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压电材料空间轴对称问题的通解及其应用*

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摘　要

本文根据横观各向同性压电材料空间轴对称问题场方程的结构特点，利用逐次引进势函数的方法，最后得到将位移分量和电势函数用满足特定偏微分方程的单一势函数表示的所谓通解，推导过程表明这种形式的通解是完备的。作为应用举例，文中用通解求解了压电材料半无限体表面受集中力的问题，得到位移、应力、电位移分量及电势函数的解析表达式。本文所提供的通解可作为分析含空腔、夹杂或币形裂纹等缺陷的压电材料的机-电耦合行为的工具，算例所得结果可直接用于求解压电体相互间或压电体与普通弹性体间的接触问题。

关键词　压电材料　机-电耦合　空间轴对称问题　完备性通解

一、引　　言

压电材料由于具有力学变形及电场的耦合效应（简称机-电耦合），已被广泛地用来制做各种换能器及传感器；近年来在诸如声纳发射器、脉冲发生器、流体鉴控器、压电声表面波器等高新科技器件中，压电材料作为功能部件起着关键性的作用。为了适应现代科学技术迅速发展的需要，各种新老器件对压电材料的性能提出了愈来愈高的要求，因而需要从机-电耦合的角度上更加深入地研究其有关行为。一类广泛应用并极具发展潜力的压电材料即压电陶瓷，因在力学性能上固有的脆性，在工作状态下由机械或电载荷引起的应力集中会导致裂纹的产生和扩展；除裂纹外材料本身往往还有空洞、夹杂、分层等缺陷，它们也是使器件失效的不可忽视的因素。因而压电陶瓷器件可靠工作的寿命预估要求人们必须充分认识并精确描述其损伤和断裂过程。近年来文献[1]、[2]先后研究了压电材料的平面及反平面裂纹问题，文献[3]进一步给出了具有缺陷压电介质平面问题的一般解法。文献[4]虽然用三维的分析方法讨论了电场对横观各向同性压电体中裂纹尖端附近应力场分布的影响，但具体只考虑了裂纹端线平行于介质横观各向同性轴的特殊情况。压电器件以各种不同的大小形状工作在各种各样的载荷条件下，为分析较复杂的空洞、夹杂或裂纹问题，发展压电材料三维问题的一般解法是必要的，本文的工作可以说是朝着这个方向的进步，因为它给出了一类特殊的但却具有广泛应用背景的三维问题的普通解法。

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二、控制微分方程

当不计体力和不存在自由电荷时，三维压电理论的控制方程为

$$\sigma_{ij,j}=0 \tag{2.1}$$

$$D_{i,i}=0 \tag{2.2}$$

$$\sigma_{ij}=c_{ijkl}\varepsilon_{kl}-e_{kij}E_k \tag{2.3}$$

$$D_i=e_{ikl}\varepsilon_{kl}+\epsilon_{ik}E_k \tag{2.4}$$

$$\varepsilon_{ij}=\frac{1}{2}(u_{j,i}+u_{i,j}) \tag{2.5}$$

$$E_i=-\Phi_{,i} \tag{2.6}$$

其中　i，j，k，$l=1$，2，3，而σ_{ij}，D_i，ε_{ij}，u_i，E_i和Φ分别是应力、电位移、应变、位移、电场分量和电势，c_{ijkl}，ϵ_{ij}和e_{kij}分别为弹性、介电和压电常数。在极端各向异性下压电介质独立的性能常数为21＋6＋18＝45个，对横观各向同性体则为5＋2＋3＝10个。

上述(2.1)、(2.2)两式是以应力和电位移分量表示的平衡方程，(2.3)、(2.4)两式为本构方程，对横观各向同性压电材料，若取z轴垂直各向同性平面，在柱坐标下(2.1)、(2.2)及(2.3)、(2.4)各式分别成为

$$\left.\begin{aligned}
&\frac{\partial\sigma_r}{\partial r}+\frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta}+\frac{\partial\tau_{rz}}{\partial z}+\frac{\sigma_r-\sigma_\theta}{r}=0\\
&\frac{\partial\tau_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta}+\frac{\partial\tau_{\theta z}}{\partial z}+\frac{2\tau_{r\theta}}{r}=0\\
&\frac{\partial\tau_{rz}}{\partial r}+\frac{1}{r}\frac{\partial\tau_{\theta z}}{\partial\theta}+\frac{\partial\sigma_z}{\partial z}+\frac{\tau_{rz}}{r}=0\\
&\frac{\partial D_r}{\partial r}+\frac{1}{r}\frac{\partial D_\theta}{\partial\theta}+\frac{\partial D_z}{\partial z}+\frac{D_r}{r}=0
\end{aligned}\right\} \tag{2.7}$$

$$\left.\begin{aligned}
&\sigma_r=c_{11}\frac{\partial u_r}{\partial r}+c_{12}\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r}{r}\right)+c_{13}\frac{\partial u_z}{\partial z}+e_{31}\frac{\partial\Phi}{\partial z}\\
&\sigma_\theta=c_{12}\frac{\partial u_r}{\partial r}+c_{11}\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r}{r}\right)+c_{13}\frac{\partial u_z}{\partial z}+e_{31}\frac{\partial\Phi}{\partial z}\\
&\sigma_z=c_{13}\frac{\partial u_r}{\partial r}+c_{13}\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r}{r}\right)+c_{33}\frac{\partial u_z}{\partial z}+e_{33}\frac{\partial\Phi}{\partial z}\\
&\tau_{z\theta}=c_{44}\left(\frac{\partial u_\theta}{\partial z}+\frac{1}{r}\frac{\partial u_z}{\partial\theta}\right)+e_{15}\frac{1}{r}\frac{\partial\Phi}{\partial\theta}\\
&\tau_{zr}=c_{44}\left(\frac{\partial u_z}{\partial r}+\frac{\partial u_r}{\partial z}\right)+e_{15}\frac{\partial\Phi}{\partial r}\\
&\tau_{r\theta}=\frac{1}{2}(c_{11}-c_{12})\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta}+\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r}\right)
\end{aligned}\right\} \tag{2.8}$$

$$\left.\begin{aligned}
D_r&=e_{15}\left(\frac{\partial u_z}{\partial r}+\frac{\partial u_r}{\partial z}\right)-\epsilon_{11}\frac{\partial \Phi}{\partial r}\\
D_\theta&=e_{15}\left(\frac{\partial u_\theta}{\partial z}+\frac{1}{r}\frac{\partial u_z}{\partial \theta}\right)-\epsilon_{11}\frac{1}{r}\frac{\partial \Phi}{\partial \theta}\\
D_z&=e_{31}\frac{\partial u_r}{\partial r}+e_{31}\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_r}{r}\right)+e_{33}\frac{\partial u_z}{\partial z}-\epsilon_{33}\frac{\partial \Phi}{\partial z}
\end{aligned}\right\}\tag{2.9}$$

在将(2.3)、(2.4)两式转换为(2.8)、(2.9)两式时已将应变及电场分量代换成位移分量和电势.

如果考虑的是空间轴对称问题，有关场量只是 r 和 z 的函数而与 θ 无关，同时 $u_\theta=0$，故(2.7)～(2.9)式简化为

$$\left.\begin{aligned}
&\frac{\partial \sigma_r}{\partial r}+\frac{\partial \tau_{zr}}{\partial z}+\frac{\tau_r-\sigma_\theta}{r}=0\\
&\frac{\partial \tau_{zr}}{\partial r}+\frac{\partial \sigma_z}{\partial z}+\frac{\tau_{zr}}{r}=0\\
&\frac{\partial D_r}{\partial r}+\frac{\partial D_z}{\partial z}+\frac{D_r}{r}=0
\end{aligned}\right\}\tag{2.10}$$

$$\left.\begin{aligned}
\sigma_r&=c_{11}\frac{\partial u_r}{\partial r}+c_{12}\frac{u_r}{r}+c_{13}\frac{\partial u_z}{\partial z}+e_{31}\frac{\partial \Phi}{\partial z}\\
\sigma_\theta&=c_{12}\frac{\partial u_r}{\partial r}+c_{11}\frac{u_r}{r}+c_{13}\frac{\partial u_z}{\partial z}+e_{31}\frac{\partial \Phi}{\partial z}\\
\sigma_z&=c_{13}\frac{\partial u_r}{\partial r}+c_{13}\frac{u_r}{r}+c_{33}\frac{\partial u_z}{\partial z}+e_{33}\frac{\partial \Phi}{\partial z}\\
\tau_{zr}&=c_{44}\left(\frac{\partial u_z}{\partial r}+\frac{\partial u_r}{\partial z}\right)+e_{15}\frac{\partial \Phi}{\partial r}
\end{aligned}\right\}\tag{2.11}$$

$$\left.\begin{aligned}
D_r&=e_{15}\left(\frac{\partial u_z}{\partial r}+\frac{\partial u_r}{\partial z}\right)-\epsilon_{11}\frac{\partial \Phi}{\partial r}\\
D_z&=e_{31}\frac{\partial u_r}{\partial r}+e_{31}\frac{u_r}{r}+e_{33}\frac{\partial u_z}{\partial z}-\epsilon_{33}\frac{\partial \Phi}{\partial z}
\end{aligned}\right\}\tag{2.12}$$

将(2.11)、(2.12)代入(2.10)式，得

$$\left.\begin{aligned}
&c_{11}\frac{\partial^2 u_r}{\partial r^2}+c_{11}\frac{1}{r}\frac{\partial u_r}{\partial r}+c_{44}\frac{\partial^2 u_r}{\partial z^2}-c_{11}\frac{u_r}{r}+(c_{13}+c_{44})\frac{\partial^2 u_z}{\partial r\partial z}\\
&\quad+(e_{31}+e_{15})\frac{\partial^2 \Phi}{\partial r\partial z}=0\\
&(c_{13}+c_{44})\frac{\partial^2 u_r}{\partial r\partial z}+(c_{13}+c_{44})\frac{1}{r}\frac{\partial u_r}{\partial z}+c_{44}\frac{\partial^2 u_z}{\partial r^2}+c_{44}\frac{1}{r}\frac{\partial u_z}{\partial r}\\
&\quad+c_{33}\frac{\partial^2 u_z}{\partial z^2}+e_{15}\frac{\partial^2 \Phi}{\partial r^2}+e_{15}\frac{1}{r}\frac{\partial \Phi}{\partial r}+e_{33}\frac{\partial^2 \Phi}{\partial z^2}=0\\
&(e_{15}+e_{31})\frac{\partial^2 u_r}{\partial r\partial z}+(e_{15}+e_{31})\frac{1}{r}\frac{\partial u_r}{\partial z}+e_{15}\frac{\partial^2 u_z}{\partial r^2}+e_{15}\frac{1}{r}\frac{\partial u_z}{\partial r}\\
&\quad+e_{33}\frac{\partial^2 u_z}{\partial z^2}-\epsilon_{11}\frac{\partial^2 \Phi}{\partial r^2}-\epsilon_{11}\frac{1}{r}\frac{\partial \Phi}{\partial r}-\epsilon_{33}\frac{\partial^2 \Phi}{\partial z^2}=0
\end{aligned}\right\}\tag{2.13}$$

上式即是以位移u_r，u_z及电势Φ为基本未知量时横观各向同性压电材料空间轴对称问题的控制微分方程。

三、位移及电势通解

用逐次引入势函数的方法将(2.13)式联立积分，最后可得到一个满足特殊形式的六阶偏微分方程的函数$\psi(z, r)$，而u_r，u_z及Φ均可由$\psi(z, r)$求得。(2.13)式的一、二两式可改写为

$$\left.\begin{aligned}&\frac{\partial^2 u_r}{\partial r^2}+\frac{\partial}{\partial r}\left(\frac{u_r}{r}\right)+\gamma_1\frac{\partial^2 u_r}{\partial z^2}+\alpha_1\frac{\partial^2 u_r}{\partial r\partial z}+\beta_1\frac{\partial^2\Phi}{\partial r\partial z}=0\\&\frac{\partial^2 u_r}{\partial r\partial z}+\frac{1}{r}\frac{\partial u_r}{\partial z}+\alpha_2\left(\frac{\partial^2 u_z}{\partial r^2}+\frac{1}{r}\frac{\partial u_z}{\partial r}\right)+\gamma_2\frac{\partial^2 u_z}{\partial z^2}\\&\quad+\beta_2\left(\frac{\partial^2\Phi}{\partial r^2}+\frac{1}{r}\frac{\partial\Phi}{\partial r}\right)+\delta_2\frac{\partial^2\Phi}{\partial z^2}=0\end{aligned}\right\}\tag{3.1}$$

其中

$$\alpha_1=\frac{c_{13}+c_{44}}{c_{11}},\quad \beta_1=\frac{e_{31}+e_{15}}{c_{11}},\quad \gamma_1=\frac{c_{44}}{c_{11}}$$

$$\alpha_2=\frac{c_{44}}{c_{13}+c_{44}},\quad \beta_2=\frac{e_{15}}{c_{13}+c_{44}},\quad \gamma_2=\frac{c_{33}}{c_{13}+c_{44}},\quad \delta_2=\frac{e_{33}}{c_{13}+c_{44}}$$

为便于积分，(3.1)式经整理后成为

$$\frac{\partial}{\partial r}\left(\frac{\partial u_r}{\partial r}+\frac{u_r}{r}+\alpha_1\frac{\partial u_z}{\partial z}+\beta_1\frac{\partial\Phi}{\partial z}\right)=-\gamma_1\frac{\partial}{\partial z}\left(\frac{\partial u_r}{\partial z}\right)\tag{3.2}$$

$$\begin{aligned}&\frac{\partial}{\partial r}\left[\left(r\frac{\partial u_r}{\partial r}\right)+\alpha_2\left(r\frac{\partial u_z}{\partial r}\right)+\beta_2\left(r\frac{\partial\Phi}{\partial r}\right)\right]\\&\quad=-\frac{\partial}{\partial z}\left[\gamma_2\frac{\partial(ru_z)}{\partial z}+\delta_2\frac{\partial(r\Phi)}{\partial z}\right]\end{aligned}\tag{3.3}$$

现对(3.2)式引进函数$M(z, r)$，对(3.3)式引进函数$U(z, r)$，分别使

$$\left.\begin{aligned}&\frac{\partial u_r}{\partial r}+\frac{u_r}{r}+\alpha_1\frac{\partial u_z}{\partial z}+\beta_1\frac{\partial\Phi}{\partial z}=-\frac{\partial M}{\partial z}\\&\gamma_1\frac{\partial u_r}{\partial z}=\frac{\partial M}{\partial r}\end{aligned}\right\}\tag{3.4}$$

$$\left.\begin{aligned}&r\frac{\partial u_r}{\partial z}+\alpha_2\left(r\frac{\partial u_z}{\partial r}\right)+\beta_2\left(r\frac{\partial\Phi}{\partial r}\right)=-\frac{\partial U}{\partial z}\\&\gamma_2\frac{\partial(ru_z)}{\partial z}+\delta_2\frac{\partial(r\Phi)}{\partial z}=\frac{\partial U}{\partial r}\end{aligned}\right\}\tag{3.5}$$

则(3.2)、(3.3)式恒被满足。(3.4)、(3.5)两式的第一式可改写如下

$$\frac{1}{r}\left[\frac{\partial}{\partial r}(ru_r)\right]=-\alpha_1\frac{\partial u_z}{\partial z}-\beta_1\frac{\partial\Phi}{\partial z}-\frac{\partial M}{\partial z}$$

$$\alpha_2\frac{\partial u_z}{\partial r}+\beta_2\frac{\partial\Phi}{\partial r}=-\frac{\partial u_r}{\partial z}-\frac{1}{r}\frac{\partial U}{\partial z}$$

为使上两式成立，可再次引入两个函数$N(z, r)$，$V(z, r)$，并使

$$\left.\begin{aligned} u_r&=-\frac{1}{r}\frac{\partial N}{\partial z}\\ \alpha_1 u_z+\beta_1\Phi+M&=\frac{1}{r}\frac{\partial N}{\partial r}\end{aligned}\right\} \tag{3.6}$$

$$\left.\begin{aligned} u_r+\frac{1}{r}U&=\frac{\partial V}{\partial r}\\ \alpha_2 u_z+\beta_2\Phi&=-\frac{\partial V}{\partial z}\end{aligned}\right\} \tag{3.7}$$

同样，对(3.4)、(3.5)的第二两个式子引进函数$P(z, r)$和$W(z, r)$，取

$$\gamma_1 u_r=\frac{\partial P}{\partial r},\quad M=\frac{\partial P}{\partial z} \tag{3.8}$$

$$\gamma_2 r u_z+\delta_2 r\Phi=\frac{\partial W}{\partial r},\quad U=\frac{\partial W}{\partial z} \tag{3.9}$$

它们即被满足．以上为积分方程(3.1)式已引进了六个新的函数，显然它们并非相互 独立，而是通过u_r，u_z及Φ由(3.6)～(3.9)共八个方程相关联，若使以上诸式中消去u_r，u_z及Φ，必将揭示这些函数之间的内在联系．

将(3.6)、(3.7)的第二两式联立求解，得

$$\left.\begin{aligned} u_z&=-\frac{\beta_2}{\alpha_1\beta_2-\alpha_2\beta_1}\left(\frac{\partial P}{\partial z}-\frac{1}{r}\frac{\partial N}{\partial r}\right)+\frac{\beta_1}{\alpha_1\beta_2-\alpha_2\beta_1}\frac{\partial V}{\partial z}\\ \Phi&=\frac{\alpha_2}{\alpha_1\beta_2-\alpha_2\beta_1}\left(\frac{\partial P}{\partial z}-\frac{1}{r}\frac{\partial N}{\partial r}\right)-\frac{\alpha_1}{\alpha_1\beta_2-\alpha_2\beta_1}\frac{\partial V}{\partial z}\end{aligned}\right\} \tag{3.10}$$

上式中已利用了(3.8)式的第二 式$M=\partial P/\partial z$．(3.6)、(3.7)的第一两式应给出相同的u_r，故有

$$\frac{1}{r}U=\frac{\partial V}{\partial r}+\frac{1}{r}\frac{\partial N}{\partial z}$$

利用(3.9)的第二式，上式成为

$$\frac{1}{r}\frac{\partial W}{\partial z}-\frac{1}{r}\frac{\partial N}{\partial z}=\frac{\partial V}{\partial r} \tag{3.11}$$

将(3.6)的第一式代入(3.8)的第一式，得

$$-\frac{1}{r}\frac{\partial N}{\partial z}=\frac{1}{\gamma_1}\frac{\partial P}{\partial r}$$

上式可改写为

$$-\gamma_1\frac{\partial}{\partial z}\left(\frac{N}{r}\right)=\frac{\partial P}{\partial r} \tag{3.12}$$

再将(3.10)式代入(3.9)的第一式，经运算整理得

$$a_2\frac{\partial(rP)}{\partial z}+b_2\frac{\partial(rV)}{\partial z}=\frac{\partial W}{\partial r}+a_2\frac{\partial N}{\partial r} \tag{3.13}$$

其中

$$a_2=\frac{\alpha_2\delta_2-\beta_2\gamma_1}{\alpha_1\beta_2-\alpha_2\beta_1},\quad b_2=\frac{\beta_1\gamma_2-\alpha_1\delta_2}{\alpha_1\beta_2-\alpha_2\beta_1}$$

对(3.11)～(3.13)式可进一步进行积分，为此分别引进新的函数$R(z, r)$、$Y(z, r)$和$X(z, r)$，并依次使

$$\left.\begin{aligned}&\frac{W}{r}-\frac{N}{r}=\frac{\partial R}{\partial r},\ V=\frac{\partial R}{\partial z}\\&\gamma_1\frac{N}{r}=\frac{\partial Y}{\partial r},\ P=-\frac{\partial Y}{\partial z}\\&a_2rP+b_2rV=\frac{\partial X}{\partial r},\ W+a_2N=\frac{\partial X}{\partial z}\end{aligned}\right\}\tag{3.14}$$

后四式给出

$$\left.\begin{aligned}&N=\frac{r}{\gamma_1}\frac{\partial Y}{\partial r}\\&V=\frac{a_2}{b_2}\frac{\partial Y}{\partial z}+\frac{1}{b_2}\frac{1}{r}\frac{\partial X}{\partial r}\\&P=-\frac{\partial Y}{\partial z}\\&W=\frac{\partial X}{\partial z}-\frac{a_2}{\gamma_1}r\frac{\partial Y}{\partial r}\end{aligned}\right\}\tag{3.15}$$

将上式中N，V和W的表达式代入(3.14)式，得

$$\frac{1}{r}\frac{\partial X}{\partial z}-\frac{a_2}{\gamma_1}\frac{\partial Y}{\partial r}-\frac{1}{\gamma_1}\frac{\partial Y}{\partial r}=\frac{\partial R}{\partial r}\tag{3.16}$$

$$\frac{a_2}{b_2}\frac{\partial Y}{\partial z}+\frac{1}{b_2}\frac{1}{r}\frac{\partial X}{\partial r}=\frac{\partial R}{\partial z}\tag{3.17}$$

对(3.16)式引入函数$A(z, r)$，使

$$\frac{X}{r}=\frac{\partial A}{\partial r},\ R+\frac{a_2+1}{\gamma_1}Y=\frac{\partial A}{\partial z}\tag{3.18}$$

对(3.17)式引入函数$B(z, r)$，使

$$X=\frac{\partial B}{\partial z},\ b_2R-a_2Y=\frac{1}{r}\frac{\partial B}{\partial r}\tag{3.19}$$

据(3.18)、(3.19)的第一两式应有

$$\frac{\partial A}{\partial r}=\frac{1}{r}\frac{\partial B}{\partial z}$$

为使上式成立，可引入函数$\psi(z, r)$，使

$$A=\frac{\partial\psi}{\partial z},\ B=r\frac{\partial\psi}{\partial r}\tag{3.20}$$

至此，以上推导过程中引入的全部辅助函数均可用最后引入的函数$\psi(z, r)$表示，将(3.20)代入(3.18)、(3.19)求得X，Y用ψ的表达式，再将结果代入(3.15)式求得N，V，P和W，最后由(3.6)的第一式及(3.10)式即求得u_r，u_z和Φ为

$$
\left.\begin{aligned}
u_r&=\xi_1\frac{\partial^2}{\partial r\partial z}\left[b_2\frac{\partial^2\psi}{\partial z^2}-\left(\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}\right)\right]\\
u_z&=\zeta_1\frac{\partial^4\psi}{\partial z^4}+\zeta_2\frac{\partial^2}{\partial z^2}\left(\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}\right)\\
&\quad-\xi_2\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)\left(\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}\right)\\
\Phi&=\zeta_3\frac{\partial^4\psi}{\partial z^4}+\zeta_4\frac{\partial^2}{\partial z^2}\left(\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}\right)\\
&\quad-\xi_3\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)\left(\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}\right)
\end{aligned}\right\}\tag{3.21}
$$

其中 $\xi_i(i=1,\ 2,\ 3)$，$\zeta_i(i=1,\ 2,\ 3,\ 4)$为由压电体材料性能参数确定的常量，具体表达式见附录A。

为了得到$\psi(z,\ r)$所满足的方程，需将(3.21)式代入控制微分方程(2.13)式的第三式，经运算整理得

$$
\begin{aligned}
&k_1\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)\left(\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}\right)\\
&\quad+k_2\frac{\partial^2}{\partial z_2}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)\left(\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}\right)\\
&\quad+k_3\frac{\partial^4}{\partial z^4}\left(\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}\right)+k_4\frac{\partial^6\psi}{\partial z^6}=0
\end{aligned}\tag{3.22}
$$

上式的特征方程为

$$C\lambda^6-B\lambda^4+A\lambda^2-1=0$$

令$\lambda^2=\rho$，则上式成为

$$C\rho^3-B\rho^2+A\rho-1=0$$

设该方程的三个根分别为$\rho_1=s_1^2$，$\rho_2=s_2^2$，$\rho_3=s_3^2$，则(3.22)式可写为如下形式

$$
\begin{aligned}
&\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{s_1^2}\frac{\partial^2}{\partial z^2}\right)\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right.\\
&\quad\left.+\frac{1}{s_2^2}\frac{\partial^2}{\partial z^2}\right)\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{s_3^2}\frac{\partial^2}{\partial z^2}\right)\psi=0
\end{aligned}\tag{3.23}
$$

若令

$$\nabla_i^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{s_i^2}\frac{\partial^2}{\partial z^2}\qquad(i=1,\ 2,\ 3)$$

(3.23)式可简写为

$$\nabla_1^2\nabla_2^2\nabla_3^2\psi=0\tag{3.24}$$

以上各式中$k_i(i=1,\ 2,\ 3)$，A，B，C均为常数，具体表达式见附录B。

综上所述，横观各向同性压电材料空间轴对称问题的求解被归结为由(3.23)式求得函数$\psi(z,\ r)$，再据(3.21)式即可求得位移分量及电势，对实际问题当然还要考虑边界条件。因而(3.21)与(3.23)式结合起来就构成了问题的所谓通解，由于能满足(3.23)式的$\psi(z,\ r)$通过(3.21)式求得的u_r，u_z及Φ必使(2.13)式得到满足，同时上述推导过程已表明(2.13)式的解总可以表示成(3.21)的形式，因而这样的通解是完备的。

值得指出的是，在联立求解方程组(2.13)式时，若先通过引入势函数的方法积分其第二

和第三式，然后再使最终引入的单一势函数满足第一式，则得形式不同的完备通解如下

$$\left.\begin{aligned} u_r&=\xi_1\frac{1}{r}\left(\frac{\partial^2}{\partial r^2}-\frac{1}{r}\frac{\partial}{\partial r}\right)\left(\frac{\partial^2\psi}{\partial r^2}-\frac{1}{r}\frac{\partial\psi}{\partial r}\right)\\ &\quad+\zeta_2\frac{1}{r}\frac{\partial^2}{\partial z^2}\left(\frac{\partial^2\psi}{\partial r^2}-\frac{1}{r}\frac{\partial\psi}{\partial r}\right)+\zeta_3\frac{1}{r}\frac{\partial^4\psi}{\partial z^4}\\ u_z&=\xi_1\frac{1}{r}\frac{\partial^2}{\partial r\partial z}\left(\frac{\partial^2\psi}{\partial r^2}-\frac{1}{r}\frac{\partial\psi}{\partial r}\right)+\eta_1\frac{1}{r}\frac{\partial^4\psi}{\partial r\partial z^3}\\ \Phi&=\xi_2\frac{1}{r}\frac{\partial^2}{\partial r\partial z}\left(\frac{\partial^2\psi}{\partial r^2}-\frac{1}{r}\frac{\partial\psi}{\partial r}\right)+\eta_2\frac{1}{r}\frac{\partial^4\psi}{\partial r\partial z^3} \end{aligned}\right\} \tag{3.25}$$

其中 $\psi(z,\ r)$满足

$$\nabla_1^2\nabla_2^2\nabla_3^2\psi=0 \tag{3.26}$$

$$\nabla_i^2=\frac{\partial^2}{\partial r^2}-\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{s_i^2}\frac{\partial^2}{\partial z^2}\qquad (i=1,\ 2,\ 3)$$

这里s_i^2与前述完备性通解表达式中的相同。

四、通 解 应 用 举 例

上面获得的所谓通解，为横观各向同性压电材料空间轴对称问题的分析求解提供了有力的工具，作为例子以下将求解半无限体表面受轴对称分布载荷或集中载荷两个具体问题。

首先考虑图 1 所示问题，设一横观各向同性压电半无限体在平行各向同性面的表面受以O为中心的轴对称法向分布载荷$p(r)$的作用，取O为柱坐标系的原点，Oz轴垂直表面向下。若$p(r)$的合力向量为有限值或零，则可用Fourier-Bessel积分表示如下

$$p(r)=\int_0^\infty \chi(t)tJ_0(tr)dt \tag{4.1}$$

其中

$$\chi(t)=\int_0^\infty p(\xi)\xi J_0(\xi t)d\xi \tag{4.2}$$

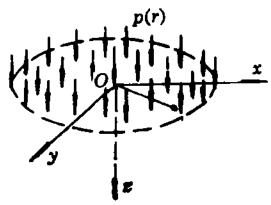

图 1

J_0为零阶贝塞尔函数。上述问题的力学边界条件为

$$\left.\begin{aligned}\sigma_z|_{z=0}&=-p(r)=-\int_0^\infty\chi(t)\,tJ_0(tr)dt\\ \tau_{zr}|_{z=0}&=0\end{aligned}\right\} \tag{4.3}$$

根据外电场为零时电位移分量与应力分量间的关系，在本问题中应有$D_z=d_{33}\sigma_z$，d_{33}为压电应变常数，故电学边界条件为

$$D_z|_{z=0}=-d_{33}p(r)=-d_{33}\int_0^\infty\chi(t)tJ_0(tr)dt \tag{4.4}$$

显而易见，在无穷远处（即$\sqrt{r^2+x^2}\to\infty$）还有

$$\sigma_z=\sigma_\theta=\sigma_r=\tau_{zr}=0,\quad D_z=D_r=0 \tag{4.5}$$

采用第一种形式的通解，方程(3.23)式可用分离变量法求解，设$\psi=Z(tz)J_0(tr)$，解得

$$\psi=(A\exp[s_1tz]+B\exp[s_2tz]+C\exp[s_3tz]+D\exp[-s_1tz]+E\exp[-s_2tz]+F\exp[-s_3tz]J_0(tr)$$

可以证明将上式对t积分得到的函数仍为原方程的解，注意到边界条件(4.3)、(4.4)的具体

形式，这里可直接取(3.23)式的解答为

$$\Psi=\int_0^\infty (A\exp[s_1tz]+B\exp[s_2tz]+C\exp[s_3tz]+D\exp[-s_1tz]+E\exp[-s_2tz]+F\exp[-s_3tz])J_0(tr)dt$$

A，B，C，D，E，F为特定常数，据(4.5)式应有 $A=B=C=0$．D，E，F 则由边界条件(4.2)、(4.3)式确定．经运算得位移分量及电势的表达式为

$$\left.\begin{aligned}
u_r&=\int_0^\infty (I_1\exp[-s_1tz]+J_1\exp[-s_2tz]+K_1\exp[-s_3tz])\chi(t)J_1(tr)dt\\
u_z&=\int_0^\infty (T_1\exp[-s_1tz]+P_1\exp[-s_2tz]+Q_1\exp[-s_3tz])\chi(t)J_0(tr)dt\\
\Phi&=\int_0^\infty (T_2\exp[-s_1tz]+P_2\exp[-s_2tz]+Q_2\exp[-s_3tz])\chi(t)J_0(tr)dt
\end{aligned}\right\}\tag{4.6}$$

将上式代入本构方程(2.11)，(2.12)式，得相应的应力及电位移分量为

$$\left.\begin{aligned}
\sigma_r&=-\int_0^\infty (I_2\exp[-s_1tz]+J_2\exp[-s_2tz]+K_2\exp[-s_3tz])\chi(t)\frac{1}{r}J_1(tr)dt\\
&\quad+\int_0^\infty (T_3\exp[-s_1tz]+P_3\exp[-s_2tz]+Q_3\exp[-s_3tz])\chi(t)tJ_0(tr)dt\\
\sigma_\theta&=-\int_0^\infty (I_3\exp[-s_1tz]+J_3\exp[-s_2tz]+K_3\exp[-s_3tz])\chi(t)\frac{1}{r}J_1(tr)dt\\
&\quad+\int_0^\infty (T_4\exp[-s_1tz]+P_4\exp[-s_2tz]+Q_4\exp[-s_3tz])\chi(t)tJ_0(tr)dt\\
\sigma_z&=-\int_0^\infty (T_5\exp[-s_1tz]+P_5\exp[-s_2tz]+Q_5\exp[-s_3tz])\chi(t)tJ_0(tr)dt\\
\tau_{zr}&=-\int_0^\infty (T_6\exp[-s_1tz]+P_6\exp[-s_2tz]+Q_6\exp[-s_3tz])\chi(t)tJ_1(tr)dt\\
D_z&=-\int_0^\infty (T_7\exp[-s_1tz]+P_7\exp[-s_2tz]+Q_7\exp[-s_3tz])\chi(t)tJ_0(tr)dt\\
D_r&=-\int_0^\infty (T_8\exp[-s_1tz]+P_8\exp[-s_2tz]+Q_8\exp[-s_3tz])\chi(t)tJ_1(tr)dt
\end{aligned}\right\}\tag{4.7}$$

其中 I_i，J_i，$K_i(i=1, 2, 3)$，T_i，P_i，$Q_i(i=1, 2, \cdots, 8)$均为材料性能参数决定的常数，因篇幅所限这里不予列出．

为了得到横观各向同性压电材料半无限体表面受集中力P的解答，如图2所示，可设想在O点以无穷小量ε为半径的圆面积上作用衤集度为$p=P/\pi\varepsilon^2$的分布载荷，当ε趋于零时此分布载荷合力的极限即为P，这样问题即被转化为前面已经讨论的情况，据(4.2)式应有

$$\chi(t)=\int_0^\infty p(\xi)\xi J_0(\xi t)d\xi=\lim_{\varepsilon\to 0}\int_0^\varepsilon \frac{P}{\pi\varepsilon^2}\xi J_0(\xi t)d\xi$$

$$=\lim_{\varepsilon\to 0}\frac{P}{\pi}\frac{J_1(t\varepsilon)}{t\varepsilon}=\frac{P}{2\pi}$$

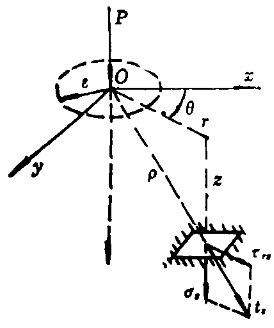

图 2

把上式代入(4.6)、(4.7)两式，并利用如下关系

$$\int_0^\infty \exp[-stz]J_0(tr)dt=\frac{1}{\sqrt{r^2+s^2z^2}}$$

$$\int_0^\infty \exp[-stz]tJ_0(tr)dt=\frac{sz}{(r^2+s^2z^2)^{3/2}}$$

$$\int_0^\infty \exp[-stz]J_1(tr)dt=\frac{1}{r}\left(1-\frac{sz}{\sqrt{r^2+s^2z^2}}\right)$$

$$\int_0^\infty \exp[-stz]tJ_1(tr)dt=\frac{r}{(r^2+s^2z^2)^{3/2}}$$

可得

$$\left.\begin{aligned}
u_r&=\frac{P}{2\pi r}\left[(I_1+J_1+K_1)-z\left(\frac{I_1s_1}{\sqrt{r^2+s_1^2z^2}}+\frac{J_1s_2}{\sqrt{r^2+s_2^2z^2}}+\frac{K_1s_3}{\sqrt{r^2+s_3^2z^2}}\right)\right]\\
u_z&=\frac{P}{2\pi}\left[\frac{T_1}{\sqrt{r^2+s_1^2z^2}}+\frac{P_1}{\sqrt{r^2+s_2^2z^2}}+\frac{Q_1}{\sqrt{r^2+s_3^2z^2}}\right]\\
\Phi&=\frac{P}{2\pi}\left[\frac{T_2}{\sqrt{r^2+s_1^2z^2}}+\frac{P_2}{\sqrt{r^2+s_2^2z^2}}+\frac{Q_2}{\sqrt{r^2+s_3^2z^2}}\right]
\end{aligned}\right\}\quad(4.8)$$

$$\left.\begin{aligned}
\sigma_r&=-\frac{P}{2\pi r^2}\left[(I_2+J_2+K_2)-z\left(\frac{I_2s_1}{\sqrt{r^2+s_1^2z^2}}+\frac{J_2s_2}{\sqrt{r^2+s_2^2z^2}}\right.\right.\\
&\quad\left.\left.+\frac{K_2s_3}{\sqrt{r^2+s_3^2z^2}}\right)\right]-\frac{Pz}{2\pi}\left[\frac{T_3s_1}{(r^2+s_1^2z^2)^{3/2}}+\frac{P_3s_2}{(r^2+s_2^2z^2)^{3/2}}\right.\\
&\quad\left.+\frac{Q_3s_3}{(r^2+s_3^2z^2)^{3/2}}\right]\\
\sigma_\theta&=-\frac{P}{2\pi r^2}\left[(I_3+J_3+K_3)-z\left(\frac{I_3s_1}{\sqrt{r^2+s_1^2z^2}}\right.\right.\\
&\quad\left.\left.+\frac{J_3s_2}{\sqrt{r^2+s_2^2z^2}}+\frac{K_3s_3}{\sqrt{r^2+s_3^2z^2}}\right)\right]-\frac{Pz}{2\pi}\left[\frac{T_4s_1}{(r^2+s_1^2z^2)^{3/2}}\right.\\
&\quad\left.+\frac{P_4s_2}{(r^2+s_2^2z^2)^{3/2}}+\frac{Q_4s_3}{(r^2+s_3^2z^2)^{3/2}}\right]\\
\sigma_z&=-\frac{Pz}{2\pi}\left[\frac{T_5s_1}{(r^2+s_1^2z^2)^{3/2}}+\frac{P_5s_2}{(r^2+s_2^2z^2)^{3/2}}+\frac{Q_5s_3}{(r^2+s_3^2z^2)^{3/2}}\right]\\
\tau_{zr}&=-\frac{Pr}{2\pi}\left[\frac{T_6s_1}{(r^2+s_1^2z^2)^{3/2}}+\frac{P_6s_2}{(r^2+s_2^2z^2)^{3/2}}+\frac{Q_6s_3}{(r^2+s_3^2z^2)^{3/2}}\right]
\end{aligned}\right\}\quad(4.9)$$

$$
\left.\begin{aligned}
D_z&=-\frac{Pz}{2\pi}\left[\frac{T_7 s_1}{(r^2+s_1^2 z^2)^{3/2}}+\frac{P_7 s_2}{(r^2+s_2^2 z^2)^{3/2}}+\frac{Q_7 s_3}{(r^2+s_3^2 z^2)^{3/2}}\right]\\
D_r&=-\frac{Pr}{2\pi}\left[\frac{T_8 s_1}{(r^2+s_2^1 z^2)^{3/2}}+\frac{P_8 s_2}{(r^2+s_2^2 z^2)^{3/2}}+\frac{Q_8 s_3}{(r^2+s_3^2 z^2)^{3/2}}\right]
\end{aligned}\right\}
$$

正像经典弹性力学中由布希涅斯克解答通过叠加法可求得一些有用问题的解答一样，上述解答也可作为求解有关问题的基础，例如可用于求解压电体与压电体或压电体与通常的弹性体之间的接触问题。

上述解例说明了本文所得通解对解决横观各向同性压电材料的空间轴对称问题是极为有用的，灵活运用这个通解还可以分析压电陶瓷中孔洞、夹杂及硬币形裂纹等缺陷在机-电耦合过程中的性态，为压电器件可靠性寿命预估提供科学的理论依据。

附 录 A

$\xi_1=-h^{-1}(c_{13}e_{15}-c_{44}e_{31})$, $\xi_2=h^{-1}c_{11}e_{15}$

$\xi_3=-h^{-1}c_{11}c_{44}$, $\zeta_1=-h^{-1}c_{44}e_{33}$

$\zeta_2=h^{-1}[c_{13}(e_{31}+e_{15})+c_{44}e_{31}-c_{11}e_{33}]$, $\zeta_3=h^{-1}c_{33}c_{44}$

$\zeta_4=-h^{-1}[c_{13}(c_{13}+c_{44})+c_{44}c_{13}-c_{11}c_{33}]$

$$h=\frac{c_{44}(c_{44}e_{33}-c_{33}e_{15})}{c_{13}+c_{44}}+[c_{33}(e_{31}+e_{15})-e_{33}(c_{13}+c_{44})]\left[\frac{c_{11}(c_{44}e_{33}-c_{33}e_{15})}{(c_{13}+c_{44})(c_{13}e_{15}-c_{44}e_{31})}+1\right]$$

附 录 B

$k_1=\epsilon_{11}\xi_3-e_{15}\xi_2$

$k_2=-(e_{31}+e_{15})\xi_1+e_{15}\xi_2(b_2-\gamma_1)+e_{15}\eta_1(a_2+1)-e_{33}\xi_2-\epsilon_{11}\xi_3(b_2-\gamma_1)$
$\quad-\epsilon_{11}\eta_2(a_2+1)+\epsilon_{33}\xi_3$

$k_3=(e_{31}+e_{15})\xi_1 b_2+e_{15}\xi_1 b_2\gamma_1+e_{15}\eta_1 a_2\gamma_1+e_{33}\xi_2(b_2-\gamma_1)+e_{33}\eta_1(a_2+1)$
$\quad-\epsilon_{11}\xi_3 b_2\gamma_1-\epsilon_{11}\eta_2 a_2\gamma_1-\epsilon_{33}\xi_3(b_2-\gamma_1)-\epsilon_{33}\eta_2(a_2+1)$

$k_4=e_{33}\xi_2 b_2\gamma_1+e_{33}\eta_1 a_2\gamma_1-\epsilon_{33}\xi_3 b_2\gamma_1-\epsilon_{33}\eta_2 a_2\gamma_1$

$A=k_2/k_1$, $B=k_3/k_1$, $C=k_4/k_1$

$\eta_1=\beta_1/(\alpha_1\beta_2-\alpha_2\beta_1)[a_2\gamma_1+b_2(a_2+1)]$

$\eta_2=-\alpha_1/(\alpha_1(\alpha_1\beta_2-\alpha_2\beta_1)[a_2\gamma_1+b_2(a_2+1)]$

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A General Solution And the Application of Space Axisymmetric Problem in Piezoelectric Material

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Abstract

According to the structure feature of the governing equations of space axisymmetric problem in transversely isotropic piezoelectric material, using the method of introducing potential function one by one, in this paper we obtain the so-called general solution of displacement and electric potential function denoted by uinque potential function which satisfies specific partiality equations. As an applying example of the general solution, we solve problem of semi-infinite body made of piezoelectric material, on the surface of the semi-infinite body a concentrative force is applied, and get the analytic formulations of stress and electric displacement components. The general solution provided by this paper can be used as a tool to analyse the mechanical-electrical coupling behavior of piezoelectric material which contains defects such as cavity, inclusion, penuy-shape crack, and so on. The result of the solved problem can be used directly to analyse contact problems which take place between two piezoelectric bodies or piezoelectric body and elastic body.

Key words piezoelectric material, mechanical-electrical coupling, space axisymmetric problem, general solution