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厚环壳在内压作用下的应力分析

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摘 要

本文利用由三维弹性力学方程，通过几何小参数$\alpha=r_0/R_0$摄动得到的厚环壳渐近方程，求得了厚环壳在内压q作用下的应力和变形解.

关键词 厚环壳 内压 渐近解 应力分析

一、引 言

在文献[1]中，利用小参数$\alpha=r_0/R_0$(图1)，从弹性力学基本方程，通过摄动展开，得到了厚环壳的各级渐近求解方程．同时指出这些方程可以逐级进行求解，而且在每一级可以分成两组独立的方程组（分别代表平面问题和扭转问题），上一级的解只影响下一级的体积力项和应变修正项．因此对各级渐近方程，可以逐级求解下去．

本文则在此基础上，利用这些方程，求得了厚环壳承受内压q作用时的应力和变形，导出了前几级渐近解的解析表达式．数值结果表明：厚环壳的应力分布与圆环和曲杆的应力分布十分相似，而且这种渐近解收敛很快．一般只要取前二级近似解就能满足要求．

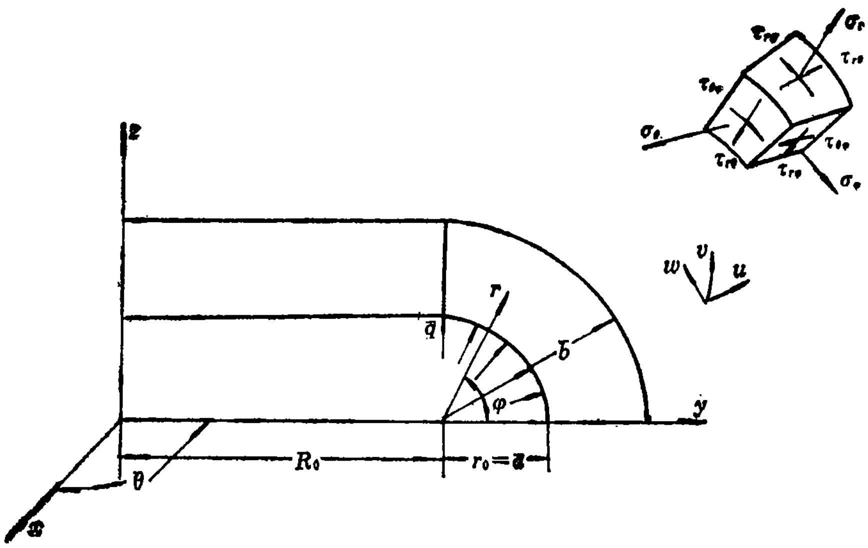

图 1

二、厚环壳的渐近求解方程

文献[1]中，利用小参数$\alpha=r_0/R_0$（环壳子午方向圆弧内径与圆 环 半径之比），从三维弹性力学基本方程出发，设边界应力已知值从α^p 量级开始，将应力、应变和位移均表达成α的幂级数（从α^{p-2}级开始），通过摄动展开得到厚环壳的渐近求解方程。

对α^{p-2}级的平衡方程为：

$$\left.\begin{aligned}&\frac{\partial\sigma_r^{-2}}{\partial r}+\frac{1}{r}\ \frac{\partial\tau_{r\varphi}^{-2}}{\partial\varphi}+\frac{1}{r}(\sigma_r^{-2}-\sigma_\varphi^{-2})=0\\&\frac{1}{r}\ \frac{\partial\sigma_\varphi^{-2}}{\partial\varphi}+\frac{\partial\tau_{r\varphi}^{-2}}{\partial r}+\frac{2}{r}\ \tau_{r\varphi}^{-2}=0\\&\frac{\partial\tau_{r\theta}^{-2}}{\partial r}+\frac{1}{r}\ \frac{\partial\tau_{\theta\varphi}^{-2}}{\partial\varphi}+\frac{1}{r}\ \tau_{r\theta}^{-2}=0\end{aligned}\right\}\tag{2.1}$$

几何关系

$$\left.\begin{aligned}&\varepsilon_r^{-2}=\frac{\partial u^{-2}}{\partial r}, && \varepsilon_{r\varphi}^{-2}=\frac{\partial w^{-2}}{\partial r}+\frac{1}{r}\ \frac{\partial u^{-2}}{\partial\varphi}-\frac{w^{-2}}{r}\\&\varepsilon_\theta^{-2}=0, && \varepsilon_{r\theta}^{-2}=\partial v^{-2}/\partial r\\&\varepsilon_\varphi^{-2}=\frac{1}{r}\ \frac{\partial w^{-2}}{\partial\varphi}+\frac{u^{-2}}{r}, && \varepsilon_{\theta\varphi}^{-2}=\frac{1}{r}\ \frac{\partial v^{-2}}{\partial\varphi}\end{aligned}\right\}\tag{2.2}$$

弹性关系

$$\left.\begin{aligned}&\varepsilon_r^{-2}=\sigma_r^{-2}-\mu(\sigma_\theta^{-2}+\sigma_\varphi^{-2}), && \varepsilon_{r\varphi}^{-2}=2(1+\mu)\tau_{r\varphi}^{-2}\\&\varepsilon_\varphi^{-2}=\sigma_\varphi^{-2}-\mu(\sigma_r^{-2}+\sigma_\theta^{-2}), && \varepsilon_{\theta\varphi}^{-2}=2(1+\mu)\tau_{\theta\varphi}^{-2}\\&\varepsilon_\theta^{-2}=\sigma_\theta^{-2}-\mu(\sigma_r^{-2}+\sigma_\varphi^{-2}), && \varepsilon_{r\theta}^{-2}=2(1+\mu)\tau_{r\theta}^{-2}\end{aligned}\right\}\tag{2.3}$$

式中位移、应力均已无量纲化[1]，各符号意义见图1.

对α^{p+n}级渐近方程可分成两大组，即

A组：

$$\left.\begin{aligned}&\frac{\partial\sigma_r^n}{\partial r}+\frac{1}{r}\ \frac{\partial\tau_{r\varphi}^n}{\partial\varphi}+\frac{1}{r}(\sigma_r^n-\sigma_\varphi^n)+R^n(r,\theta,\varphi)=0\\&\frac{1}{r}\ \frac{\partial\sigma_\varphi^n}{\partial\varphi}+\frac{\partial\tau_{r\varphi}^n}{\partial r}+\frac{2}{r}\tau_{r\varphi}^n+\psi^n(r,\theta,\varphi)=0\\&\varepsilon_r^n=\frac{\partial u^n}{\partial r},\ \ \varepsilon_\varphi^n=\frac{1}{r}\ \frac{\partial w^n}{\partial\varphi}+\frac{u^n}{r}\\&\varepsilon_\theta^n=e_\theta^n,\quad \varepsilon_{r\varphi}^n=\frac{\partial w^n}{\partial r}+\frac{1}{r}\ \frac{\partial u^n}{\partial\varphi}-\frac{w^n}{r}\\&\varepsilon_r^n=\sigma_r^n-\mu(\sigma_\theta^n+\sigma_\varphi^n),\ \ \varepsilon_\varphi^n=\sigma_\varphi^n-\mu(\sigma_r^n+\sigma_\theta^n)\\&\varepsilon_\theta^n=\sigma_\theta^n-\mu(\sigma_r^n+\sigma_\varphi^n),\ \ \varepsilon_{r\varphi}^n=2(1+\mu)\tau_{r\varphi}^n\end{aligned}\right\}\tag{2.4}$$

B组：

$$\left.\begin{aligned}&\frac{\partial\tau^n_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial\tau^n_{\theta\varphi}}{\partial\varphi}+\frac{1}{r}\tau^n_{r\theta}+Q^n(r,\theta,\varphi)=0\\&\varepsilon^n_{r\theta}=\frac{\partial v^n}{\partial r}+e^n_{r\theta},\quad \varepsilon^n_{\theta\varphi}=\frac{1}{r}\frac{\partial v^n}{\partial\varphi}+e^n_{\theta\varphi}\\&\varepsilon^n_{r\theta}=2(1+\mu)\tau^n_{r\theta},\quad \varepsilon^n_{\theta\varphi}=2(1+\mu)\tau^n_{\theta\varphi}\end{aligned}\right\}\tag{2.5}$$

其中

$$\left.\begin{aligned}R^n(r,\theta,\varphi)=&\left[\frac{\partial\tau^{n-1}_{r\theta}}{\partial\theta}+(\sigma^{n-1}_r-\sigma^{n-1}_\theta)\cos\varphi-\tau^{n-1}_{r\varphi}\sin\varphi\right]\\&+\left[\frac{\partial\tau^{n-2}_{r\theta}}{\partial\theta}+(\sigma^{n-2}_r-\sigma^{n-2}_\theta)\cos\varphi-\tau^{n-2}_{r\varphi}\sin\varphi\right](-r\cos\varphi)\\&+\cdots\\&+\left[\frac{\partial\tau^{-2}_{r\theta}}{\partial\theta}+(\sigma^{-2}_r-\sigma^{-2}_\theta)\cos\varphi-\tau^{-2}_{r\varphi}\sin\varphi\right](-r\cos\varphi)^{n+1}\\\psi^n(r,\theta,\varphi)=&\left[\frac{\partial\tau^{n-1}_{\theta\varphi}}{\partial\theta}+\tau^{n-1}_{r\varphi}\cos\varphi+(\sigma^{n-1}_\theta-\sigma^{n-1}_\varphi)\sin\varphi\right]\\&+\left[\frac{\partial\tau^{n-2}_{\theta\varphi}}{\partial\theta}+\tau^{n-2}_{r\varphi}\cos\varphi+(\sigma^{n-2}_\theta-\sigma^{n-2}_\varphi)\sin\varphi\right](-r\cos\varphi)\\&+\cdots\\&+\left[\frac{\partial\tau^{-2}_{\theta\varphi}}{\partial\theta}+\tau^{-2}_{r\varphi}\cos\varphi+(\sigma^{n-2}_\theta-\sigma^{n-2}_\varphi)\sin\varphi\right](-r\cos\varphi)^{n+1}\\Q^n(r,\theta,\varphi)=&\left[\frac{\partial\sigma^{n-1}_\theta}{\partial\theta}+2\tau^{n-1}_{r\theta}\cos\varphi-2\tau^{n-1}_{\theta\varphi}\sin\varphi\right]\\&+\left[\frac{\partial\sigma^{n-2}_\theta}{\partial\theta}+2\tau^{n-2}_{r\theta}\cos\varphi-2\tau^{n-2}_{\theta\varphi}\sin\varphi\right](-r\cos\varphi)\\&+\cdots\\&+\left[\frac{\partial\sigma^{-2}_\theta}{\partial\theta}+2\tau^{-2}_{r\theta}\cos\varphi-2\tau^{-2}_{\theta\varphi}\sin\varphi\right](-r\cos\varphi)^{n+1}\\e^n_\theta=&\left[\frac{\partial v^{n-1}}{\partial\theta}+u^{n-1}\cos\varphi-w^{n-1}\sin\varphi\right]+\left[\frac{\partial v^{n-2}}{\partial\theta}+u^{n-2}\cos\varphi-w^{n-2}\sin\varphi\right]\\&\cdot(-r\cos\varphi)\\&+\cdots+\left[\frac{\partial v^{-2}}{\partial\theta}+u^{-2}\cos\varphi-w^{-2}\sin\varphi\right](-r\cos\varphi)^{n+1}\\e^n_{r\theta}=&\left[\frac{\partial u^{n-1}}{\partial\theta}-v^{n-1}\cos\varphi\right]+\left[\frac{\partial u^{n-2}}{\partial\theta}-v^{n-2}\cos\varphi\right](-r\cos\varphi)\\&+\cdots+\left[\frac{\partial u^{-2}}{\partial\theta}-v^{-2}\cos\varphi\right](-r\cos\varphi)^{n+1}\\e^n_{\theta\varphi}=&\left[\frac{\partial w^{n-1}}{\partial\theta}+v^{n-1}\sin\varphi\right]+\left[\frac{\partial w^{n-2}}{\partial\theta}+v^{n-2}\sin\varphi\right](-r\cos\varphi)\\&+\cdots+\left[\frac{\partial w^{-2}}{\partial\theta}+v^{-2}\sin\varphi\right](-r\cos\varphi)^{n+1}\\&(n=-1,\ 0,\ 1,\ 2,\ 3,\ \cdots)\end{aligned}\right\}\tag{2.6}$$

A组方程具有平面应变问题方程的性质，B组方程具有扭转问题方程的性质．R^n，ψ^n，

Q^n相当于体积力，由前几级近似解得到．这两组方程可以独立求解，在同一级互不相关，但在下一级通过体积力项和应变修正项发生耦合作用．对给定应力部分边界条件的量值从α^p级开始，它们由已给值确定．此外还必须满足位移边界条件．

上述方程对弯环各种受力情况都适用，对环的厚度、曲率、边界形状并无限制，它是求解$\alpha=r_0/R_0\ll 1$这类厚壳问题的一般方程．

三、弯环壳受内压q作用的解

对图1所示弯环壳，在内边界($r_0=\bar{a}$)作用内压$q=\bar{q}/E$(无量纲形式，E为弹性模量)，其余边界上应力为零．边界条件为

$$\left.\begin{array}{lll} \varphi=0: & \tau_{r\varphi}=0,\ w=0 & \\ \varphi=\pi/2: & \sigma_\varphi=0,\ \tau_{r\varphi}=0 & \\ r=b=\bar{b}/r_0: & \sigma_r=0,\ \tau_{r\varphi}=0 & \\ r=a=\bar{a}/r_0: & \tau_{r\varphi}=0,\quad \sigma_r=\begin{cases}-q, & (\text{对}\alpha^p\text{级}) \\ -q\cos\varphi, & (\text{对}\alpha^{p+1}\text{级}) \\ 0, & (\text{其余各级})\end{cases} \end{array}\right\} \tag{3.1}$$

采用无量纲形式求解．

1. α^{p-2}级的解

由于α^{p-2}级边界各点应力均为零，根据方程(2.1)～(2.3)和条件(3.1)，解得各应力、应变分量均为零．

仅存在位移

$$\left.\begin{array}{l} u^{-2}=-A^{-2}\cos\varphi \\ w^{-2}=A^{-2}\sin\varphi \end{array}\right\} \tag{3.2}$$

A^{-2}为位移待定常数．

2. α^{p-1}级的解

边界各点的应力同样也为零．方程(2.4)～(2.6)在条件(3.1)，(3.2)之下的解为

$$\left.\begin{array}{l} \sigma_r^{-1}=\sigma_\varphi^{-1}=\tau_{r\varphi}^{-1}=0 \\ \sigma_\theta^{-1}=-A^{-2},\ \varepsilon_\theta^{-1}=-A^{-2} \\ u^{-1}=-A^{-1}\cos\varphi+\mu A^{-2}r \\ w^{-1}=A^{-1}\sin\varphi \end{array}\right\} \tag{3.3}$$

A^{-1}为α^{p-1}级中出现的待定常数．

3. α^p级的解

由方程(2.4)～(2.6)在条件(3.1)、(3.2)、(3.3)之下得到下列方程：

$$\left.\begin{aligned}
&\frac{\partial\sigma_r^0}{\partial r}+\frac{1}{r}\frac{\partial\tau_{r\varphi}^0}{\partial\varphi}+\frac{1}{r}(\sigma_r^0-\sigma_\varphi^0)+A^{-2}\cos\varphi=0\\
&\frac{1}{r}\frac{\partial\sigma_\varphi^0}{\partial\varphi}+\frac{\partial\tau_{r\varphi}^0}{\partial r}+\frac{2}{r}\tau_{r\varphi}^0-A^{-2}\sin\varphi=0\\
&\varepsilon_r^0=\partial u^0/\partial r,\qquad \varepsilon_\theta^0=-A^{-1}+(1+\mu)A^{-2}r\cos\varphi\\
&\varepsilon_\varphi^0=\frac{1}{r}\frac{\partial w^0}{\partial\varphi}+\frac{u^0}{r},\quad \varepsilon_{r\varphi}^0=\frac{\partial w^0}{\partial r}+\frac{1}{r}\frac{\partial u^0}{\partial\varphi}-\frac{w^0}{r}\\
&\varepsilon_r^0=\sigma_r^0-\mu(\sigma_\theta^0+\sigma_\varphi^0),\ \varepsilon_\varphi^0=\sigma_\varphi^0-\mu(\sigma_r^0+\sigma_\theta^0)\\
&\varepsilon_\theta^0=\sigma_\theta^0-\mu(\sigma_r^0+\sigma_\varphi^0),\ \varepsilon_{r\varphi}^0=2(1+\mu)\tau_{r\varphi}^0
\end{aligned}\right\}\tag{3.4}$$

另外$\tau_{r\theta}^0$，$\tau_{\theta\varphi}^0$，$\varepsilon_{r\theta}^0$，$\varepsilon_{\theta\varphi}^0=0$。边界条件(3.1)变为

$$\left.\begin{aligned}
&r=a:\ \sigma_r^0=-q,\ \tau_{r\varphi}^0=0\\
&r=b:\ \sigma_r^0=0,\ \tau_{r\varphi}^0=0\\
&\varphi=0:\ \tau_{r\varphi}^0=0,\ w^0=0\\
&\varphi=\frac{\pi}{2}:\ \int_a^b\sigma_\varphi^0dr=0,\ \int_a^b\sigma_\varphi^0rdr=0,\ \int_a^b\tau_{r\varphi}^0dr=0
\end{aligned}\right\}\tag{3.5}$$

引进满足平衡方程((3.4)式的前2式)的应力函数Φ^0，则

$$\left.\begin{aligned}
&\sigma_r^0=\frac{1}{r}\frac{\partial\Phi^0}{\partial r}+\frac{1}{r^2}\frac{\partial^2\Phi^0}{\partial\varphi^2}-A^{-2}r\cos\varphi\\
&\sigma_\varphi^0=\frac{\partial^2\Phi^0}{\partial r^2}-A^{-2}r\cos\varphi\\
&\tau_{r\varphi}^0=\frac{1}{r^2}\frac{\partial\Phi^0}{\partial\varphi}-\frac{1}{r}\frac{\partial^2\Phi^0}{\partial r\partial\varphi}
\end{aligned}\right\}\tag{3.6}$$

由(3.4)式中的3～6式，Φ^0还应满足协调方程

$$\nabla^2\nabla^2\Phi^0=0\tag{3.7}$$

在此 $$\nabla^2=\frac{\partial}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}$$

取(3.7)式的解为[2]

$$\begin{aligned}
\Phi^0=&A_0\ln r+B_0r^2\ln r+C_0r^2\\
&+A_1r\varphi\sin\varphi+[B_1r^3+C_1/r+D_1r\ln r]\cos\varphi\\
&+E_1\left\{\left[\frac{a^2b^2}{r}\ln r+r^3\ln r\right]\cos\varphi+\left[\frac{a^2b^2}{r}+2(a^2+b^2)r\ln r-r^3\right]\varphi\sin\varphi\right\}
\end{aligned}\tag{3.8}$$

Φ^0中的最后一项是新近得到的一个新的应力函数。将Φ^0代入(3.6)式及(3.4)式中的应力应变关系式，得

$$\left.\begin{aligned}
\sigma_r^0 &= A_0\frac{1}{r^2}+B_0[2\ln r+1]+2C_0+\left[A_1\frac{2}{r}+2B_1r+D_1\frac{1}{r}-C_1\frac{2}{r^3}\right]\cos\varphi \\
&\quad -A^{-2}r\cos\varphi+E_1\left\{\left[-\frac{2a^2b^2}{r^3}\ln r+\frac{3a^2b^2}{r^3}+2r\ln r-r\right.\right. \\
&\quad \left.+\frac{4(a^2+b^2)}{r}\ln r\right]\cos\varphi \\
&\quad \left.+\left[-\frac{2a^2b^2}{r^3}-\frac{2(a^2+b^2)}{r}-2r\right]\varphi\sin\varphi\right\} \\
\sigma_\varphi^0 &= -A_0\frac{1}{r^2}+B_0[2\ln r+3]+2C_0+\left[6B_1r+C_1\frac{2}{r^3}+D_1\frac{1}{r}\right]\cos\varphi-A^{-2}r\cos\varphi \\
&\quad +E_1\left\{\left[\frac{2a^2b^2}{r^3}\ln r-\frac{3a^2b^2}{r^3}+6r\ln r+5r\right]\cos\varphi+\left[\frac{2a^2b^2}{r^3}\right.\right. \\
&\quad \left.\left.+\frac{2(a^2+b^2)}{r}-6r\right]\varphi\sin\varphi\right\} \\
\tau_{r\varphi}^0 &= \left[2B_1r-C_1\frac{2}{r^3}+D_1\frac{1}{r}\right]\sin\varphi \\
&\quad +E_1\left\{\left[-\frac{2a^2b^2}{r^3}\ln r+\frac{3a^2b^2}{r^3}+2r\ln r+3r-\frac{2(a^2+b^2)}{r}\right]\sin\varphi\right. \\
&\quad \left.+\left[\frac{2a^2b^2}{r^3}-\frac{2(a^2+b^2)}{r}+2r\right]\varphi\cos\varphi\right\} \\
\sigma_\theta^0 &= \mu(\sigma_\varphi^0+\sigma_r^0)+(1+\mu)A^{-2}r\cos\varphi-A^{-1}
\end{aligned}\right\}\tag{3.9}$$

其中待定系数$A_0, B_0, \cdots, E_1, A^{-2}$等可以由$\alpha^p$级的边界条件来确定。将(3.9)式代入(3.5)式，解得各系数为：

$$\left.\begin{aligned}
A^{-2} &= -\frac{4qa}{\pi(b^2-a^2)}, \qquad A_0=-\frac{a^2b^2}{\varDelta}\left[b^2-a^2-2a^2\ln\frac{a}{b}\right]q \\
B_0 &= -\frac{a^2}{\varDelta}\left[b^2-a^2-2b^2\ln\frac{a}{b}\right]q \\
C_0 &= \frac{a^2}{\varDelta}\left[b^2-a^2+b^2\ln b-a^2\ln a-(1+2\ln b)b^2\ln\frac{a}{b}\right]q \\
A_1 &= -\frac{a^2+b^2}{\pi H(b^2-a^2)}[2(b^2\ln a-a^2\ln b)+b^2-a^2]qa \\
B_1 &= -\frac{1}{\pi H^2}\left[(a^2\ln a+b^2\ln b+a^2+b^2)\ln\frac{a}{b}\right. \\
&\quad \left.+(b^2-a^2)\left(1+\frac{1}{2}\ln a\cdot b\right)\right]qa \\
C_1 &= \frac{a^2b^2}{\pi H^2}\left[(-b^2\ln a-a^2\ln b+a^2+b^2)\ln\frac{a}{b}\right. \\
&\quad \left.+(b^2-a^2)\left(1-\frac{1}{2}\ln a\cdot b\right)\right]qa
\end{aligned}\right\}\tag{3.10}$$

$$D_1=\frac{1}{\pi H^2}\left[b^4-a^4+2(a^4+b^4)\ln\frac{a}{b}\right]qa$$

$$E_1=\frac{1}{\pi H}qa$$

$$H=(a^2+b^2)\ln\frac{a}{b}+b^2-a^2,\quad \Delta=(b^2-a^2)^2-4a^2b^2\left(\ln\frac{a}{b}\right)^2$$

4． α^{p+1}，α^{p+2}级解

对α^{p+1}级，在$r=a$处边界应力为$\sigma_r^1=-q\cos\varphi$，$\tau_{r\varphi}^1=0$。在$r=b$处为$\sigma_r^1=\tau_{r\varphi}^1=0$。基本方程(3.4)在满足边界条件(3.1)之下的求解，可以同样地进行，但十分冗繁。在此仅将A^{-1}的值列出如下：

$$\begin{aligned}A^{-1}=&\frac{4}{\pi(b^2-a^2)}\Big\{B_0\pi\left(\mu-\frac{1}{2}\right)\left[b^2\ln b-a^2\ln a+\frac{1}{2}(b^2-a^2)\right]\\&+C_0\pi\left(\mu-\frac{1}{2}\right)(b^2-a^2)+A_1(b-a)\left(2\mu-\frac{4}{3}\right)+B_1\frac{2}{3}(b^3-a^3)(4\mu-1)\\&+C_1\frac{2}{3ab}(a-b)+D_1(b-a)\left(2\mu-\frac{1}{3}\right)+A^{-2}\left(1-\frac{\mu}{3}\right)(b^3-a^3)\\&+E_1\Big[-\frac{2}{3}(b^2-a^2)(b\ln b-a\ln a)+\frac{8}{3}\mu(b^3\ln b-a^3\ln a)\\&+4(a^2+b^2)(b\ln b-a\ln a)\left(\mu-\frac{2}{3}\right)+\frac{5}{3}ab(b-a)\\&+(b^3-a^3)\left(1-\frac{20}{9}\mu\right)\Big]\Big\}-\frac{qa}{b^2-a^2}\end{aligned}\tag{3.11}$$

（A^{-1}实际上可以根据α^{p+1}级的体积力，在$r\varphi$平面内，满足x方向的整体平衡条件求得）。

通过对α^{p+2}级的求解，可以求得A^0，略去小量后有：

$$\begin{aligned}A^0=&\frac{4(1+\mu)}{\pi(b^2-a^2)}\Big\{-A_0(b-a)+B_0\Big[\frac{2}{3}(1-2\mu)(b^3\ln b-a^3\ln a)\\&-\left(\frac{17}{9}-\frac{16}{9}\mu\right)(b^3-a^3)\Big]\Big\}+A_1(1+\mu)\Big\{\frac{2(1-\mu)}{b^2-a^2}(b^2\ln b-a^2\ln a)\\&-\frac{1}{2}(1-2\mu)\Big\}-B_1(1+\mu)(a^2+b^2)\\&+D_1(1+\mu)\Big\{\frac{1-2\mu}{b^2-a^2}(b^2\ln b-a^2\ln a)+\mu\Big\}\\&-\frac{1}{4}A^{-2}(a^2+b^2)+\frac{4(1+\mu)(b^3-a^3)}{3\pi(b^2-a^2)}A^{-1}\\&+E_1\frac{1+\mu}{b^2-a^2}\Big\{-a^2b^2\ln\frac{a}{b}-b^4\ln b+a^4\ln a-b^4+a^4\\&+2(1-\mu)(a^2+b^2)[b^2(\ln b)^2-a^2(\ln a)^2]\end{aligned}$$

$$+(1-2\mu)(a+b)(b^2\ln b-a^2\ln a)-\frac{\pi^2}{6}(1-\mu)(b^4-a^4)\Big\}$$

$$+\frac{8(1+\mu)}{3\pi(b^2-a^2)}(1-2\mu)(b^3-a^3)C_0 \qquad (3.12)$$

于是由(3.10)、(3.11)、(3.12)式和(3.2)、(3.3)、(3.4)式就得到α^{p-2}，α^{p-1}，α^p级的位移解．由(3.3)、(3.9)、(3.10)式可得到α^p，α^{p-1}级的应力解．用此法也可以得到更高级的近似解，但一般α^p级的精度就已经足够了．

从求解结果可以看出：受内压q作用厚环壳的应力解，相当于环向为均布应力σ_θ^{-1}的解，和受q力及分布体积力作用下平面应变曲杆解的叠加．其物理模型是十分清楚的．

四、数 值 结 果

当$r_0=a=1$，$b=2$，$\mu=0.3$，$r_0/R_0=1/15$时，表2、图2给出了作用内压q时，弯环壳

表1 **在q作用下，$\varphi=0$，$\varphi=\pi/2$处的位移** 单位：q

各级近似	$\varphi=0$		$\varphi=\pi/2$		误差
	u	w	u	w	
α^{p-2}级	95.49	0	0	− 95.49	17%
$\alpha^{p-2}+\alpha^{p-1}$级	109.71	0	−2.87	−112.58	3%
$\alpha^{p-2}+\alpha^{p-1}+\alpha^p$级	107.25	0	5.65	−115.66	—

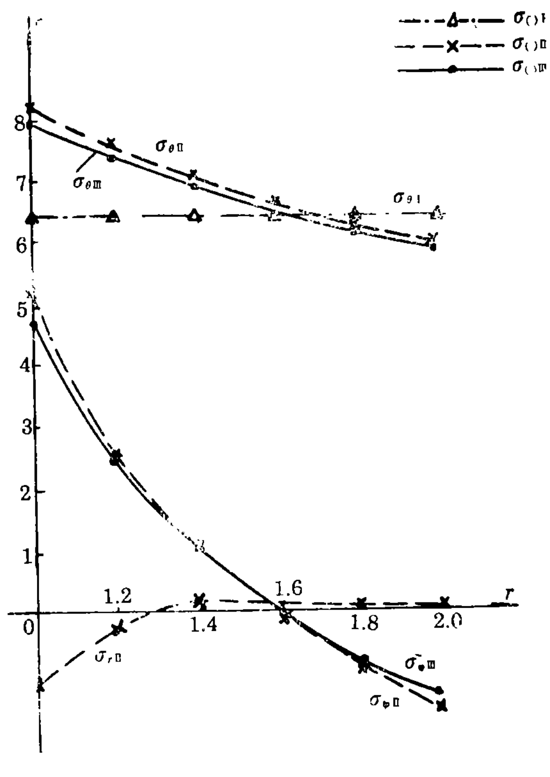

图2 弯环$\varphi=0$截面上的应力分布(单位q)

$\varphi=0$截面上σ_r，σ_θ，σ_φ的各级近似解。可以看出它们具有曲杆应 力分布形式，且收敛很快。表 1 给出了$\varphi=0$，$\varphi=\pi/2$处的位移。

表2　　作用内压q时，弯环$\varphi=0$截面上的应力分布(无量纲)

r	σ_θ各级之解			$r_0/R_0=1/15$，$\sigma_r,\sigma_\theta,\sigma_\varphi$各级近似解* 单位：$q$					
	$\alpha^{p-1}\times\left(\frac{R_0}{r_0}\right)q$	$\alpha^p\times q$	$\alpha^{p+1}\times\left(\frac{r_0}{R_0}\right)q$	$\sigma_{\theta\,\mathrm{I}}$	$\sigma_{\theta\,\mathrm{II}}$	$\sigma_{\theta\,\mathrm{III}}$	$\sigma_{\varphi\,\mathrm{II}}$	$\sigma_{\varphi\,\mathrm{III}}$	$\sigma_{r\,\mathrm{II}}$
1	0.4244	1.842	−4.347	6.366	8.208	7.918	5.182	4.748	−1.0
1.2	0.4244	1.236	−3.738	6.366	7.602	7.353	2.735	2.529	−0.207
1.4	0.4244	0.749	−3.136	6.366	7.115	6.906	1.232	1.188	0.040
1.6	0.4244	0.326	−2.541	6.366	6.692	6.523	0.144	0.222	0.088
1.8	0.4244	−0.061	−1.941	6.366	6.305	6.176	−0.743	−0.571	0.053
2.0	0.4244	−0.471	−1.320	6.366	5.949	5.861	−1.509	−1.261	0
误差				19.5%	3.6%	—	9%	—	

*注：$\sigma(\ )_{\mathrm{I}}=\alpha^{p-1}$级；$\sigma(\ )_{\mathrm{II}}=\alpha^{p-1}+\alpha^p$级；$\sigma(\ )_{\mathrm{III}}=\alpha^{p-1}+\alpha^p+\alpha^{p+1}$级。

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Stress Analysis of Thick Ring Shell Submitted to the Action of Internal Pressure

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Abstract

In this paper, from asymptotic equations of thick ring shell obtained on the basis of the equations of three dimensional elastic mechanics using geometric small parameter $\alpha=r_0/R_0$, we find the solutions of the stresses and the deformations of thick ring shell submitted to the action of internal pressure q.

Key words thick ring shell, internal pressure, asymptotic solution, stress analysis