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两对边简支中间有任意多个单向弹性线支矩形板横向振动的一个解析解法*

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摘 要

本文给出了两对边简支另两对边任意支承的中间有任意多个单向弹性线支矩形板横向振动的一个新的解析解法、将弹性线支反力看作是作用于板上的待求外力，求得了含有未知的弹性线支反力的两对边简支矩形板的运动方程的解析解；利用弹性线支反力与板横向位移之间的线性关系导出频率方程；频率方程及振型函数的表述均与已有方法不同。

关键词 矩形板 振动频率 弹性线支 解析解

一、引 言

对两对边简支中间有单向弹性线支及刚性线支矩形板横向振动的研究已有一些报道[1~5]，其求解方法均是将矩形板沿线支处划分成若干个子板，在分别求得各子板运动方程的解后，利用两相连子板在线支处位移、转角和弯矩相等及剪力之差等于线支反力的连接条件求出频率方程，振型函数则需分段表示，当线支个数较多时，计算是比较复杂的。本文提出了一个求解两对边简支另两对边任意支承的中间有任意个单向弹性线支矩形板横向振动的一个新的解析解法，频率方程是以一个阶数等于中间线支个数的矩阵表示的，振型函数则可用一个统一的解析式表示。

二、运动微分方程及其解

由常微分方程理论可知，对任意的线性微分方程组

$$\frac{d\boldsymbol{x}}{dt}=A(t)\boldsymbol{x}+f(t) \tag{2.1}$$

其全解为[6]

$$\left.\begin{aligned}\boldsymbol{x}(t)&=X(t)C+F(t)\\F(t)&=\int_{t_0}^{t}X(t)X^{-1}(s)f(s)ds\end{aligned}\right\} \tag{2.2}$$

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式中，$X(t)$为方程(2.1)的齐次通解，C为待定积分常数，$F(t)$为方程(2.1)的非齐次特解．下面将利用(2.2)式来分析两对边简支中间有任意多个单向弹性线支矩形板的横向振动．

考虑如图1所示的两对边简支另两对边任意支承中间有多个单向弹性线支矩形板，在直角坐标系Oxy中，任意外力作用下等厚度弹性薄板横向振动的微分方程为[7]

$$D\left(\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}\right)+\rho h\frac{\partial^2 w}{\partial t^2}=p(x,y,t) \tag{2.3}$$

$$0\leqslant x\leqslant a,\quad 0\leqslant y\leqslant b$$

式中，$w(x,y,t)$是矩形板的横向位移，t是时间，$D=Eh^3/12(1-\nu^2)$为板的弯曲刚度，E为材料的弹性模量，h为板的厚度，ν为泊松比，ρ为板的体密度，$p(x,y,t)$为作用于板上的外力，a，b分别是矩形板的长度和宽度．

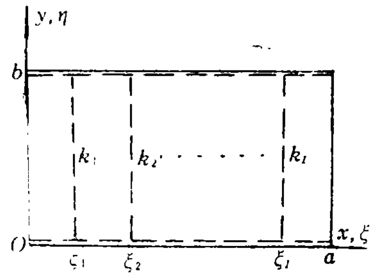

图1　两对边简支另两对边任意支承的中间有多个单向弹性线支矩形板

当矩形板作自由振动时，若将$p(x,y,t)$看作是作用于板上的弹性线支反力，则$p(x,y,t)$随时间的变化频率与矩形板的固有频率是一致的，此时，矩形板的横向位移和弹性线支反力可分别写成

$$w(x,y,t)=W(x,y)\sin\omega t,\quad p(x,y,t)=P(x,y)\sin\omega t \tag{2.4}$$

式中，ω是矩形板的固有圆频率．若设

$$\xi=\frac{x}{a},\quad \eta=\frac{y}{b} \tag{2.5}$$

将(2.4)式代入方程(2.3)，得到

$$\frac{\partial^4 W}{\partial\xi^4}+2\frac{a^2}{b^2}\frac{\partial^4 W}{\partial\xi^2\partial\eta^2}+\frac{a^4}{b^4}\frac{\partial^4 W}{\partial\eta^4}-\frac{\rho h}{D}a^4\omega^2 W=\frac{a^4}{D}P(\xi,\eta) \tag{2.6}$$

$$0\leqslant\xi\leqslant 1,\quad 0\leqslant\eta\leqslant 1$$

将$P(\xi,\eta)$展成关于η的 Fourier 正弦级数，有

$$P(\xi,\eta)=2\sum_{n=1}^{\infty}P_n(\xi)\sin n\pi\eta,\quad P_n(\xi)=\int_0^1 P(\xi,\eta)\sin n\pi\eta d\eta \tag{2.7}$$

对于两对边($\eta=0$，$\eta=1$)简支的矩形板，其位移函数可写成

$$W(\xi,\eta)=\sum_{n=1}^{\infty}W_n(\xi)\sin n\pi\eta \tag{2.8}$$

将(2.7)、(2.8)两式代入(2.6)式，得到

$$\frac{d^4 W_n(\xi)}{d\xi^4}-2\frac{a^2}{b^2}n^2\pi^2\frac{d^2 W_n(\xi)}{d\xi^2}+\left(\frac{a^4}{b^4}n^4\pi^4-k_n^4\right)W_n(\xi)=2\frac{a^4}{D}P_n(\xi) \tag{2.9}$$

式中，$k_n^4=(\rho h/D)\omega_n^2 a^4$．

由(2.2)式可知，当$k_n^2<(a^2/b^2)n^2\pi^2$时，可求得(2.9)式的全解为

$$\left.\begin{aligned}&W_n(\xi)=C_1\sinh\lambda_{n1}\xi+C_2\cosh\lambda_{n1}\xi+C_3\sinh\lambda_{n2}\xi+C_4\cosh\lambda_{n2}\xi+F_n(\xi)\\&F_n(\xi)=\frac{a^4}{D\lambda_{n1}\lambda_{n2}k_n^2}\int_0^{\xi}P_n(s)[\lambda_{n1}\sinh\lambda_{n2}(\xi-s)-\lambda_{n2}\sinh\lambda_{n1}(\xi-s)]ds\end{aligned}\right\} \tag{2.10}$$

式中

$$\lambda_{n1}=\sqrt{(a^2/b^2)n^2\pi^2-k_n^2},\qquad \lambda_{n2}=\sqrt{(a^2/b^2)n^2\pi^2+k_n^2} \tag{2.11}$$

当$k_n^2>(a^2/b^2)n^2\pi^2$时，(2.9)式的全解为

$$\left.\begin{aligned}W_n(\xi)&=C_1\sin\lambda_{n1}\xi+C_2\cos\lambda_{n1}\xi+C_3\sinh\lambda_{n2}\xi+C_4\cosh\lambda_{n2}\xi+F_n(\xi)\\F_n(\xi)&=\frac{a^4}{D\lambda_{n1}\lambda_{n2}k_n^2}\int_0^\xi P_n(s)[\lambda_{n1}\sinh\lambda_{n2}(\xi-s)-\lambda_{n2}\sin\lambda_{n1}(\xi-s)]ds\end{aligned}\right\}\tag{2.12}$$

这里，$\lambda_{n1}=\sqrt{k_n^2-(a^2/b^2)n^2\pi^2}$.

(2.10)、(2.12)两式中，$C_i(i=1,2,3,4)$为待定积分常数，可利用矩形板另两边($\xi=0$, $\xi=1$)的边界条件求出.

三、积分常数与线支反力

下面分别以两对边($\xi=0$，$\xi=1$)简支和固支的矩形板为例，说明本文方法的应用.

两对边简支的边界条件为

$$\left.\begin{aligned}&(w)_{\xi=0}=0,\quad\left(\frac{\partial^2 w}{\partial\xi^2}\right)_{\xi=0}=0\\&(w)_{\xi=1}=0,\quad\left(\frac{\partial^2 w}{\partial\xi^2}\right)_{\xi=1}=0\end{aligned}\right\}\tag{3.1}$$

对于两对边($\xi=0$，$\xi=1$)简支或固支的矩形板,已有的分析表明,总有$k_n^2>(a^2/b^2)n^2\pi^2$,因而$W_n(\xi)$的解可只取(2.12)式，将(2.12)式代入上式，得到

$$\left.\begin{aligned}&C_1=\frac{a^4}{D\lambda_{n1}k_n^2\sin\lambda_{n1}}\int_0^1 P_n(s)\sin\lambda_{n1}(1-s)ds,\quad C_2=0\\&C_3=-\frac{a^4}{D\lambda_{n2}k_n^2\sinh\lambda_{n2}}\int_0^1 P_n(s)\sinh\lambda_{n2}(1-s)ds,\quad C_4=0\end{aligned}\right\}\tag{3.2}$$

两对边固支的边界条件为

$$\left.\begin{aligned}&(w)_{\xi=0}=0,\quad\left(\frac{\partial w}{\partial\xi}\right)_{\xi=0}=0\\&(w)_{\xi=1}=0,\quad\left(\frac{\partial w}{\partial\xi}\right)_{\xi=1}=0\end{aligned}\right\}\tag{3.3}$$

将(2.12)式代入上式，得到

$$\left.\begin{aligned}C_1=&\frac{a^4}{Dk_n^2H_n}\int_0^1 P_n(s)\Big\{(\cos\lambda_{n1}-\cosh\lambda_{n2})[\cosh\lambda_{n2}(1-s)-\cos\lambda_{n1}(1-s)]\\&+\left(\sin\lambda_{n1}+\frac{\lambda_{n2}}{\lambda_{n1}}\sinh\lambda_{n2}\right)\left[\frac{\lambda_{n1}}{\lambda_{n2}}\sinh\lambda_{n2}(1-s)-\sin\lambda_{n1}(1-s)\right]\Big\}ds\\C_2=&\frac{a^4}{Dk_n^2H_n}\int_0^1 P_n(s)\ \Big\{(\cos\lambda_{n1}-\cosh\lambda_{n2})\left[\frac{\lambda_{n1}}{\lambda_{n2}}\sinh\lambda_{n2}(1-s)-\sin\lambda_{n1}(1-s)\right]\\&-\left(\sin\lambda_{n1}-\frac{\lambda_{n1}}{\lambda_{n2}}\sinh\lambda_{n2}\right)[\cosh\lambda_{n2}(1-s)-\cos\lambda_{n1}(1-s)]\Big\}ds\\C_3=&-\frac{\lambda_{n1}}{\lambda_{n2}}C_1,\qquad C_4=-C_2\end{aligned}\right\}\tag{3.4}$$

式中

$$\begin{aligned}H_n=&-\left(\sin\lambda_{n1}-\frac{\lambda_{n1}}{\lambda_{n2}}\sinh\lambda_{n2}\right)(\lambda_{n1}\sin\lambda_{n1}+\lambda_{n2}\sinh\lambda_{n2})\\&-\lambda_{n1}(\cos\lambda_{n1}-\cosh\lambda_{n2})^2\end{aligned}\tag{3.5}$$

类似地，可求得其它边界条件下的积分常数。

设矩形板共有I个平行于y轴的单向弹性线支，则作用于板上的弹性线支反力可写成

$$P(x,y)=\begin{cases}0, & x\neq x_i,\\ P'_i(y), & x=x_i,\end{cases}\qquad (i=1,2,\cdots,I) \tag{3.6}$$

式中，$P'_i(y)$为作用于x_i处的第i个弹性线支的支承反力．令$\xi_i=x_i/a$，则有

$$P(\xi,\eta)=\begin{cases}0, & \xi\neq\xi_i,\\ P'_i(\eta)/a, & \xi=\xi_i,\end{cases}\qquad (i=1,2,\cdots,I) \tag{3.7}$$

由(2.7)式可知

$$P'_i(\eta)=2\sum_{n=1}^{\infty}P_{ni}\sin n\pi\eta \tag{3.8}$$

将上式分别代入(2.12)、(3.2)、(3.4)三式，得到

$$F_n(\xi)=\frac{a^3}{D\lambda_{n1}\lambda_{n2}k_n^2}\sum_{j=1}^{I}P_{nj}[\lambda_{n1}\sinh\lambda_{n2}(\xi-\xi_j)-\lambda_{n2}\sin\lambda_{n1}(\xi-\xi_j)]u(\xi-\xi_j) \tag{3.9}$$

式中，$u(\xi-\xi_j)=\begin{cases}0, & \xi<\xi_j\\ 1, & \xi>\xi_j\end{cases}$

为单位阶跃函数．

若两对边简支，有

$$\left.\begin{aligned}C_1&=\frac{a^3}{D\lambda_{n1}k_n^2\sin\lambda_{n1}}\sum_{j=1}^{I}P_{nj}\sin\lambda_{n1}(1-\xi_j)\\ C_2&=-\frac{a^3}{D\lambda_{n2}k_n^2\sinh\lambda_{n2}}\sum_{j=1}^{I}P_{nj}\sinh\lambda_{n2}(1-\xi_j)\end{aligned}\right\} \tag{3.10}$$

若两对边固支，有

$$\left.\begin{aligned}C_1=&\frac{a^3}{Dk_n^2H_n}\sum_{j=1}^{I}P_{nj}\Big\{(\cos\lambda_{n1}-\cosh\lambda_{n2})[\cosh\lambda_{n2}(1-\xi_j)-\cos\lambda_{n1}(1-\xi_j)]\\ &+\Big(\sin\lambda_{n1}+\frac{\lambda_{n2}}{\lambda_{n1}}\sinh\lambda_{n2}\Big)\Big[\frac{\lambda_{n1}}{\lambda_{n2}}\sinh\lambda_{n2}(1-\xi_j)-\sin\lambda_{n1}(1-\xi_j)\Big]\Big\}\\ C_2=&\frac{a^3}{Dk_n^2H_n}\sum_{j=1}^{I}P_{nj}\Big\{(\cos\lambda_{n1}-\cosh\lambda_{n2})\Big[\frac{\lambda_{n1}}{\lambda_{n2}}\sinh\lambda_{n2}(1-\xi_j)-\sin\lambda_{n1}(1-\xi_j)\Big]\\ &-\Big(\sin\lambda_{n1}-\frac{\lambda_{n1}}{\lambda_{n2}}\sinh\lambda_{n2}\Big)[\cosh\lambda_{n2}(1-\xi_j)-\cos\lambda_{n1}(1-\xi_j)]\Big\}\end{aligned}\right\} \tag{3.11}$$

四、频 率 方 程

在第i个弹性线支处，矩形板的横向位移与弹性线支反力的关系为

$$P'_i(\eta)=-K_iW(\xi_i,\eta) \tag{4.1}$$

式中，K_i为第i个弹性线支的刚度。

将(2.8)式和(3.8)式代入上式，得频率方程，可写成统一的矩阵形式

$$\begin{bmatrix} B_{11}^n+2\gamma_1^n & B_{12}^n & \cdots & B_{1I}^n \\ B_{21}^n & B_{22}^n+2\gamma_2^n & & B_{2I}^n \\ \vdots & \vdots & \ddots & \vdots \\ B_{I1}^n & B_{I2}^n & \cdots & B_{II}^n+2\gamma_I^n \end{bmatrix}\begin{bmatrix} P_{n1} \\ P_{n2} \\ \vdots \\ P_{nI} \end{bmatrix}=0 \tag{4.2}$$

若两对边简支，有

$$\left.\begin{aligned} &\gamma_i^n=\frac{D\lambda_{n1}\lambda_{n2}k_n^2}{K_ia^3} \\ &B_{ij}^n=\frac{\lambda_{n2}\sin\lambda_{n1}\xi_i}{\sin\lambda_{n1}}\sin\lambda_{n1}(1-\xi_j)-\frac{\lambda_{n1}\sinh\lambda_{n2}\xi_i}{\sinh\lambda_{n2}}\sinh\lambda_{n2}(1-\xi_j) \\ &\qquad+\{\lambda_{n1}\sinh\lambda_{n2}(\xi_i-\xi_j)-\lambda_{n2}\sin\lambda_{n1}(\xi_i-\xi_j)\}u(\xi_i-\xi_j) \\ &\qquad\qquad(n=1,2,\cdots,\infty;\ i,j=1,2,\cdots,I) \end{aligned}\right\} \tag{4.3}$$

若两对边固支，有

$$\left.\begin{aligned} &\gamma_i^n=\frac{Dk_n^2H_n}{a^3K_i} \\ &B_{ij}^n=\left(\sin\lambda_{n1}\xi_i-\frac{\lambda_{n1}}{\lambda_{n2}}\sinh\lambda_{n2}\xi_i\right)\Big\{(\cos\lambda_{n1}-\cosh\lambda_{n2})[\cosh\lambda_{n2}(1-\xi_j) \\ &\qquad-\cos\lambda_{n1}(1-\xi_j)]+\left(\sin\lambda_{n1}+\frac{\lambda_{n2}}{\lambda_{n1}}\sinh\lambda_{n2}\right) \\ &\qquad\cdot\left[\frac{\lambda_{n1}}{\lambda_{n2}}\sinh\lambda_{n2}(1-\xi_j)-\sin\lambda_{n1}(1-\xi_j)\right]\Big\} \\ &\qquad+(\cos\lambda_{n1}\xi_i-\cosh\lambda_{n2}\xi_i)\Big\{(\cos\lambda_{n1}-\cosh\lambda_{n2})\Big[\frac{\lambda_{n1}}{\lambda_{n2}}\sinh\lambda_{n2}(1-\xi_j) \\ &\qquad-\sin\lambda_{n1}(1-\xi_j)\Big]-\left(\sin\lambda_{n1}-\frac{\lambda_{n1}}{\lambda_{n2}}\sinh\lambda_{n2}\right) \\ &\qquad\cdot[\cosh\lambda_{n2}(1-\xi_j)-\cos\lambda_{n1}(1-\xi_j)]\Big\}+\frac{H_n}{\lambda_{n1}\lambda_{n2}}\{\lambda_{n1}\sinh\lambda_{n2}(\xi_i-\xi_j) \\ &\qquad-\lambda_{n2}\sin\lambda_{n1}(\xi_i-\xi_j)\}u(\xi_i-\xi_j) \\ &\qquad\qquad(n=1,2,\cdots,\infty;\ i,j=1,2,\cdots,I) \end{aligned}\right\} \tag{4.4}$$

由(4.2)式可精确地求出各阶固有频率及与之对应的$P_{ni}(i=1,2,\cdots,I)$诸常数的比值，将计算结果代回(2.8)式可得各阶振型函数。对于其它边界条件下的频率方程和振型函数，可推导出类似的结果，这里不再一一推演。如果第i个线支为刚性支承，即$K_i\to\infty$，此时，仅需令(4.2)式中的$\gamma_i^n=0$即可。

由(4.3)式可知，当矩形板四边简支且中间仅有一个($I=1$)刚度为K的弹性线支时，可很容易地写出其频率方程为

$$2\frac{D\lambda_{n1}\lambda_{n2}k_n^2}{Ka^3}+\frac{\lambda_{n2}\sin\lambda_{n1}\xi_1}{\sin\lambda_{n1}}\sin\lambda_{n1}(1-\xi_1) -\frac{\lambda_{n1}\sinh\lambda_{n2}\xi_1}{\sinh\lambda_{n2}}\sinh\lambda_{n2}(1-\xi_1)=0 \tag{4.5}$$

五、结 束 语

本文提供了一个求解两对边简支中间有任意个单向弹性线支矩形板横向振动的新方法，频率方程及振型函数的表述均与已有方法不同。频率方程的阶数比分块求解法降低四倍，而振型函数可写成一个统一的解析式，因而比已有方法更简单适用，尤其是当线支个数较多时，更能显示出本文方法的优越性。

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An Analytical Solution of Transverse Vibration of Rectangular Plates Simply Supported at Two Opposite Edges with Arbitrary Number of Elastic Line Supports in One Way

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Abstract

This paper presents a new analytical solution of transverse vibration of rectangular plates simply supported at two opposite edges with arbitrary number of elastic line supports in one way. The reaction forces of the elastic line supports are regarded as the unknown external forces acted on the plate. The analytical solution of the differential equation of motion of the rectangular plate, which includes the unknown reaction forces, is gained. The frequency equation is derived by using the linear relationships between the reacion forces of the elastic line supports and the transverse displacements of the plate along the elastic line supports. The representations of the frequency equation and the mode shape functions are different from those obtained by other methods.

Key words rectangular plate, eigen-frequency, elastic line support, analytical solution