应用数学和力学编委会编
重庆出版社出版

# 广义Sawyer-Eliassen方程解的存在性*

余庆余 许 秦

（兰州大学）（CIMMS研究所，美国）

## 摘 要

本文讨论了广义Sawyer-Eliassen方程组的解的存在性问题.

**关键词** 解 广义Sawyer-Eliassen方程 地转风

在大气锋面环流的研究中，Sawyer-Eliassen方程起着重要的作用，可参见[1]～[3]. 在[4]中，作者之一把它推广到广义Sawyer-Eliassen方程，以包含涡动粘滞性以及负的湿位涡度的影响，在本文中我们考虑这个方程组解的存在性.

设$\Omega\subset R^3$为长方体，即

$\Omega=\{(x,y,z)\in R^3,\ -L<x<L,\ -K<y<K,\ 0<z<H\}$，方程组为

$$\left.\begin{array}{l}((u\partial_x+v\partial_y+w\partial_z)-(\gamma_1\partial_x^2+\gamma_2\partial_y^2+\gamma_3\partial_z^2)-x\partial_z)u-fv=0\\ ((u\partial_x+v\partial_y+w\partial_z)-(\gamma_1\partial_x^2+\gamma_2\partial_y^2+\gamma_3\partial_z^2)-x\partial_z)fv+T^2u+S^2w=A\\ ((u\partial_x+v\partial_y+w\partial_z)-(\gamma_1\partial_x^2+\gamma_2\partial_y^2+\gamma_3\partial_z^2)-x\partial_z)w-(g/\Theta)\theta=0\\ ((u\partial_x+v\partial_y+w\partial_z)-(\gamma_1\partial_x^2+\gamma_2\partial_y^2+\gamma_3\partial_z^2)-x\partial_z)(g/\Theta)\theta+S^2u+N^2w=B\\ \partial_xu+\partial_yv+\partial_zw=0\end{array}\right\}\qquad(1)$$

边界条件为$u|_{\partial\Omega}=v|_{\partial\Omega}=w|_{\partial\Omega}=\theta|_{\partial\Omega}=0$. 方程中$u$，$v$，$w$是非地转风的分量，$\theta$是位温，$f$是柯氏参数，$fv$是柯氏力在$x$方向上的分量，$g$为重力加速度，$(g/\Theta)\theta$一项表示浮力扰动，$S^2u+N^2w$是熵的平流影响项，$A$，$B$是地转强迫项，$F^2=f(f+\eta)$，$S^2=f\partial_zv_g$，$\eta=\partial_xv_g$，其中$v_g$是基本地转风，$\theta_g$是满足热成风关系的位温，$\Theta$是参考温度，$\gamma_i\geqslant0$，$i=1,2,3$，所以$f$，$g$，$\Theta$，$A$，$B$，$F$，$S$，$N$，$\eta$均是给定的常数或连续函数.

如记$fv$为新的$v$，$(g/\Theta)\theta$为新$\theta$，并记$D_N=u\partial_x+v\partial_y+w\partial_z$，$D=\gamma_1\partial_x^2+\gamma_2\partial_y^2+\gamma_3\partial_z^2$，$D_z=x\partial_z$，则$D_N$为地转风平流项，$D$代表耗散项，$D_z$为给定地转风$u_g=-x$的平流. 此时(1)可简记为

$$\left.\begin{array}{l}(D_N-D-D_z)u-v=0\\ (D_N-D-D_z)v+T^2u+S^2w=A\\ (D_N-D-D_z)w-\theta=0\\ (D_N-D-D_z)\theta+S^2u+N^2w=B\\ \partial_xu+\partial_yv+\partial_zw=0\end{array}\right\}\qquad(2)$$

如更记$\varphi=(u,\ v,\ w,\ \theta)$，$\varphi^\circ=(u,\ v,\ w)$，$\gamma=(\gamma_1,\ \gamma_2,\ \gamma_3)$，

* 钱伟长推荐. 1992年6月21日收到第一稿，1994年2月5日收到修改稿.

$\gamma\nabla=(\gamma_1\partial_x,\ \gamma_2\partial_y,\ \gamma_3\partial_z)$,

$$M=\begin{pmatrix}0 & -1 & 0 & 0\\ T^2 & 0 & S^2 & 0\\ 0 & 0 & 0 & -1\\ S^2 & 0 & N^2 & 0\end{pmatrix}\qquad P=\begin{pmatrix}0\\ A\\ 0\\ B\end{pmatrix}$$

则(2)还可改写成更简明的形式

$$\left.\begin{aligned}&(\varphi^\circ\cdot\nabla)\varphi+(\gamma\nabla\cdot\nabla)\varphi-D_z\varphi+M\varphi=P\\ &\nabla\cdot\varphi^\circ=0\end{aligned}\right\}\tag{3}$$

引入如下的函数空间

$X=\{\varphi^\circ\in C_0^\infty(\Omega)\times C_0^\infty(\Omega)\times C_0^\infty(\Omega)\,|\,\nabla\cdot\varphi^\circ=0\}$, $X_T=X\times C_0^\infty(\Omega)$,

$V=X$在$H_0^1(\Omega)\times H_0^1(\Omega)\times H_0^1(\Omega)$中的闭包，$V_T=V\times H_0^1(\Omega)$. 在$V_T$中按下式定义元素$\varphi$的范数$\|\varphi\|$

$$\|\varphi\|^2=\int_\Omega|\nabla\varphi|^2dxdydz=\int_\Omega(|\nabla u|^2+|\nabla v|^2+|\nabla w|^2+|\nabla\theta|^2)dxdydz.$$

对任何$\varphi_1\in V_T$，和(3)中第一式作$L_2$内积$\langle,\ \rangle$，再分部积分可得

$$\langle(\varphi^\circ\cdot\nabla)\varphi,\ \varphi_1\rangle+\langle\gamma\nabla\varphi,\ \nabla\varphi_1\rangle-\langle D_z\varphi,\ \varphi_1\rangle+\langle M\varphi,\ \varphi_1\rangle=\langle P,\ \varphi_1\rangle\tag{4}$$

我们有

**定理** 当 $k\leqslant\frac{1}{2}$时，(3)在$V_T$中至少有一弱解，即存在$\varphi_0\in V_T$，使对一切$\varphi_1\in V_T$，(4)成立，其中$k=\max\max\limits_{\substack{-L\leqslant x\leqslant L\\ -K\leqslant y\leqslant K\\ z=0}}(|F^2-1|,\ S^2,\ |N^2-1|)$.

记 $b_1(\varphi,\ \varphi_1,\ \varphi_2)=\langle(\varphi^\circ\cdot\nabla)\varphi,\ \varphi_1\rangle$,

$b_2(\varphi,\ \varphi_1)=\langle\gamma\nabla\varphi,\ \nabla\varphi_1\rangle$,

$b_3(\varphi,\ \varphi_1)=\langle D_z\varphi,\ \varphi_1\rangle$,

$b_4(\varphi,\ \varphi_1)=\langle M\varphi,\ \varphi_1\rangle$.

为证定理，先需要如下引理

**引理1** 对任$\varphi$, $\varphi_1$, $\varphi_2\in V_T$，有

i) $|b_1(\varphi,\ \varphi_1,\ \varphi_2)|\leqslant c_1\|\varphi\|\|\varphi_1\|\|\varphi_2\|$ (对某$c_1>0$),

ii) $b_1(\varphi,\ \varphi_1,\ \varphi_2)=-b_1(\varphi,\ \varphi_2,\ \varphi_1)$,

iii) $b_1(\varphi,\ \varphi_1,\ \varphi_1)=0$

**证明** i) 由Hölder不等式可得

ii) 利用分部积分及边界条件可证.

iii) 由ii)立得.

**引理2** 对任$\varphi\in V_T$, $c_2\|\varphi\|^2\leqslant b_2(\varphi,\ \varphi)\leqslant c_3\|\varphi\|^2$，其中$c_2=\min(\gamma_1,\ \gamma_2,\ \gamma_3)$, $c_3=\max(\gamma_1,\ \gamma_2,\ \gamma_3)$.

**引理3** 对任$\varphi\in V_T$,

$$b_3(\varphi,\ \varphi)=-\frac{1}{2}\int_\Omega(u^2+v^2+w^2+\theta^2)dxdydz=-\frac{1}{2}\|\varphi\|^2_{L^2(\Omega)}.$$

**证明** 由分部积分、边界条件和Cauchy-Schwartz不等式得出.

**引理4** 对任$\varphi\in V_T$, $|b_4(\varphi,\ \varphi)|\leqslant k\|\varphi\|^2_{L^2(\Omega)}$.

**证明** 用Cauchy-Schwartz不等式可证.

**引理 5** 如$\varphi_n$在$V_T$中弱收敛于$\varphi_0$，那末对任何$\varphi\in V_T$，有

i) $b_1(\varphi_n, \varphi_n, \varphi)\to b_1(\varphi_0, \varphi_0, \varphi)$,

ii) $b_i(\varphi_n, \varphi)\to b_i(\varphi_0, \varphi)\qquad (i=2, 3, 4)$.

**证明** 只证i)，ii)类似可证.

$$b_1(\varphi_n, \varphi_n, \varphi)=-b_1(\varphi_n, \varphi, \varphi_n)$$

$$=-\int_\Omega[(u_n\partial_x u+v_n\partial_y u+w_n\partial_z u)u_n+(u_n\partial_x v+v_n\partial_y v+w_n\partial_z v)v_n$$

$$+(u_n\partial_x w+v_n\partial_z w+w_n\partial_z w)w_n+(u_n\partial_x\theta+v_n\partial_y\theta+w_n\partial_z\theta)\theta_n]dxdydz,$$

当$\varphi_n$在$V_T$中弱收敛于$\varphi_0$时，在$L^2(\Omega)$中分别有$u_n\to u_0$，$v_n\to v_0$，$w_n\to w_0$，$\theta_n\to\theta_0$(参见[5])，再应用不等式([6])

$$\int_\Omega w\frac{\partial v_2}{\partial p}v_1dxdydz\leqslant\|w\|_{L^2(\Omega)}\left\|\frac{\partial v_2}{\partial p}\right\|_{L^6(\Omega)}\|v_1\|_{L^3(\Omega)},$$

可得 $b_1(\varphi_n, \varphi_n, \varphi)\to -b_1(\varphi_0, \varphi, \varphi_0)=b_1(\varphi_0, \varphi_0, \varphi)$.

现在可以证明本文的定理.

**定理的证明** 设$\{e_i\}$是$V_T$的基，并记

$$E_n=\mathrm{span}\{e_1, e_2, \cdots, e_n\}\qquad (n=1, 2, 3, \cdots)$$

显然 $V_T=\bigcup_{n=1}^{\infty}E_n$,

记 $B_R=\{\varphi\in V_T\,|\,\|\varphi\|\leqslant R\}$，$S_R=\partial B_R$,

$B_R^n=\{\varphi\in E_n\,|\,\|\varphi\|\leqslant R\}$，$S_R^{(n)}=\partial B_R^{(n)}$.

由下式定义映射$G$: $V_T\to V_T$

$$\langle G\varphi, \varphi_1\rangle=b_1(\varphi, \varphi, \varphi_1)+b_2(\varphi, \varphi_1)-b_3(\varphi, \varphi_1)+b_4(\varphi, \varphi_1)-\langle P, \varphi\rangle$$

当$k\leqslant\frac{1}{2}$且$R$充分大时，对任意$\varphi\in S_R^{(n)}$

$$\langle P_nG\varphi, \varphi\rangle=\langle G\varphi, \varphi\rangle=b_1(\varphi, \varphi, \varphi)+b_2(\varphi, \varphi)-b_3(\varphi, \varphi)+b_4(\varphi, \varphi)-\langle P, \varphi\rangle$$

$$\geqslant C_2\|\varphi\|^2+\frac{1}{2}\|\varphi\|_{L^2}^2-k\|\varphi\|_{L^2}^2-\|P\|\|\varphi\|$$

$$=(C_2\|\varphi\|-\|P\|)\|\varphi\|+\left(\frac{1}{2}-k\right)\|\varphi\|_{L^2}^2>0,$$

其中$P_n$: $V_T\to E_n$是正交投影算子. 于是由Brouwer度的锐角原理([7])，存在$\varphi_n\in B_R^{(n)}$，使对一切$\xi_k\in E_n$

$$\langle G\varphi_n, \xi_k\rangle=\langle P_nG\varphi_n, \xi_k\rangle=0,\qquad (\text{对}k\leqslant n),$$

即 $b_1(\varphi_n, \varphi_n, \xi_k)+b_2(\varphi_n, \xi_k)-b_3(\varphi_n, \xi_k)+b_4(\varphi_n, \xi_k)=\langle P, \xi_k\rangle$.

因为$V_T$是Hilbert空间，$\{\varphi_n\}$有弱收敛子列，我们仍记为$\varphi_n$，并记其弱极限为$\varphi_0$，那末由引理5，对一切$\xi_k\in E_n$，$k\leqslant n$有

$$b_1(\varphi_0, \varphi_0, \xi_k)+b_2(\varphi_0, \xi_k)-b_3(\varphi_0, \xi_k)+b_4(\varphi_0, \xi_k)=\langle P, \xi_k\rangle.$$

又因 $V_T=\bigcup_{n=1}^{\infty}E_n$，故对一切$\varphi\in V_T$有

$$b_1(\varphi_0, \varphi_0, \varphi)+b_2(\varphi_0, \varphi)-b_3(\varphi_0, \varphi)+b_4(\varphi_0, \varphi)=\langle P, \varphi\rangle,$$

从而定理得证。

**注记**　当$L=K=H=\infty$时，可由修改引理5的证明得到类似结论.

## 参 考 文 献

[1] Hoskins, B. J., *Ann. Rev. Fluid Mech.*, 14(1982), 131—151.
[2] Emanuel, K. A., *J. Atmos. Sci.*, 42(1985), 1062—1071.
[3] Thorpe, A. J., and K. A. Emanuel, *J. Atmos. Sci.*, 42(1985), 1809—1824.
[4] Xu Q., *J. Atmos, Sci.*, 46(17) (1989), 2671—2683.
[5] 张恭庆，《临界点理论及其应用》，上海科学技术出版社 (1986).
[6] 汪守宏，博士论文"大尺度大气运动方程的可解性"，兰州大学 (1988).
[7] 陈文塬，《非线性泛函分析》，兰州甘肃人民出版社，(1982).

# An Exsistence Theorem of Generalized Sawyer-Eliassen Equation

Yu Qing-yu

*(Lanzhou University, Lanzhou)*

Xu Qin

*(CLMMS Insitute, U. S. A.)*

**Abstract**

In this paper, we obtain an existence theorem for generalized Sawyer-Eliassen equation, which has played an important role in the study of frontal circulation of atmosphere.

**Key words** solution, generalized Sawyer-Eliassen equation