilon)-f(\varepsilon\tau,X,Y,\varepsilon)\\&\frac{dW}{d\tau}=g(\varepsilon\tau,X+U,Y+W,\varepsilon)-g(\varepsilon\tau,X,Y,\varepsilon)\\&X(0,\varepsilon)+U(0,\varepsilon)=A(\xi_1,\xi_2,V_1,V_2,\varepsilon)\\&Y(0,\varepsilon)+W(0,\varepsilon)=B(\xi_1,\xi_2,V_1,V_2,\varepsilon)\end{aligned}\right\}\tag{2.9}$$

这里，考虑到U，W具有初始层性质，记

$$V_1=X(1,\varepsilon)-X(0,\varepsilon)-U(0,\varepsilon)$$
$$V_2=Y(1,\varepsilon)-Y(0,\varepsilon)-W(0,\varepsilon)$$

令(2.9)中$\varepsilon=0$得

$$\begin{aligned}\frac{du_0}{d\tau}&=f(0,x_0+u_0,y_0+w_0,0)-f(0,x_0,y_0,0)\\&=f_x(0,x_0+\theta_1u_0,y_0+w_0,0)u_0+f_y(0,x_0,y_0+\theta_2w_0,0)w_0\\\frac{dw_0}{d\tau}&=g_x(0,x_0+\theta_3u_0,y_0+w_0,0)u_0+g_y(0,x_0,y_0+\theta_4w_0,0)w_0\end{aligned}$$

$$(0\leqslant\theta_1,\theta_2,\theta_3,\theta_4\leqslant1)$$

$$x_0(0)+u_0(0)=A(\xi_1(0),\xi_2(0),x_0(1)-x_0(0)-u_0(0),y_0(1)-y_0(0)-w_0(0),0)$$
$$y_0(0)+w_0(0)=B(\xi_1(0),\xi_2(0),x_0(1)-x_0(0)-u_0(0),y_0(1)-y_0(0)-w_0(0),0)$$

由假设[H3]，[H4]及文献[6]，该方程组有一组具初始层性质的解

$$\left.\begin{aligned}u_0(\tau)&=O(\exp[-\eta_{10}\tau])\\w_0(\tau)&=O(\exp[-\eta_{20}\tau])\end{aligned}\right\}\quad(\eta_{10},\eta_{20}\text{为某正常数})$$

再将(2.9)中等号右端的函数作关于ε的幂级数展开，并令等号两端ε的同次幂系数相等得下述方程组列

$$\frac{du_n}{d\tau}=f_x(0,x_0+u_0,y_0+w_0,0)u_n+f_y(\text{——})w_n+r_n'$$

$$\frac{dw_n}{d\tau}=g_x(0,x_0+u_0,y_0+w_0,0)u_n+g_y(\text{——})w_n+r_n''$$

$$\begin{aligned}x_n(0)+u_n(0)=&A_{\xi_1}(\xi_1(0),\xi_2(0),x_0(1)-x_0(0)-u_0(0),y_0(1)-y_0(0)\\&-w_0(0),0)\xi_{1n}+A_{\xi_2}(\text{——})\xi_{2n}+A_{v_1}(\text{——})v_{1n}+A_{v_2}(\text{——})v_{2n}+A_n\\y_n(0)+w_n(0)=&B_{\xi_1}(\text{——})\xi_{1n}+B_{\xi_2}(\text{——})\xi_{2n}+B_{v_1}(\text{——})v_{1n}\\&+B_{v_2}(\text{——})v_{2n}+B_n\end{aligned}$$

上面的r_n',r_n''为$u_0,u_1,\cdots,u_{n-1},w_0,w_1,\cdots,w_{n-1}$的已知函数，$A_n,B_n$为$A_i,B_i(i=0,1,\cdots,n-1)$的已知函数.

同理可知，此方程组列有下述形式的解：

$$u_n(\tau)=O(\exp[-\eta_{1n}\tau]),\quad w_n(\tau)=O(\exp[-\eta_{2n}\tau]) \qquad (\eta_{1n},\eta_{2n}\text{为正常数})$$

至此，将求得的u_n,w_n代入(2.7)，(2.8)即得形式展式，下面将证明其一致有效性。

三、余项估计

定理 在假设[H1]～[H4]下，问题(1.1)～(1.3)存在解$x(t,\varepsilon),y(t,\varepsilon)$，且满足如下估计式

$$x(t,\varepsilon)=\sum_{i=0}^{m}(x_i(t)+u_i(\tau))\varepsilon^i+O(\varepsilon^{m+1})\equiv X_m+O(\varepsilon^{m+1}) \tag{3.1a}$$

$$y(t,\varepsilon)=\sum_{i=0}^{m}(y_i(t)+w_i(\tau))\varepsilon^i+O(\varepsilon^{m+1})\equiv Y_m+O(\varepsilon^{m+1}) \tag{3.1b}$$

证明

首先构造上、下函数组

$$\bar{x}=X_m+\gamma_1\varepsilon^{m+1},\quad \underline{x}=X_m-\gamma_1\varepsilon^{m+1}$$

$$\bar{y}=Y_m+\gamma_1\varepsilon^{m+1},\quad \underline{y}=Y_m-\gamma_1\varepsilon^{m+1}$$

$$\bar{\xi}_1=\zeta_{1m}+\gamma_2\varepsilon^{m+1},\quad \underline{\xi}_1=\zeta_{1m}-\gamma_2\varepsilon^{m+1}$$

$$\bar{\xi}_2=\zeta_{2m}+\gamma_3\varepsilon^{m+1},\quad \underline{\xi}_2=\zeta_{2m}-\gamma_3\varepsilon^{m+1}$$

其中 $\gamma_1,\gamma_2,\gamma_3$为某正数，$\zeta_{1m},\zeta_{2m}$满足

$$\xi_i(\varepsilon)=\sum_{j=0}^{m}\xi_{ij}\varepsilon^j+O(\varepsilon^{m+1})\equiv\zeta_{im}+O(\varepsilon^{m+1}) \quad (i=1,2)$$

不难证明[7]，存在正常数$\delta_1,\delta_2,\delta_3$，使得

$$\left|\varepsilon\frac{dX_m}{dt}-f(t,X_m,Y_m,\varepsilon)\right|<\delta_1\varepsilon^{m+1} \tag{3.2a}$$

$$\left|\varepsilon\frac{dY_m}{dt}-g(t,X_m,Y_m,\varepsilon)\right|<\delta_2\varepsilon^{m+1} \tag{3.2b}$$

$$|X_n(0)-A(\zeta_{1m},\zeta_{2m},X_m(1)-X_m(0),Y_m(1)-Y_m(0),\varepsilon)|<\delta_3\varepsilon^{m+1} \tag{3.2c}$$

从而，γ_1当取值充分大时有：

$$\left.\begin{aligned}
&\varepsilon\frac{d\underline{x}}{dt}-\inf_{\underline{y}\leq k\leq\bar{y}}f(t,\underline{x},k,\varepsilon)=\varepsilon\frac{dX_m}{dt}-f(t,X_m,Y_m,\varepsilon)\\
&\quad-f_x(\text{——})(\underline{x}-X_m)-\inf_{\underline{y}\leq k\leq\bar{y}}f_y(\text{——})(k-Y_m)\\
&\quad\leq\left(\delta_1-l_1\gamma_1+\frac{l_1}{2}\gamma_1\right)\varepsilon^{m+1}\leq 0\\
&\varepsilon\frac{d\underline{y}}{dt}-\inf_{\underline{x}\leq l\leq\bar{x}}g(t,l,\underline{y},\varepsilon)=\varepsilon\frac{dY_m}{dt}-g(t,X_m,Y_m,\varepsilon)\\
&\quad-\inf_{\underline{x}\leq l\leq\bar{x}}g_x(\text{——})(l-X_m)-g_y(\text{——})(\underline{y}-Y_m)\\
&\quad\leq\left(\delta_2+\frac{l_2}{2}\gamma_1-l_2\gamma_1\right)\varepsilon^{m+1}\leq 0
\end{aligned}\right\} \tag{3.3}$$

同理可证

$$\varepsilon\frac{d\bar{x}}{dt}-\sup_{\underline{y}<k<\bar{y}}f(t,\bar{x},k,\varepsilon)\geqslant 0$$

$$\varepsilon\frac{d\bar{y}}{dt}-\sup_{\underline{x}<l<\bar{x}}g(t,l,\bar{y},\varepsilon)\geqslant 0$$

下面给出

$$\underline{x}(0)\leqslant\inf A(\xi_1,\xi_2,v_1,v_2,\varepsilon),\ \underline{\xi}_i\leqslant\xi_i\leqslant\bar{\xi}_i,\underline{v}_i\leqslant v_i\leqslant\bar{v}_i\qquad (i=1,2)\tag{3.4}$$

的证明。由文[5]中记号，有

$$\bar{v}_1=\int_0^1\bar{f}(s,\underline{x},\underline{y},\bar{x},\bar{y},\varepsilon)ds\leqslant\int_0^1\frac{d\bar{x}}{ds}ds=\bar{x}(1)-\bar{x}(0)$$

$$=X_m(1)-X_m(0)$$

$$\bar{v}_2=\int_0^1\bar{g}(s,\underline{x},\underline{y},\bar{x},\bar{y},\varepsilon)ds\leqslant\int_0^1\frac{d\bar{y}}{ds}ds=\bar{y}(1)-\bar{y}(0)$$

$$=Y_m(1)-Y_m(0)$$

所以

$$\inf_{\substack{\underline{\xi}_i\leqslant\xi_i\leqslant\bar{\xi}_i\\ \underline{v}_i\leqslant v_i\leqslant\bar{v}_i}}A(\xi_1,\xi_2,v_1,v_2,\varepsilon)\geqslant\inf_{\underline{\xi}_i\leqslant\xi_i\leqslant\bar{\xi}_i}A(\xi_1,\xi_2,\bar{v}_1,\bar{v}_2,\varepsilon)$$

$$\geqslant\inf_{\underline{\xi}_i\leqslant\xi_i\leqslant\bar{\xi}_i}A(\xi_1,\xi_2,X_m(1)-X_m(0),Y_m(1)-Y_m(0),\varepsilon)$$

$$\geqslant A(\zeta_{1m},\zeta_{2m},X_m(1)-X_m(0),Y_m(1)-Y_m(0),\varepsilon)-k\gamma_2\varepsilon^{m+1}$$

这里k的取得由A关于ξ_1,ξ_2的可微性保证。从而，由(3.2c)便可取γ_1充分大而使(3.4)成立。

至于

$$\underline{y}(0)\leqslant\inf B(\xi_1,\xi_2,v_1,v_2,\varepsilon)\quad \underline{\xi}_i\leqslant\xi_i\leqslant\bar{\xi}$$

$$\bar{x}(0)\geqslant\sup A(\xi_1,\xi_2,v_1,v_2,\varepsilon)\quad (\underline{v}_i\leqslant v_i\leqslant\bar{v}_i)$$

$$\bar{y}(0)\geqslant\sup B(\xi_1,\xi_2,v_1,v_2,\varepsilon)\quad (i=1,2)$$

的证明仿(3.4)的步骤给出，这样，由[5]中定理1的微分形式得证本定理。文中只就二维情形给出了证明，对于高维情形也有类似的结论。

参考文献

[1] F. A. Howes, Differential inequalities of higher order and the asymptotic solution of nonlinear boundary value problems, *SIAM J. Meth. Anal.*, 13 (1982), 61—80.

[2] 康盛亮、陈琪，非线性向量边值问题的奇摄动，应用数学和力学，9(9) (1988), 817—824.

[3] 江福汝，关于常微分方程非线性边值问题的奇异摄动，中国科学（A辑），3 (1985),232—242.

[4] A. McNabb, Comparison theorems for differential equations, *J. Math. Anal. Appl.*, 19 (1986), 417—428.

[5] A. McNabb and G. Weir, Comparison theorems for ordinary differential equations with general boundary conditions, *J. Math. Anal. Appl.*, 130(1988), 144—154.

[6] J. K. Hale, *Ordinary Differeutial Equations*, Robert E. Krieger Pub. Co., Huntington (1980).

[7] Mo Jiaqi(莫嘉琪), Singular perturbation for a boundary value problem of forth

order nonlinear differential equation, *Chin, Ann. of Math.*, 8B(1987), 80—88.

Singular Perturbation for a Nonlinear Boundary Value Problem of First Order System

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Abstract

In this paper, we study the following perturbed nonlinear boundary value problem of the form:

$$\varepsilon x' = f(t,x,y,\varepsilon)$$
$$\varepsilon y' = g(t,x,y,\varepsilon)$$
$$x(0) = A(\xi_1,\xi_2,x(1)-x(0),\ y(1)-y(0),\varepsilon)$$
$$y(0) = B(\xi_1,\xi_2,x(1)-x(0),\ y(1)-y(0),\varepsilon)$$

where ξ_1, ξ_2 are functions of ε, $0<\varepsilon\ll 1$. Under some Suitable Conditions, we give the asymptotic expansion of solution of any order, and obtain the estimation of remainder term by using the comparison theorem.

Key words nonlinear boundary value, singular perturbation, comparison theorem, asymptotic expansion