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受轴向压缩圆柱壳塑性屈曲的内时分析

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摘　要

采用内时塑性本构方程的增量和全量表达式分析了受轴向压缩圆柱壳的塑性屈曲，得到了塑性屈曲临界应力σ_{cr}与圆柱壳特征尺寸R/h间的关系．对AMΓ和Д1T铝合金圆柱壳塑性屈曲进行了分析，与实验结果的比较表明：除对于AMΓ圆柱壳由内时塑性本构方程的全量表达式给出了较经典塑性理论全量分析略为保守的结果外，在其它场合下，内时分析均给出了较经典塑性理论更符合实验数据的结果．

关键词　塑性屈曲　圆柱壳　内时

一、引　　言

薄壁结构被广泛地应用于航空、航天、核电、化工、舰船、汽车、建筑等众多的工业部门中，对其弹塑性稳定性的研究一直受到力学工作者和结构设计工程师们的关注．科学技术和现代工业的发展一方面要求对结构的弹塑性稳定性进行更深入可靠和合乎实际的分析，另一方面也为分析提供了有力的实验测试和计算的手段．

由于结构的弹塑性失稳涉及非比例弹塑性加卸载以及由于不可避免地存在着的缺陷所引起的变形局部化等复杂的过程，因此采取精确的本构关系进行结构的弹塑性分析显得十分重要．70年代以来，材料本构关系的研究取得了重要进展，出现了一批能更精确描述材料在复杂加载条件的响应特性的本构方程如Mroz的多屈服面模型[1]，Dafalias 和 Popov 的切线刚度法[2]，Chabohe的考虑回复的背应力演化[3]以及Valanis的内时塑性本构方程[4,5]等．尽管它们在许多方面获得了应用并被证明是行之有效的，但在弹塑性稳定性方面的应用却鲜见报道．

内时塑性本构方程由 Valanis 于 1971 是提出[4]并于 1980 年进行了完善[5]．以后，Watanabe 和 Atluri[6]指出内时塑性本构方程可包含上述其它本构方程为其特例．本文分别采用内时塑性本构方程的增量和全量表达式对受轴向压缩AMΓ和Д1T铝合金圆柱壳的塑性屈曲临界载荷进行了分析，取得了与实验数据[7]较为吻合的结果．

二、内时塑性本构方程

在等温小变形条件下，初始各向同性和塑性不可压缩材料的内时塑性本构方程可表示如

下[5]：

$$s_{ij}=\int_0^z \rho(z-z')\frac{de^p_{ij}}{dz'}dz' \tag{2.1}$$

式中s_{ij}和e^p_{ij}分别为偏应力和塑性应变张量，z为内时标度，它与内时量度ζ和强化函数 $f(z)$间存在下列关系

$$dz=\frac{d\zeta}{f(z)},\quad d\zeta=\|de^p_{ij}\| \tag{2.2}$$

$\|\cdot\|$表示欧几里德模．核函数$\rho(z)$取为

$$\rho(z)=\sum_{r=1}^{3}C_r\exp[-\alpha_r z] \tag{2.3}$$

如果取$f(z)=1$，且假设比例加载，则可得下述全量表达式[8]

$$s_{ij}=n_{ij}\sum_{r=0}^{3}\frac{C_r}{\alpha_r}(1-\exp[-\alpha_r z]) \tag{2.4}$$

式中n_{ij}表示加载方向上的单位矢量，它满足

$$n_{ij}n_{ij}=1 \tag{2.5}$$

且

$$\left.\begin{array}{l} ds_{ij}=\overline{ds}n_{ij},\quad de^p_{ij}=\overline{de}^p n_{ij},\quad s_{ij}=\bar{s}n_{ij},\quad e^p_{ij}=\bar{e}^p_{ij}n_{ij} \\ \overline{ds}=\sqrt{ds_{ij}ds_{ij}},\quad \overline{de}^p\sqrt{de^p_{ij}de^p_{ij}} \end{array}\right\} \tag{2.6}$$

从上述关系容易看出

$$\overline{de}=ds_{ij}n_{ij},\quad \overline{de}^p=de^p_{ij}n_{ij} \tag{2.7}$$

利用关系式

$$de^p_{ij}=\left(\overline{de}-\frac{\overline{ds}}{2G}\right)n_{ij} \tag{2.8}$$

式中e_{ij}和G分别为偏应变张量和剪切弹性模量，不难得到比例加载条件下

$$dz=\overline{de}-\frac{\overline{ds}}{2G},\quad z=\bar{e}-\frac{\bar{s}}{2G} \tag{2.9}$$

于是由式(2.4)可得

$$\bar{s}=\frac{1}{z}\left(\bar{s}-\frac{\bar{s}}{2G}\right)\sum_{r=1}^{3}\frac{C_r}{\alpha_r}(1-\exp[-\alpha_r z]) \tag{2.10}$$

设$\bar{\varepsilon}$和$\bar{\sigma}$分别为有效应力和有效应变，则

$$\bar{e}=\sqrt{3/2}\,\bar{\varepsilon},\quad \bar{s}=\sqrt{2/3}\,\bar{\sigma} \tag{2.11}$$

联立式(2.10)和(2.11)可得割线模量

$$E_s=\frac{\bar{\sigma}}{\bar{\varepsilon}}=\frac{3}{2}\left[2G\sum_{r=1}^{3}\frac{C_r}{\alpha_r}(1-\exp[-\alpha_r z])\Big/\left(2Gz+\sum_{r=1}^{3}\frac{C_r}{\alpha_r}(1-\exp[-\alpha_r z])\right)\right] \tag{2.12}$$

对式(2.4)微分且注意到式(2.9)和(2.11)可得切线模量

$$E_t=\frac{\overline{d\sigma}}{\overline{d\varepsilon}}=\frac{3}{2}\left[2G\sum_{r=1}^{3}C_r\exp[-\alpha_r z]\Big/\left(2G+\sum_{r=1}^{3}C_r\exp[-\alpha_r z]\right)\right] \tag{2.13}$$

设　　$E^0_s=\sigma_x/\varepsilon_x$和$E^0_t=d\sigma_x/d\varepsilon_x$

分别为单向拉伸应力应变曲线上(σ_x,ε_x)点的割线和切线模量，不难求得

$$\frac{1}{E_s^0}=\frac{1-2\nu}{3E}+\frac{1}{E_s},\quad \frac{1}{E_t^0}=\frac{1-2\nu}{3E}+\frac{1}{E_t} \tag{2.14}$$

三、圆柱壳塑性屈曲方程

采用扁壳理论的基本方程，采用小变形假设并认为壳体处于平面应力状态，即$\sigma_z=\tau_{xz}=\tau_{\theta z}=0$。如果中面应变增量为$\Delta\varepsilon_{x0}$，$\Delta\varepsilon_{\theta0}$和$\Delta\gamma_{x\theta0}$，中面位移增量为$\Delta u$，$\Delta v$和$\Delta w$，则距中面为$z$的一点处的应变增量为[9]：

$$\Delta\varepsilon_x=\Delta\varepsilon_{x0}-z\frac{\partial^2(\Delta w)}{\partial x^2},\quad \Delta\varepsilon_\theta=\Delta\varepsilon_{\theta0}-z\frac{\partial^2(\Delta w)}{R^2\partial\theta^2},\ \Delta\gamma_{x\theta}=\Delta\gamma_{x\theta0}-2z\frac{\partial^2(\Delta w)}{R\partial x\partial\theta} \tag{3.1}$$

式中

$$\Delta\varepsilon_{x0}=\frac{\partial(\Delta u)}{\partial x},\quad \Delta\varepsilon_{\theta0}=\frac{\partial(\Delta v)}{R\partial\theta}-\frac{\Delta w}{R},\ \Delta\gamma_{x\theta0}=\frac{\partial(\Delta u)}{R\partial\theta}+\frac{\partial(\Delta v)}{\partial x} \tag{3.2}$$

圆柱壳材料元的平衡方程为[9]

$$\left.\begin{aligned}&\frac{\partial(\Delta N_x)}{\partial x}+\frac{\partial(\Delta N_{x\theta})}{R\partial\theta}=0,\ \frac{\partial(\Delta N_{x\theta})}{\partial x}+\frac{\partial(\Delta N_\theta)}{R\partial\theta}=0\\&\frac{\partial^2(\Delta M_x)}{\partial x^2}+2\frac{\partial^2(\Delta M_{x\theta})}{R\partial x\partial\theta}+\frac{\partial^2(\Delta M_\theta)}{R^2\partial\theta^2}+\frac{\Delta N_\theta}{R}\\&\quad+h\left(\sigma_x\frac{\partial^2(\Delta w)}{\partial x^2}+\sigma_\theta\frac{\partial^2(\Delta w)}{R^2\partial\theta^2}+2\tau_{x\theta}\frac{\partial^2(\Delta w)}{R\partial\theta\partial x}\right)=0\end{aligned}\right\} \tag{3.3}$$

式中σ_x，σ_θ和$\tau_{x\theta}$表示屈曲前的中面应力，ΔN_x，ΔN_θ，$\Delta N_{x\theta}$和ΔM_x，ΔM_θ，$\Delta M_{x\theta}$分别表示屈曲时中面力和弯扭矩的变化，它们由下式定义

$$\left.\begin{aligned}&\Delta N_x=\int_{-h/2}^{h/2}\Delta\sigma_x dz,\quad \Delta N_\theta=\int_{-h/2}^{h/2}\Delta\sigma_\theta dz,\quad \Delta N_{x\theta}=\int_{-h/2}^{h/2}\Delta\tau_{x\theta}dz\\&\Delta M_x=\int_{-h/2}^{h/2}\Delta\sigma_x zdz,\ \Delta M_\theta=\int_{-h/2}^{h/2}\Delta\sigma_\theta zdz,\ \Delta M_{x\theta}=\int_{-h/2}^{h/2}\Delta\tau_{x\theta}zdz\end{aligned}\right\} \tag{3.4}$$

式中h为圆柱壳壁厚。若引入应力函数ϕ使得

$$\Delta N_x=\frac{\partial^2\phi}{R^2\partial\theta^2},\quad \Delta N_\theta=\frac{\partial^2\phi}{\partial x^2},\ \Delta N_{x\theta}=-\frac{\partial^2\phi}{R\partial x\partial\theta} \tag{3.5}$$

则式(3.3)中的前两式可得到满足。从式(3.2)中消去Δu和Δv可得协调方程

$$\frac{\partial^2(\Delta\varepsilon_{x0})}{R^2\partial\theta^2}+\frac{\partial^2(\Delta\varepsilon_{\theta0})}{\partial x^2}-\frac{\partial^2(\Delta\gamma_{x\theta0})}{R\partial\theta\partial x}+\frac{\partial^2(\Delta w)}{R\partial x^2}=0 \tag{3.6}$$

四、受轴向压缩圆柱壳塑性屈曲临界载荷的内时全量理论分析

在平面应力问题中，一点处的应力应变关系可表示为

$$\left.\begin{aligned}&\varepsilon_x=\frac{1-2\nu}{3E}(\sigma_x+\sigma_\theta)+\frac{1}{E_s}\left(\sigma_x-\frac{1}{2}\sigma_\theta\right)\\&\varepsilon_\theta=\frac{1-2\nu}{3E}(\sigma_x+\sigma_\theta)+\frac{1}{E_s}\left(\sigma_\theta-\frac{1}{2}\sigma_x\right)\\&\gamma_{x\theta}=3\tau_{x\theta}/E_s\end{aligned}\right\} \tag{4.1}$$

其增量形式

$$
\left.\begin{aligned}
\Delta\varepsilon_x &= \frac{1-2\nu}{3E}(\Delta\sigma_x+\Delta\sigma_\theta)+\frac{1}{E_s}\left(\Delta\sigma_x-\frac{1}{2}\Delta\sigma_\theta\right)+\left(\sigma_x-\frac{1}{2}\sigma_\theta\right)\Delta\tilde{\sigma}^* \\
\Delta\varepsilon_\theta &= \frac{1-2\nu}{3E}(\Delta\sigma_x+\Delta\sigma_\theta)+\frac{1}{E_s}\left(\Delta\sigma_\theta-\frac{1}{2}\Delta\sigma_x\right)+\left(\sigma_\theta-\frac{1}{2}\sigma_x\right)\Delta\tilde{\sigma}^* \\
\Delta\gamma_{x\theta} &= \frac{3}{E_s}\Delta\tau_{x\theta}+3\tau_{x\theta}\Delta\tilde{\sigma}^*
\end{aligned}\right\} \tag{4.2}
$$

式中

$$
\left.\begin{aligned}
&\Delta\tilde{\sigma}^*=\left(\frac{1}{E_t}-\frac{1}{E_s}\right)\frac{\Delta\tilde{\sigma}}{\tilde{\sigma}}, \quad \tilde{\sigma}=\sqrt{\sigma_x^2+\sigma_\theta^2-\sigma_x\sigma_\theta+3\tau_{x\theta}^2} \\
&\Delta\tilde{\sigma}=\frac{1}{\tilde{\sigma}}\left[\left(\sigma_x-\frac{1}{2}\sigma_\theta\right)\Delta\sigma_x+\left(\sigma_\theta-\frac{1}{2}\sigma_x\right)\Delta\sigma_\theta+3\tau_{x\theta}\Delta\tau_{x\theta}\right]
\end{aligned}\right\} \tag{4.3}
$$

等效应变$\tilde{\varepsilon}$及其增量$\Delta\tilde{\varepsilon}$由下式决定

$$
\tilde{\varepsilon}=\frac{1}{\sqrt{3}}\sqrt{\frac{1}{k_0}(\varepsilon_x+\varepsilon_\theta)^2+3(\varepsilon_x+\varepsilon_\theta)^2+\gamma_{x\theta}^2}\,, \quad \Delta\tilde{\varepsilon}=\frac{1}{k}[k_1\Delta\varepsilon_x+k_2\Delta\varepsilon_\theta+k_3\Delta\gamma_{x\theta}] \tag{4.4}
$$

其中

$$
\left.\begin{aligned}
&k_0=\left[1+\frac{4(1-2\nu)}{3E}E_s\right]^2, \quad k=1-\frac{(E_s-E_t)(\sigma_x+\sigma_\theta)^2}{3c_1\tilde{\sigma}^2} \\
&k_1=\frac{1}{c_1\tilde{\sigma}}\left[c_2\sigma_x-\frac{2}{3}E_s\sigma_\theta\right], \quad k_2=\frac{1}{c_1\tilde{\sigma}}\left[c_2\sigma_\theta-\frac{2}{3}E_s\sigma_x\right] \\
&k_3=\frac{\tau_{x\theta}}{\tilde{\sigma}}, \quad c_1=\frac{E}{1-2\nu}+\frac{4}{3}E_s, \quad c_2=c_1-\frac{2}{3}E_s
\end{aligned}\right\} \tag{4.5}
$$

将式(3.1)代入(4.2)式并沿壁厚积分，设屈曲前横截面上各点应力状态相反，屈曲发生时扰动充分小以致于沿壁厚各点E_t和E_s的变化可以忽略，则得

$$
\left.\begin{aligned}
h\varepsilon_{x0} &= \frac{1-2\nu}{3E}(\Delta N_x+\Delta N_\theta)+\frac{1}{E_s}\left(\Delta N_x-\frac{1}{2}\Delta N_\theta\right) \\
&\quad +c_3\left(\frac{1}{E_t}-\frac{1}{E_s}\right)\left(\sigma_x-\frac{1}{2}\sigma_\theta\right) \\
h\varepsilon_{\theta0} &= \frac{1-2\nu}{3E}(\Delta N_x+\Delta N_\theta)+\frac{1}{E_s}\left(\Delta N_\theta-\frac{1}{2}\Delta N_x\right) \\
&\quad +c_3\left(\frac{1}{E_t}-\frac{1}{E_s}\right)\left(\sigma_\theta-\frac{1}{2}\sigma_x\right) \\
h\gamma_{x\theta0} &= \frac{3}{E_s}\Delta N_{x\theta}+3c_3\left(\frac{1}{E_t}-\frac{1}{E_s}\right)\tau_{x\theta}
\end{aligned}\right\} \tag{4.6}
$$

式中 $c_3=\frac{1}{\tilde{\sigma}^2}\left[\left(\sigma_x-\frac{1}{2}\sigma_\theta\right)\Delta N_x+\left(\sigma_\theta-\frac{1}{2}\sigma_x\right)\Delta N_\theta+3\tau_{x\theta}\Delta N_{x\theta}\right]$

若设 $c_4=\frac{c_1-2E_s}{c_1}, \quad c_5=\frac{c_1-E_s}{c_1}, \quad c_6=\frac{E_s-E_t}{E_s}$

则从式(4.1)解得

$$
\sigma_x=\frac{2}{3}E_s[\varepsilon_x+c_4(\varepsilon_x+\varepsilon_\theta)], \quad \sigma_\theta=\frac{2}{3}E_s[\varepsilon_\theta+c_4(\varepsilon_x+\varepsilon_\theta)], \quad \tau_{x\theta}=\frac{1}{3}E_s\gamma_{x\theta} \tag{4.7}
$$

由此可求得

$$
\left.\begin{aligned}
\Delta\sigma_x&=\frac{2}{3}E_s[\Delta\varepsilon_x+c_4(\Delta\varepsilon_x+\Delta\varepsilon_\theta)]-\frac{k_1}{k}(E_s-E_t)(k_1\Delta\varepsilon_x+k_2\Delta\varepsilon_\theta+k_3\Delta\gamma_{x\theta})\\
\Delta\sigma_\theta&=\frac{2}{3}E_s[\Delta\varepsilon_\theta+c_4(\Delta\varepsilon_x+\Delta\varepsilon_\theta)]-\frac{k_2}{k}(E_s-E_t)(k_1\Delta\varepsilon_x+k_2\Delta\varepsilon_\theta+k_3\Delta\gamma_{x\theta})\\
\Delta\tau_{x\theta}&=\frac{1}{3}E_s\Delta\gamma_{x\theta}-k_3(E_s-E_t)\Delta\bar{\varepsilon}
\end{aligned}\right\}\tag{4.8}
$$

将式(4.8)代入(3.4)式的后三式得

$$
\left.\begin{aligned}
\Delta M_x=&-\frac{E_sh^3}{9}\left[c_5-\frac{3k_1^2}{4k}c_6\right]\frac{\partial^2(\Delta w)}{\partial x^2}\\
&-\frac{E_sh^3}{18}\left[c_4-\frac{3k_1k_2}{2k}c_6\right]\frac{\partial^2(\Delta w)}{R^2\partial\theta^2}+\frac{E_s-E_t}{6}\frac{k_1}{k}h^3k_3\frac{\partial^2(\Delta w)}{R\partial x\partial\theta}\\
\Delta M_\theta=&-\frac{E_sh^3}{18}\left[c_4-\frac{3k_1k_2}{2k}c_6\right]\frac{\partial^2(\Delta w)}{\partial x^2}\\
&-\frac{E_sh^3}{9}\left[c_5-\frac{3k_2^2}{4k}c_6\right]\frac{\partial^2(\Delta w)}{R^2\partial\theta^2}+\frac{E_s-E_t}{6}\frac{k^2}{k}h^3k_3\frac{\partial^2(\Delta w)}{R\partial x\partial\theta}\\
\Delta M_{x\theta}=&\frac{E_s-E_t}{12}\frac{k_1}{k}h^3k_3\frac{\partial^2(\Delta w)}{\partial x^2}+\frac{E_s-E_t}{12}\frac{k_2}{k}h^3k_3\frac{\partial^2(\Delta w)}{R^2\partial\theta^2}\\
&-\frac{E_sh^3}{18}\left(1-3k_3^2\frac{c_6}{k}\right)\frac{\partial^2(\Delta w)}{R\partial x\partial\theta}
\end{aligned}\right\}\tag{4.9}
$$

联立式(4.6)和协调方程式(3.6)给出

$$
\begin{aligned}
&\left(\frac{1}{c_7}-\frac{3c_9k_1^2}{4c_8k}\right)\frac{\partial^4(\Delta w)}{\partial x^4}+2\left(\frac{1}{c_7}-\frac{3c_9}{4c_8}\frac{k_1k_2+2k_3^2}{k}\right)\frac{\partial^4(\Delta w)}{R^2\partial x^2\partial\theta^2}\\
&\quad+\left(\frac{1}{c_7}-\frac{3c_9k_2^2}{4c_8k}\right)\frac{\partial^4(\Delta w)}{R^4\partial\theta^4}-\frac{3c_9}{c_8}k_3\left(\frac{k_1}{k}\frac{\partial^4(\Delta w)}{R\partial x^3\partial\theta}+\frac{k_2}{k}\frac{\partial^4(\Delta w)}{R^3\partial x\partial\theta^3}\right)\\
&=\frac{9}{h^3}\left(\frac{1}{E_s^0}-\frac{1-2\nu}{3E}\right)\left[N_x^0\frac{\partial^2(\Delta w)}{\partial x^2}+N_\theta^0\frac{\partial^2(\Delta w)}{R^2\partial\theta^2}+2N_{x\theta}^0\frac{\partial^2(\Delta w)}{R\partial x\partial\theta}+\frac{\partial^2\phi}{R\partial x^2}\right]
\end{aligned}\tag{4.10}
$$

式中 $N_x^0=h\sigma_x,\ N_\theta^0=h\sigma_\theta,\ N_{x\theta}^0=h\tau_{x\theta}$

$$
c_7=\frac{(1-2\nu)E_s}{E},\quad c_8=\frac{1}{E_t}-\frac{1-2\nu}{3E},\quad c_9=\frac{1}{E_t^0}-\frac{1}{E_s^0}
$$

同理，联立式(4.9)和式(3.3)的第三式可得

$$
\begin{aligned}
&\frac{1}{E_s}\nabla^4\phi+\frac{c_9}{\bar{\sigma}^2}\left[\left(\sigma_x-\frac{1}{2}\sigma_\theta\right)\frac{\partial^2F(\sigma,\phi)}{R^2\partial\theta^2}\right.\\
&\quad\left.+\left(\sigma_\theta-\frac{1}{2}\sigma_x\right)\frac{\partial^2F(\sigma,\phi)}{\partial x^2}-3\tau_{x\theta}\frac{\partial^2F(\sigma,\phi)}{R\partial x\partial\theta}\right]=-\frac{h}{R}\frac{\partial^2(\Delta w)}{R\partial x^2}
\end{aligned}\tag{4.11}
$$

式中 $F(\sigma,\phi)=\left(\sigma_x-\frac{1}{2}\sigma_\theta\right)\frac{\partial^2\phi}{R^2\partial\theta^2}+\left(\sigma_\theta-\frac{1}{2}\sigma_x\right)\frac{\partial^2\phi}{\partial x^2}-3\tau_{x\theta}\frac{\partial^2\phi}{R\partial x\partial\theta}$

在轴向压力P作用下，屈曲前的应力为

$$
\sigma_x=-\frac{P}{2\pi Rh},\quad\sigma_\theta=\tau_{x\theta}=0,\quad\bar{\sigma}=\frac{P}{2\pi Rh}\tag{4.12}
$$

在[7]中，沈立等通过实验观察发现塑性屈曲的初始屈曲形式是轴对称的，故可取应力

函数ϕ和中面位移函数Δw如下

$$\phi=\phi_m\sin\frac{m\pi x}{L},\ \Delta w=w_m\sin\frac{m\pi x}{L} \tag{4.13}$$

则由式(4.10)和(4.11)可得

$$\nabla^4\phi+\frac{1}{4}\left(\frac{\varphi_s}{\varphi_t}-1\right)\frac{\partial^4\phi}{\partial x^4}=-\frac{h}{R}E_s^0\frac{\partial^2(\Delta w)}{\partial x^2} \tag{4.14}$$

$$a_1\frac{\partial^4(\Delta w)}{\partial x^4}=\left(1-\frac{1-2\nu}{3}\varphi_s\right)\frac{9}{E_s^0h}\left(N_x^0\frac{\partial^2(\Delta w)}{\partial x^2}+\frac{1}{R}\frac{\partial^2\phi}{\partial x^2}\right) \tag{4.15}$$

式中 $\varphi_s=\dfrac{E_s}{E}$, $\varphi_t=\dfrac{E_t}{E}$, $a_1=\dfrac{1}{1+(1-2\nu)\varphi_s}-\dfrac{3}{4}\dfrac{3}{3-(1-2\nu)\varphi_t}\left(1-\dfrac{\varphi_t}{\varphi_s}\right)\dfrac{k_1^2}{k}$

k和k_1亦可简化为

$$k=1-\frac{(1-2\nu)(\varphi_s-\varphi_t)}{3+4(1-2\nu)\varphi_s},\quad k_1=\frac{3+2(1-2\nu)\varphi_s}{3+4(1-2\nu)\varphi_s}$$

联立式(4.13)、(4.14)和(4.15)可得

$$\frac{\phi_m}{w_m}=4hE_s^0/R\left(3+\frac{\varphi_s}{\varphi_t}\right)\left(\frac{m\pi}{L}\right)^2 \tag{4.16}$$

$$\sigma_x=-\left[A\left(\frac{m}{L}\right)^2+B\left(\frac{L}{m}\right)^2\right] \tag{4.17}$$

式中 $A=\dfrac{a_1\pi^2h^2\varphi_sE}{9-3(1-2\nu)\varphi_s}$, $B=\dfrac{4\phi_sE}{\pi^2R^2(3-\varphi_s/\varphi_t)}$

对于长圆柱，沿轴线方向可能形成许多个波，故可将轴向载荷σ_x考虑为$\lambda=L/m$的连续函数[10]，其临界值可由σ_x的最小值决定，令

$$\partial\sigma_x/\partial\lambda=0$$

可得当

$$\lambda=\left(\frac{A}{B}\right)^{\frac{1}{4}}=\left[\frac{a_1\pi^4R^2h^2(3+\varphi_s/\varphi_t)\varphi_s}{(36-12(1-2\nu)\varphi_s)\varphi_s}\right]^{\frac{1}{4}} \tag{4.18}$$

时，存在临界应力

$$\sigma_{cr}=|\sigma_x|_{\min}=2\sqrt{AB}=\frac{4}{3}\varphi_sE\frac{h}{R}\sqrt{\frac{3a_1}{(3-(1-2\nu)\varphi_s)(3+\varphi_s/\varphi_t)}} \tag{4.19}$$

为便于计算，可由上式解出

$$\frac{R}{h}=\frac{4E_s^0}{3\sigma_{cr}}\sqrt{\frac{3a_1}{(3-(1-2\nu)\varphi_s)(3+\varphi_s/\varphi_t)}} \tag{4.20}$$

利用本节的结果容易确定R/h与σ_{cr}间的关系。设给定σ_{cr}，首先由式(2.4)确定z，进而利用式(2.12)和(2.13)分别确定E_s和E_t，由式(2.14)可得E_s^0和E_t^0，在求得φ_s和φ_t后，由式(4.20)可求得与σ_{cr}相对应的R/h。

五、受轴向压缩圆柱壳塑性屈曲临界载荷的内时增量理论分析

在[11]中Peng和Fan提出了下述增量内时弹塑性本构方程：

$$\Delta s_{ij}=2G_p\Delta e_{ij}+T_pB_{ij}\Delta z \tag{5.1}$$

式中 $T_p=\left(1+\frac{A}{2G}\right)^{-1}$, $2G_p=AT_p$, $A=\sum_{r=1}^{3}k_rC_r$

$$B_{ij}=-\sum_{r=1}^{3}k_r\alpha_r s_{ij}^{(r)}(z_n),\ k_r=\frac{1-\exp[-\alpha_r\Delta z]}{\alpha_r\Delta z}$$

$$s_{ij}^{(r)}(z_n)=\exp[-\alpha_r\ \Delta z]s_{ij}^{(r)}(z_{n-1})+\frac{\Delta e_{ij}^p}{\Delta z}\ \frac{C_r}{\alpha_r}(1-\exp[-\alpha_r\Delta z])$$

在平面应力条件下，可将式(5.1)写为矩阵形式

$$\begin{Bmatrix}\Delta\sigma_x\\ \Delta\sigma_\theta\\ \Delta\tau_{x\theta}\end{Bmatrix}=\begin{bmatrix}D_{11}' & D_{12}' & D_{14}'\\ D_{21}' & D_{22}' & D_{24}'\\ D_{41}' & D_{42}' & D_{44}'\end{bmatrix}\begin{Bmatrix}\Delta\varepsilon_x\\ \Delta\varepsilon_\theta\\ \Delta\gamma_{x\theta}\end{Bmatrix} \tag{5.2}$$

式中

$$D_{ij}'=\frac{D_{i3}D_{3j}}{D_{33}}\qquad (i,j=1,2,4) \tag{5.3}$$

$$\left.\begin{aligned}&[D]=[D_1]+\frac{2(G-G_p)}{H}[D_2],\quad H=1+\frac{T_p}{2Gf^2(z)\Delta z}B_{ij}\Delta e_{ij}^p\\ &[D_1]=\begin{bmatrix}C_1 & C_2 & C_2 & 0\\ C_2 & C_1 & C_2 & 0\\ C_2 & C_2 & C_1 & 0\\ 0 & 0 & 0 & G_p\end{bmatrix},\ C_1=K+\frac{4}{3}G_p,\ C_2=K-\frac{2}{3}G_p\\ &[D_2]=\frac{T_p}{2Gf^2(z)\Delta z}(B_x,B_\theta,B_r,B_{x\theta})^T(\Delta e_x^p,\Delta e_\theta^p,\Delta e_r,\Delta e_{x\theta}^p)\end{aligned}\right\} \tag{5.4}$$

将式(3.1)代入式(5.2)并沿壁厚积分可得

$$\begin{Bmatrix}h\Delta\varepsilon_{x0}\\ h\Delta\varepsilon_{\theta0}\\ h\Delta\gamma_{x\theta0}\end{Bmatrix}=\begin{bmatrix}D_{11}' & D_{12}' & D_{14}'\\ D_{21}' & D_{22}' & D_{24}'\\ D_{41}' & D_{42}' & D_{44}'\end{bmatrix}^{-1}\begin{Bmatrix}\Delta N_x\\ \Delta N_\theta\\ \Delta N_{x\theta}\end{Bmatrix}=\begin{bmatrix}C_{11} & C_{12} & C_{14}\\ C_{21} & C_{22} & C_{24}\\ C_{41} & C_{42} & C_{44}\end{bmatrix}\begin{Bmatrix}\Delta N_x\\ \Delta N_\theta\\ \Delta N_{x\theta}\end{Bmatrix} \tag{5.5}$$

$$\begin{Bmatrix}\Delta M_x\\ \Delta M_\theta\\ \Delta M_{x\theta}\end{Bmatrix}=-\frac{h^3}{12}\begin{bmatrix}D_{11}' & D_{12}' & D_{14}'\\ D_{21}' & D_{22}' & D_{24}'\\ D_{41}' & D_{42}' & D_{44}'\end{bmatrix}\begin{Bmatrix}\dfrac{\partial^2(\Delta w)}{\partial x^2}\\ \dfrac{\partial^2(\Delta w)}{R^2\partial\theta^2}\\ \dfrac{\partial^2(\Delta w)}{R\partial x\partial\theta}\end{Bmatrix} \tag{5.6}$$

取应力函数ϕ使满足式(4.9)，将式(5.5)代入协调方程式(3.6)可得

$$\begin{aligned}&\left(C_{11}\frac{\partial^4\phi}{R^4\partial\theta^4}+C_{12}\frac{\partial^4\phi}{R^2\partial x^2\partial\theta^2}-C_{14}\frac{\partial^4\phi}{R^3\partial x\partial\theta^3}\right)+\left(C_{21}\frac{\partial^4\phi}{R^2\partial x^2\partial\theta^2}+C_{22}\frac{\partial^4\phi}{\partial x^4}-C_{24}\frac{\partial^4\phi}{R\partial x^3\partial\theta}\right)\\ &+\left(C_{41}\frac{\partial^4\phi}{R^3\partial x\partial\theta^3}+C_{42}\frac{\partial^4\phi}{R\partial x^3\partial\theta}-C_{44}\frac{\partial^4\phi}{R^2\partial x^2\partial\theta^2}\right)+\frac{h}{R}\ \frac{\partial^2(\Delta w)}{\partial x^2}=0\end{aligned} \tag{5.7}$$

将式(5.6)代入式(3.3)的第三式且注意到式(3.5)，可得

$$\left(\frac{D_{11}'}{12}\ \frac{\partial^4(\Delta w)}{\partial x^4}+\frac{D_{12}'}{12}\ \frac{\partial^4(\Delta w)}{R^2\partial x^2\partial\theta^2}+\frac{D_{14}'}{6}\ \frac{\partial^4(\Delta w)}{R\partial x^3\partial\theta}\right)+\left(\frac{D_{21}'}{12}\ \frac{\partial^4(\Delta w)}{R^2\partial x^2\partial\theta^2}\right.$$

$$+\frac{D_{22}'}{12}\frac{\partial^4(\Delta w)}{R^4\partial\theta^4}+\frac{D_{24}'}{6}\frac{\partial^4(\Delta w)}{R^3\partial x\partial\theta^3}\Big)+\Big(\frac{D_{41}'}{6}\frac{\partial^4(\Delta w)}{R\partial x^3\partial\theta}+\frac{D_{42}'}{6}\frac{\partial^4(\Delta w)}{R^3\partial x\partial\theta^3}$$

$$+\frac{D_{44}'}{3}\frac{\partial^4(\Delta w)}{R^2\partial x^2\partial\theta^2}\Big)-\frac{1}{h^3R}\frac{\partial^2\phi}{\partial x^2}-\frac{1}{h^2}\Big(\sigma_x\frac{\partial^2(\Delta w)}{\partial x^2}+\sigma_\theta\frac{\partial^2(\Delta w)}{R^2\partial\theta^2}+2\tau_{x\theta}\frac{\partial^2(\Delta w)}{R\partial x\partial\theta}\Big)$$

$$=0 \tag{5.8}$$

当取式(4.13)所示的应力函数ϕ和位移函数Δw时，由式(5.7)和式(5.8)可解得

$$C_{22}\frac{\partial^4\phi}{\partial x^4}+\frac{h}{R}\frac{\partial^2(\Delta w)}{\partial x^2}=0,\ \frac{D_{11}'}{12}\frac{\partial^4(\Delta w)}{\partial x^4}-\frac{1}{h^3R}\frac{\partial^2\phi}{\partial x^2}-\frac{\sigma_x}{h^2}\frac{\partial^2(\Delta w)}{\partial x^2}=0 \tag{5.9}$$

即

$$\frac{\phi_m}{w_m}=\frac{h}{RC_{22}\left(\frac{m\pi}{L}\right)^2},\ \sigma_x=-\left[\frac{h^2D_{11}'}{12}\left(\frac{m\pi}{L}\right)^2+\frac{1}{R^2C_{22}}\left(\frac{L}{m\pi}\right)^2\right] \tag{5.10}$$

采用与前节相同的方法，令$\lambda=L/m$，且使$\partial\sigma_x/\partial\lambda=0$，可得当

$$\lambda=\pi\sqrt{Rh}\,(D_{11}'C_{22}/12)^{1/4} \tag{5.11}$$

时，

$$\sigma_{cr}=|\sigma_x|_{\min}=\frac{1}{\sqrt{3}}\frac{h}{R}\sqrt{\frac{D_{11}'}{C_{22}}} \tag{5.12}$$

或

$$\frac{R}{h}=\frac{1}{\sqrt{3}\,\sigma_{cr}}\sqrt{\frac{D_{11}'}{C_{22}}} \tag{5.13}$$

采用内时增量理论求解与R/h相对应的σ_{cr}的步骤如下：给定σ_{cr}及增量加载步数，采用$\Delta\sigma=\sigma_{cr}/n$及$\Delta s_{ij}=2G(\Delta e_{ij}-\Delta e_{ij}^p)$以及式(5.1)可求得$\Delta z$，$B_{ij}(z_n)$，$T_p,G_p$，$e_{ij}^p$等，进而进行下一步增量加载计算。当进行最后一步计算后，利用所求结果可依次计算[D]，[D']以及[C]，最后由式(5.13)可得与σ_{cr}对应的R/h。

六、分析结果及其与实验数据的比较

利用本文发展的方法，对AMГ防锈铝和Д1T硬铝薄壁圆柱壳在轴向压缩时塑性屈曲临界载荷进行了分析并与[7]中的实验数据进行了比较。根据[7]中提供的两种材料在简单拉伸时的$\sigma\sim\varepsilon$曲线，确定材料常数如下：

AMГ防锈铝：$C_{1,2,3}=(80654,\ 8645,\ 1686)$ MPa

$\alpha_{1,2,3}=1987,\ 227.8,\ 18.1$

$f(z)=1$

Д1T 硬铝：$C_{1,2,3}=(1003454,\ 15974,\ 216)$ MPa

$\alpha_{1,2,3}=4603,\ 411.7,\ 17.6$

$f(z)=1$

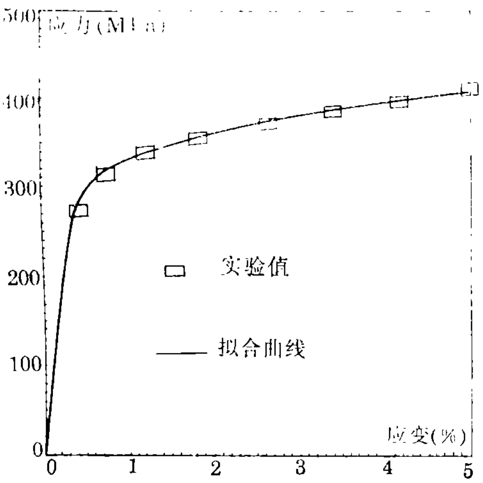
两种材料的杨氏模量均为$E=73$GPa，泊松比均为$\nu=0.28$。图1给出了两种材料的σ-ε^p拟合曲线，比较表明在所提供的实验数据的范围内，由上述材料常数确定的曲线与实验结果[7]符合得很好。

图2给出了两种材料圆柱壳的塑性屈曲临界应力σ_{cr}随圆柱壳特征尺寸R/h变化的内时全量理论的分析结果（EDT）及其实验结果（EXP）[7]和采用经典塑性全量理论的分析结果(CDT)[7]。圆柱壳的$L/R\approx3$。对于AMГ圆柱壳，由内时全量理论预言的σ_{cr}较经典塑性本构方程更接近于实验结果；而对于Д1T圆柱壳，内时全量理论则较经典全量理论给出了略

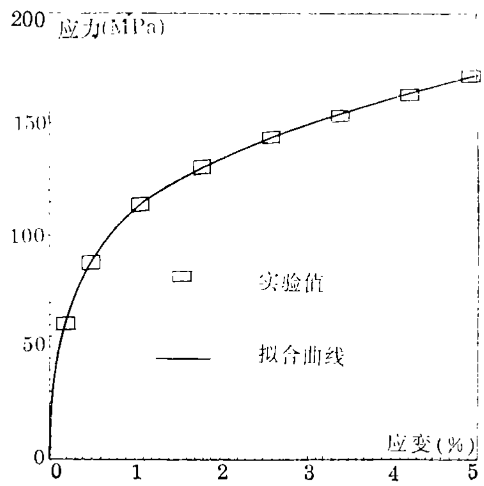
偏于保守的结果。由于内时全量本构方程能够很好地拟合简单加载条件下材料的应力应变曲线σ-ε^p，并可方便准确地确定任一应力水平下的E_t^0和E_s^0，因而使计算十分简捷。

图3给出了由增量形式的内时塑性本构方程所确定的两种材料圆柱壳受轴向压缩时σ_{cr}随R/h的变化(EIT)及其实验结果(EXP)以及采用经典的增量塑性理论的分析结果(CIT)。比较表明，对于AMΓ薄壁壳，尽管内时塑性本构方程所预言的塑性屈曲临界应力较经典增量理论更接近于实验结果，但仍给出了系统偏高的临界应力值。而对于Д1T圆柱壳，当R/h较小时，增量形式的内时塑性本构方程预言的σ_{cr}与实验结果相符。而当R/h很大时，增量形式的内时本构方程与经典增量理论一样，都预言了偏高的σ_{cr}值。

比较增量理论和全量理论的分析结果，可以看出全量形式的本构方程预言了与实验较为吻合的σ_{cr}，而增量形式的本构方程则预言了偏于危险的σ_{cr}值。其原因可能是由于未能合理地考虑由于局部塑性屈曲的出现引起的非比例加载所造成的材料承载能力的降低，与此有关的工作拟另文予以介绍。

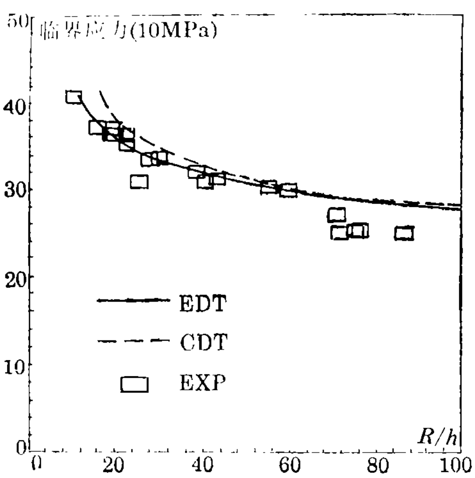


(a) Д1T硬铝

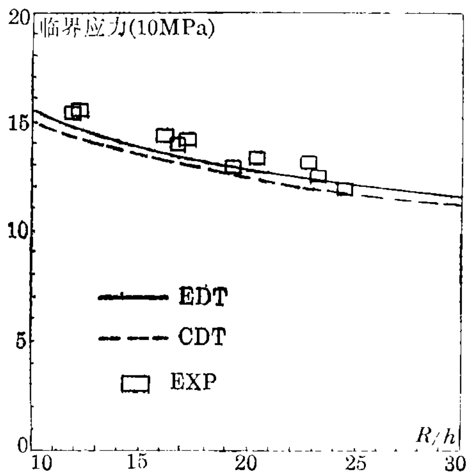


(b) AMΓ防锈铝

图1 材料的σ-ε^p拟合曲线及实验数据

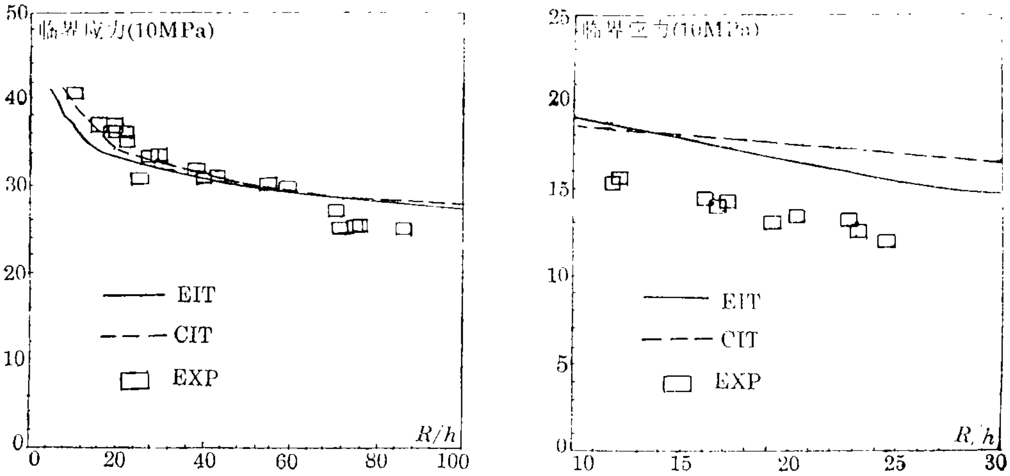


(a) Д1T圆柱壳



(b) AMΓ圆柱壳

图2 全量理论的计算结果及实验数据σ_{cr}



(a) Д1Т圆柱壳　　　　(b) AMГ圆柱壳

图3 增量理论的计算结果及实验数据σ_{cr}

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Endochronic Analysis for Compressive Buckling of Thin-Walled Cylinders in Yield Region

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Abstract

The longitudinal compressive buckling of long and thin-walled cylinders in yield region is analyzed with the incremental and finite forms of the endochronic constitutive equation, respectively. The relations between the critical stress σ_{cr} versus the ratio of R (the radius) versus h (the thickness of the wall) are derived. The critical stress of the thin-walled cylinders made of aluminum alloys AMΓ and Д1T are analyzed and compared with the experimental data and the analytical results based on the traditional theory of plasticity. It is seen that, except that the σ_{cr} of the cylinders made of Д1T predicted by the finite form of the endochronic theory seems a little more conservative than that by traditional deformation theory of plasticity, in most cases, both forms of the endochronic constitutive equation provide more satisfactory results.

Key words cylindrical shell, compressive buckling, endochronic analysis