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在边缘线布载荷作用下开顶扁球壳的非线性稳定问题的奇摄动解*

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摘　要

本文利用奇异摄动方法计算了在内边缘线布载荷作用下无刚性中心的开顶扁球壳的非线性稳定问题，得到了几何参数k值较大时本问题的一致有效的渐近解。

关键词　圆球壳　非线性稳定性　奇摄动解

一、引　言

在近代航空航海工程、精密仪器工程、自动控制和建筑结构等领域中，经常使用开顶扁薄球壳。按照设计要求，需要核算它的稳定性，从理论上导出尽量精确的计算公式或图表。

1939年，T. von Kármán 和钱学森首先指出扁球壳屈曲现象是一个非线性现象，其基本方程是一组非线性微分方程。因而，求出这类问题的精确解在数学上存在很大困难。多年来人们大都采用某种近似方法求解结构较简单的扁球壳、圆柱壳和扁锥壳的稳定性问题的近似解，而对于结构较复杂的开顶扁球壳的非线性稳定问题的研究比较少。刘人怀[1]和Tillman[2]等人先后利用修正迭代等方法研究了开顶扁球壳的非线性稳定问题，获得了一些有益的结果。对于几何参数k值较大的具有硬中心的开顶扁球壳的非线性稳定问题，1989年，康盛亮[3~4]首先利用改进多重尺度法进行了研究。

本文利用文献[5]提出的奇异摄动方法研究当几何参数k值较大时，外边缘固定夹紧，内边缘悬空，在内边缘线布载荷作用下，无刚性中心的开顶扁球壳的非线性稳定问题。求得了此边值问题的一致有效渐近解，并进行了余项误差估计。这就为决定临界载荷提供了计算精度较高的解析公式。

二、基本方程和边界条件

考虑如图1所示的无硬中心的开顶扁球壳。壳厚度为h，跨度为$2a$，内缘半径为b，中曲面半径为R，在内边缘线布载荷p的作用下，扁球壳的大挠度弯曲方程为[6]：

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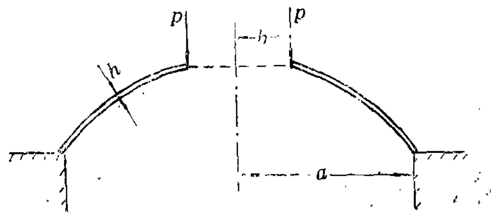

图 1

$$
\left.\begin{aligned}
&D\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}r\frac{dw}{dr}\right)-\frac{b}{r}p-N_r\left(\frac{r}{R}+\frac{dw}{dr}\right)=0\\
&\frac{1}{Eh}r\frac{d}{dr}\frac{1}{r}\frac{d}{dr}(r^2N_r)+\frac{r}{R}\frac{dw}{dr}+\frac{1}{2}\left(\frac{dw}{dr}\right)^2=0
\end{aligned}\right\} \quad (2.1)
$$

在外边缘固定夹紧、内边缘悬空的情况下，其相应的边界条件为

当$r=a$时，$w=0$，$\frac{dw}{dr}=0$，$u=0$ (2.2a)

当$r=b$时，$\frac{1}{\nu}\frac{d^2w}{dr^2}+\frac{1}{r}\frac{dw}{dr}=0$，$N_r=0$ (2.2b)

式中，$D=Eh^3/12(1-\nu^2)$为抗弯刚度，ν为泊松比，E为弹性模量，N_r为径向薄膜内力，w为球壳中曲面的挠度，r为球壳中曲面上的点至对称中心轴的距离，u为径向薄膜位移。

为了简化计算，我们引进下列无量纲量：

$$
\rho=\frac{r}{a},\ y=\sqrt{12(1-\nu^2)}\frac{w}{h},\ \theta=-\frac{dy}{d\rho},\ \alpha=\frac{b}{a},\ \bar{N}_r=\frac{a^2}{D}N_r
$$

$$
P=\sqrt{12(1-\nu^2)}\frac{a^2pb}{Dh},\ S=\rho\bar{N}_r,\ k=\sqrt{12(1-\nu^2)}\frac{a^2}{Rh}
$$

将基本方程(2.1)和边界条件(2.2)化为无量纲边值问题

$$
\left.\begin{aligned}
&\varepsilon^2\frac{d}{d\rho}\left(\frac{1}{\rho}\frac{d}{d\rho}(\rho\theta)\right)+\varepsilon^2P\rho^{-1}-\varepsilon^2\rho^{-1}S\theta+S=0\\
&\varepsilon^2\frac{d}{d\rho}\left(\frac{1}{\rho}\frac{d}{d\rho}(\rho S)\right)+\varepsilon^2\frac{\theta^2}{2\rho}-\theta=0
\end{aligned}\right\} \quad (2.3)
$$

当 $\rho=1$ 时，$y=0$，$\theta=0$，$\frac{dS}{d\rho}-\nu S=0$ (2.4a)

当 $\rho=\alpha$ 时，$S=0$，$\alpha\frac{d\theta}{d\rho}+\nu\theta=0$ (2.4b)

其中 $$\varepsilon^2=\frac{Rh}{a^2\sqrt{12(1-\nu^2)}}$$

这样，我们的问题就化为在边界条件(2.4)下求解带小参数$\varepsilon>0$的变系数的非线性微分方程组(2.3)。

三、摄动边值问题的求解

1. 外部解

先应用正则摄动法求其外部解。假设边值问题(2.3)和(2.4)的外部展开式为

$$\theta^o=\sum_{n=0}^{\infty}\varepsilon^n\theta_n(\rho),\quad S^o=\sum_{n=0}^{\infty}\varepsilon^n S_n(\rho) \tag{3.1}$$

代入方程(2.3)，令ε的各次幂系数为零，得到关于 $\theta_n(\rho)$，$S_n(\rho)(n=0,1,2,\cdots)$ 的递推方程：

$$\left.\begin{aligned}
&-\rho^2 S_0=0,\quad -\rho^2 S_1=0\\
&\rho^2\theta_0''(\rho)+\rho\theta_0'(\rho)-\theta_0(\rho)+P\rho-\rho S_0\theta_0+\rho^2 S_2=0\\
&\rho^2\theta_1''(\rho)+\rho\theta_1'(\rho)-\theta_1(\rho)-\rho(S_1\theta_0+S_0\theta_1)+\rho^2 S_3=0\\
&\cdots\cdots\\
&\rho^2\theta_{n-2}''+\rho\theta_{n-2}'-\theta_{n-2}-\rho\sum_{k=0}^{n-2}S_k\theta_{n-2-k}+\rho^2 S_n=0\qquad(n=3,4,\cdots)
\end{aligned}\right\} \tag{3.2}$$

$$\left.\begin{aligned}
&\rho^2\theta_0=0,\quad \rho^2\theta_1=0\\
&\rho^2 S_0''+\rho S_0'(\rho)-S_0(\rho)+\frac{1}{2\rho}\theta_0^2-\rho^2\theta_2=0\\
&\rho^2 S_1''+\rho S_1'-S_1+\frac{1}{2\rho}\theta_0\theta_1+\frac{1}{2\rho}\theta_1\theta_0-\rho^2\theta_3=0\\
&\cdots\cdots\\
&\rho^2 S_{n-2}''+\rho S_{n-2}'-S_{n-2}+\sum_{k=0}^{n-2}\frac{1}{2\rho}\theta_k\theta_{n-2-k}-\rho\theta_n=0\qquad(n=3,4\cdots)
\end{aligned}\right\} \tag{3.3}$$

由(3.2)和(3.3)解出θ_n，$S_n(n=0,1,2,\cdots)$后，代入(3.1)得

$$\theta^o=0,\quad S^o=-P\rho^{-1}\varepsilon^2 \tag{3.4}$$

显然，(3.4)不满足两端边界条件(2.4)，故在$\rho=1$和$\rho=\alpha$近旁出现边界层。下面应用“两变量”展开程序在$\rho=1$及$\rho=\alpha$的邻域内构造边界层校正项。

2. 边界层项

在$\rho=1$的邻域内引进两变量ξ和η：

$$\xi=\frac{u(\rho)}{\varepsilon},\quad \eta=\rho$$

将关于ρ的导数换成关于变量ξ，η的偏导数

$$\frac{d}{d\rho}=\varepsilon^{-1}(\delta_{1,0}+\varepsilon\delta_{1,1})$$

$$\frac{d^2}{d\rho^2}=\varepsilon^{-2}(\delta_{2,0}+\varepsilon\delta_{2,1}+\varepsilon^2\delta_{2,2})$$

其中 $\delta_{1,0}=u'\dfrac{\partial}{\partial\xi}$，$\delta_{1,1}=\dfrac{\partial}{\partial\eta}$，$\delta_{2,0}=(u')^2\dfrac{\partial^2}{\partial\xi^2}$

$$\delta_{2,1}=2u'\frac{\partial^2}{\partial\xi\partial\eta}+u''\frac{\partial}{\partial\xi},\quad \delta_{2,2}=\frac{\partial^2}{\partial\eta^2}$$

把(2.3)所对应的齐次方程变换成

$$\left.\begin{aligned}&(D_0+\varepsilon D_1+\varepsilon^2 D_2)\theta-\varepsilon^2\eta S\theta+S\eta^2=0\\&(D_0+\varepsilon D_1+\varepsilon^2 D_2)S+\frac{1}{2}\eta^{-1}\varepsilon^2\theta^2-\eta^2\theta=0\end{aligned}\right\}\tag{3.5}$$

其中 $D_0=\eta^2\delta_{2,0}$, $D_1=\eta^2\delta_{2,1}+\eta\delta_{1,0}$, $D_2=\delta_{2,2}+\eta\delta_{1,1}-1$

设在$\rho=1$的邻域内边界层校正项的N阶近似式为

$$\left.\begin{aligned}&V^{(1)}(\xi,\eta;\varepsilon)=\sum_{n=0}^{N}\varepsilon^{n+1}v_n(\xi,\eta)\\&\theta^{(1)}(\xi,\eta;\varepsilon)=\sum_{n=0}^{N}\varepsilon^{n+1}h_n(\xi,\eta)\end{aligned}\right\}\tag{3.6}$$

其中，v_n和h_n是在$\rho=1$的邻域内的待求的边界层型函数.

将(3.6)代入(3.5)式得

$$\left.\begin{aligned}&(D_0+\varepsilon D_1+\varepsilon^2 D_2)\sum_{n=0}^{N}\varepsilon^{n+1}h_n(\xi,\eta)-\eta\sum_{n=2}^{2N}\sum_{k=0}^{n-2}\varepsilon^{n+2}v_k h_{n-2-k}+\eta^2\sum_{n=0}^{N}\varepsilon^{n+1}v_n=0\\&(D_0+\varepsilon D_1+\varepsilon^2 D_2)\sum_{n=0}^{N}\varepsilon^{n+1}v_n(\xi,\eta)+\sum_{n=2}^{2N}\varepsilon^{n+2}\sum_{k=0}^{n-2}\frac{1}{2}\eta^{-1}h_k h_{n-2-k}-\eta^2\sum_{n=0}^{N}\varepsilon^{n+1}h_n=0\end{aligned}\right\}\tag{3.7}$$

在上式中逐次地比较ε的同次幂的系数，得到v_n, $h_n(n=1,2,\cdots)$的递推方程:

$$D_0h_0+\eta^2v_0=0\tag{3.8}$$

$$D_0h_1+D_1h_0+\eta^2v_1=0\tag{3.9}$$

$$D_0h_2+D_1h_1+D_2h_0+\eta^2v_2=0\tag{3.10}$$

……

$$D_0h_{n-1}+D_1h_{n-2}+D_2h_{n-3}-\eta\sum_{k=0}^{n-4}v_kh_{n-4-k}+\eta^2v_{n-1}=0\qquad(n=4,5,\cdots)\tag{3.11}$$

$$D_0v_0-\eta^2h_0=0\tag{3.12}$$

$$D_0v_1+D_1v_0-\eta^2h_1=0\tag{3.13}$$

$$D_0v_2+D_1v_1+D_2v_0-\eta^2h_2=0\tag{3.14}$$

……

$$D_0v_{n-1}+D_1v_{n-2}+D_2v_{n-3}+\frac{1}{2}\eta^{-1}\sum_{k=0}^{n-4}h_kh_{n-4-k}-\eta^2h_{n-1}=0\qquad(n=4,5,\cdots)\tag{3.15}$$

由(3.8)和(3.12)得

$$D_0h_0+\eta^2v_0=0\tag{3.16}$$

$$D_0v_0-\eta^2h_0=0\tag{3.17}$$

则有

$$[u'(\eta)]^4\frac{\partial^4 v_0}{\partial\xi^4}+v_0=0 \tag{3.18}$$

在方程(3.18)中，若取$u'(\eta)=-1$，即取$u(\eta)=(1-\eta)$，则得

$$\frac{\partial^4 v_0}{\partial\xi^4}+v_0=0 \tag{3.19}$$

容易求得具有边界层性质的解为

$$v_0=C_0(\eta)\exp\left[-\frac{\sqrt{2}}{2}(1-i)\xi\right]+\text{c.c.} \tag{3.20}$$

其中c.c.表示前面表示式的共轭复量。

把(3.20)式代入(3.17)式可得

$$h_0=-iC_0(\eta)\exp\left[-\frac{\sqrt{2}}{2}(1-i)\xi\right]+\text{c.c.} \tag{3.21}$$

将(3.20)和(3.21)代入(3.9)和(3.13)得

$$\left.\begin{aligned}D_0h_1+\eta^2v_1&=-i\left[\frac{\sqrt{2}}{2}(i-1)\right][2\eta^2C_0'(\eta)+\eta C_0(\eta)]\exp\left[-\frac{\sqrt{2}}{2}(1-i)\xi\right]+\text{c.c.}\\ D_0v_1-\eta^2h_1&=\left[\frac{\sqrt{2}}{2}(i-1)\right][2\eta^2C_0'(\eta)+\eta C_0(\eta)]\exp\left[-\frac{\sqrt{2}}{2}(1-i)\xi\right]+\text{c.c.}\end{aligned}\right\} \tag{3.22}$$

由消除h_1，v_1中的长期项，可得$C_0=0$，从而得

$$v_0=0,\quad h_0=0 \tag{3.23}$$

方程(3.22)化为

$$D_0h_1+\eta^2v_1=0,\quad D_0v_1-\eta^2h_1=0 \tag{3.24}$$

再由以后导出的关于v_i，h_i的边界条件，可以逐步求得v_i，$h_i(i=1,2,\cdots,N)$。

类似地，在$\rho=a$的邻域内引进两变量

$$\tilde{\xi}=\frac{\tilde{u}(\rho)}{\varepsilon},\quad \tilde{\eta}=\rho$$

可以把(2.3)对应的齐次方程变换成

$$\left.\begin{aligned}&(\tilde{D}_0+\varepsilon\tilde{D}_1+\varepsilon^2\tilde{D}_2)\theta-\varepsilon^2\tilde{\eta}S\theta+\tilde{\eta}^2S=0\\ &(\tilde{D}_0+\varepsilon\tilde{D}_1+\varepsilon^2\tilde{D}_2)S+\frac{1}{2}\eta^{-1}\varepsilon^2\theta^2-\tilde{\eta}^2\theta=0\end{aligned}\right\} \tag{3.5}$$

其中 $\tilde{D}_0=\tilde{\eta}^2\tilde{\delta}_{2,0}$，$\tilde{D}_1=\tilde{\eta}^2\tilde{\delta}_{2,1}+\tilde{\eta}\tilde{\delta}_{1,0}$，$\tilde{D}_2=\tilde{\delta}_{2,2}+\tilde{\eta}\tilde{\delta}_{1,1}-1$

$$\tilde{\delta}_{1,0}=\tilde{u}'\frac{\partial}{\partial\tilde{\xi}},\quad \tilde{\delta}_{1,1}=\frac{\partial}{\partial\tilde{\eta}},\quad \tilde{\delta}_{2,0}=(\tilde{u}')^2\frac{\partial^2}{\partial\tilde{\xi}^2}$$

$$\tilde{\delta}_{2,1}=2\tilde{u}'\frac{\partial^2}{\partial\tilde{\xi}\partial\tilde{\eta}}+\tilde{u}''\frac{\partial}{\partial\tilde{\xi}},\quad \tilde{\delta}_{2,2}=\frac{\partial^2}{\partial\tilde{\eta}^2}$$

假设在$\rho=a$的邻域内的边界层项具有下列形式的N阶近似式

$$\left.\begin{aligned}V^{(a)}(\tilde{\xi},\tilde{\eta};\varepsilon)&=\sum_{n=0}^{N}\varepsilon^{n+1}\tilde{v}_n(\tilde{\xi},\tilde{\eta})\\ \theta^{(a)}(\tilde{\xi},\tilde{\eta};\varepsilon)&=\sum_{n=0}^{N}\varepsilon^{n+1}\tilde{h}_n(\tilde{\xi},\tilde{\eta})\end{aligned}\right\} \tag{3.25}$$

其中，$\tilde{v}_n$和$\tilde{h}_n$是在$\rho=a$的邻域内的待求的边界层型函数。

与前面讨论步骤相同，可得关于$\tilde{v}_n$，$\tilde{h}_n(n=1,2,\cdots)$的递推方程

$$\tilde{D}_0\tilde{h}_0+\tilde{\eta}^2\tilde{v}_0=0 \tag{3.26}$$

$$\tilde{D}_0\tilde{h}_1+\tilde{D}_1\tilde{h}_0+\tilde{\eta}^2\tilde{v}_1=0 \tag{3.27}$$

$$\tilde{D}_0\tilde{h}_2+\tilde{D}_1\tilde{h}_1+\tilde{D}_2\tilde{h}_0+\tilde{\eta}^2\tilde{v}_2=0 \tag{3.28}$$

……

$$\tilde{D}_0\tilde{h}_{n-1}+\tilde{D}_1\tilde{h}_{n-2}+\tilde{D}_2\tilde{h}_{n-3}-\tilde{\eta}\sum_{k=0}^{n-4}\tilde{v}_k\tilde{h}_{n-4-k}+\tilde{\eta}^2\tilde{v}_{n-1}=0 \quad (n=4,5,\cdots) \tag{3.29}$$

$$\tilde{D}_0\tilde{v}_0-\tilde{\eta}^2\tilde{h}_0=0 \tag{3.30}$$

$$\tilde{D}_0\tilde{v}_1+\tilde{D}_1\tilde{v}_0-\tilde{\eta}^2\tilde{h}_1=0 \tag{3.31}$$

$$\tilde{D}_0\tilde{v}_2+\tilde{D}_1\tilde{v}_1+\tilde{D}_2\tilde{v}_0-\tilde{\eta}^2\tilde{h}_2=0 \tag{3.32}$$

……

$$\tilde{D}_0\tilde{v}_{n-1}+\tilde{D}_1\tilde{v}_{n-2}+\tilde{D}_2\tilde{v}_{n-3}+\frac{1}{2}\tilde{\eta}^{-1}\sum_{k=0}^{n-4}\tilde{h}_k\tilde{h}_{n-4-k}-\tilde{\eta}^2\tilde{h}_{n-1}=0 \quad (n=4,5,\cdots) \tag{3.33}$$

同样地，若取 $\tilde{u}'(\tilde{\eta})=1$，即取 $\tilde{u}(\tilde{\eta})=\tilde{\eta}-\alpha$，则可逐次求出上述递推方程的具有边界层型的解为

$$\tilde{v}_0=\tilde{C}_0(\tilde{\eta})\exp\left[-\frac{\sqrt{2}}{2}(1-i)\tilde{\tilde{\xi}}\right]+\text{c.c.} \tag{3.34}$$

$$\tilde{h}_0=-i\tilde{C}_0(\tilde{\eta})\exp\left[-\frac{\sqrt{2}}{2}(1-i)\tilde{\tilde{\xi}}\right]+\text{c.c.} \tag{3.35}$$

由取 $\tilde{C}_0=0$，可得

$$\tilde{v}_0=0,\quad \tilde{h}_0=0 \tag{3.36}$$

而 $\tilde{v}_1$ 和 $\tilde{h}_1$ 由下列方程

$$\tilde{D}_0\tilde{h}_1+\tilde{\eta}^2\tilde{v}_1=0,\quad \tilde{D}_0\tilde{v}_1-\tilde{\eta}^2\tilde{h}_1=0 \tag{3.37}$$

和以后导出的关于 $\tilde{v}_1$，$\tilde{h}_1$ 的边界条件确定．类似地，可逐步求得 $\tilde{v}_i$，$\tilde{h}_i(i=1,2,\cdots,N)$．

假设边值问题(2.3)和(2.4)的解 S，θ 的 N 阶近似式为

$$\left.\begin{aligned}S_N&=\sum_{n=0}^{N}\varepsilon^nS_n(\rho)+\sum_{n=0}^{N}\varepsilon^{n+1}v_n(\xi,\eta)+\sum_{n=0}^{N}\varepsilon^{n+1}\tilde{v}_n(\tilde{\tilde{\xi}},\tilde{\eta})\\ \theta_N&=\sum_{n=0}^{N}\varepsilon^n\theta_n(\rho)+\sum_{n=0}^{N}\varepsilon^{n+1}h_n(\xi,\eta)+\sum_{n=0}^{N}\varepsilon^{n+1}\tilde{h}_n(\tilde{\tilde{\xi}},\tilde{\eta})\end{aligned}\right\} \tag{3.38}$$

其中，S_n，θ_n，v_n，h_n，$\tilde{v}_n$ 和 $\tilde{h}_n$ 分别由递推方程 (3.2)，(3.3)，(3.8)～(3.15)，(3.26)～(3.33)式所确定．

将(3.38)式代入边界条件(2.4)，考虑到 $v_n(\tilde{v}_n)$ 和 $h_n(\tilde{h}_n)(n=0,1,\cdots,N)$ 的边界层性质，得到关系式：

$$\sum_{n=0}^{N}\varepsilon^n\theta_n(1)+\sum_{n=0}^{N}\varepsilon^{n+1}h_n(\xi,\eta)|_{\eta=1}=0 \tag{3.39}$$

$$\sum_{n=0}^{N}\varepsilon S_n'(1)-\nu\sum_{n=0}^{N}\varepsilon^nS_n(1)+\sum_{n=0}^{N}(\delta_{1,n}+\varepsilon\delta_{1,1})\varepsilon^nv_n|_{\eta=1}-\nu\sum_{n=0}^{N}\varepsilon^{n+1}v_n|_{\eta=1}=0 \tag{3.40}$$

$$\sum_{n=0}^{N}\varepsilon^{n}S_{n}(\alpha)+\sum_{n=0}^{N}\varepsilon^{n+1}\tilde{v}_{n}(\tilde{\xi},\tilde{\eta})\big|_{\tilde{\eta}=\alpha}=0 \tag{3.41}$$

$$\sum_{n=0}^{N}\varepsilon^{n}\theta_{n}'(\alpha)-\nu\sum_{n=0}^{N}\varepsilon^{n}\theta_{n}(\alpha)+\alpha\sum_{n=0}^{N}(\tilde{\delta}_{1,0}+\varepsilon\tilde{\delta}_{1,1})\varepsilon^{n}\tilde{h}_{n}\big|_{\tilde{\eta}=\alpha}-\nu\sum_{n=0}^{N}\varepsilon^{n+1}\tilde{h}_{n}\big|_{\tilde{\eta}=\alpha}=0 \tag{3.42}$$

从关于v_1，h_1的方程(3.24)和边界条件(3.39)、(3.40)（取$n=1$）

$$h_1|_{\eta=1}=0,\quad \frac{\partial v_1}{\partial\xi}\bigg|_{\eta=1}=0 \tag{3.43}$$

解得

$$h_1=0,\quad v_1=0 \tag{3.44}$$

代入(3.10)和(3.14)以及边界条件得

$$\left.\begin{aligned}&D_0h_2+\eta^2v_2=0,\quad h_2|_{\eta=1}=0\\&D_0v_2-\eta^2h_2=0,\quad \frac{\partial v_2}{\partial\xi}\bigg|_{\eta=1}=P(\nu+1)\end{aligned}\right\} \tag{3.45}$$

容易求得(3.45)具有边界层性质的解为

$$v_2=C_2(\eta)\exp\left[-\frac{\sqrt{2}}{2}(1-i)\xi\right]+\text{c.c.},\quad h_2=-iC_2(\eta)\exp\left[-\frac{\sqrt{2}}{2}(1-i)\xi\right]+\text{c.c.} \tag{3.46}$$

把(3.46)代入(3.11)和(3.15)（取$n=5$）得

$$\left.\begin{aligned}&D_0h_3+\eta^2v_3=i\frac{\sqrt{2}}{2}(1-i)[2\eta^2C_2'(\eta)+\eta C_2(\eta)]\exp\left[-\frac{\sqrt{2}}{2}(1-i)\xi\right]+\text{c.c.}\\&D_0v_3-\eta^2h_3=\frac{\sqrt{2}}{2}(i-1)[2\eta^2C_2'(\eta)+\eta C_2(\eta)]\exp\left[-\frac{\sqrt{2}}{2}(1-i)\xi\right]+\text{c.c.}\end{aligned}\right\} \tag{3.47}$$

由消除(3.47)的解h_3，v_3中出现长期项和(3.45)中的边界条件，可定出

$$C_2(\eta)=-\sqrt{2}P(\nu+1)\sqrt{\eta}/2 \tag{3.48}$$

将(3.48)代入(3.46)得

$$\left.\begin{aligned}&v_2=-\sqrt{2}P(\nu+1)\sqrt{\eta}\exp\left[-\frac{\sqrt{2}}{2}\xi\right]\cos\frac{\sqrt{2}}{2}\xi\\&h_2=\sqrt{2}P(\nu+1)\sqrt{\eta}\exp\left[-\frac{\sqrt{2}}{2}\xi\right]\sin\frac{\sqrt{2}}{2}\xi\end{aligned}\right\} \tag{3.49}$$

再从关于$\tilde{v}_1$和$\tilde{h}_1$的方程(3.37)以及边界条件(3.41)～(3.42)（取$n=1$）

$$\tilde{v}_1|_{\tilde{\eta}=\alpha}=P\alpha^{-1},\quad \frac{\partial\tilde{h}_1}{\partial\tilde{\xi}}\bigg|_{\tilde{\eta}=\alpha}=0 \tag{3.50}$$

求得具有边界层性质的解$\tilde{v}_1$，$\tilde{h}_1$为

$$\left.\begin{aligned}&\tilde{v}_1=P\alpha^{-3/2}\sqrt{\tilde{\eta}}\exp\left[-\frac{\sqrt{2}}{2}\tilde{\xi}\right]\left(\cos\frac{\sqrt{2}}{2}\tilde{\xi}-\sin\frac{\sqrt{2}}{2}\tilde{\xi}\right)\\&\tilde{h}_1=P\alpha^{-3/2}\sqrt{\tilde{\eta}}\exp\left[-\frac{\sqrt{2}}{2}\tilde{\xi}\right]\left(\sin\frac{\sqrt{2}}{2}\tilde{\xi}+\cos\frac{\sqrt{2}}{2}\tilde{\xi}\right)\end{aligned}\right\} \tag{3.51}$$

把(3.36)和(3.51)代入(3.28)、(3.32)以及边界条件得

$$\left.\begin{aligned}&\bar{D}_0\bar{h}_2+\bar{\eta}^2\bar{v}_2=0,\quad \bar{v}_2|_{\bar{\eta}=\alpha}=0\\&\bar{D}_0\bar{v}_2-\bar{\eta}^2\bar{h}_2=0,\quad \frac{\partial\bar{h}_2}{\partial\bar{\xi}}\bigg|_{\bar{\eta}=\alpha}=P(\nu-\alpha^2)\end{aligned}\right\}\tag{3.52}$$

容易求得(3.52)具有边界层性质的解为

$$\left.\begin{aligned}&\bar{v}_2=\tilde{C}(\bar{\eta})\exp\left[-\frac{\sqrt{2}}{2}(1-i)\bar{\xi}\right]+\text{c.c.}\\&\bar{h}_2=-i\tilde{C}_2(\bar{\eta})\exp\left[-\frac{\sqrt{2}}{2}(1-i)\bar{\xi}\right]+\text{c.c.}\end{aligned}\right\}\tag{3.53}$$

类似地，可定出

$$\tilde{C}_2(\bar{\eta})=-i(2\alpha)^{-\frac{1}{2}}P(\nu-\alpha^2)\sqrt{\bar{\eta}}$$

从而得

$$\left.\begin{aligned}&\bar{v}_2=\sqrt{2}P\alpha^{-\frac{1}{2}}(\nu-\alpha^2)\sqrt{\bar{\eta}}\exp\left[-\frac{\sqrt{2}}{2}\bar{\xi}\right]\sin\frac{\sqrt{2}}{2}\bar{\xi}\\&\bar{h}_2=-\sqrt{2}P\alpha^{-\frac{1}{2}}(\nu-\alpha^2)\sqrt{\bar{\eta}}\exp\left[-\frac{\sqrt{2}}{2}\bar{\xi}\right]\cos\frac{\sqrt{2}}{2}\bar{\xi}\end{aligned}\right\}\tag{3.54}$$

于是

$$\begin{aligned}S_N=&\varepsilon^2\left\{P\alpha^{-3/2}\sqrt{\bar{\eta}}\exp\left[-\frac{\sqrt{2}}{2}\bar{\xi}\right]\left(\cos\frac{\sqrt{2}}{2}\bar{\xi}-\sin\frac{\sqrt{2}}{2}\bar{\xi}\right)-P\rho^{-1}\right\}\\&+\varepsilon^3\left\{\sqrt{2}P\alpha^{-1/2}(\nu-\alpha^2)\sqrt{\bar{\eta}}\exp\left[-\frac{\sqrt{2}}{2}\bar{\xi}\right]\sin\frac{\sqrt{2}}{2}\bar{\xi}\right.\\&\left.-\sqrt{2\eta}P(\nu+1)\exp\left[-\frac{\sqrt{2}}{2}\xi\right]\cos\frac{\sqrt{2}}{2}\xi\right\}+O(\varepsilon^4)\end{aligned}\tag{3.55}$$

$$\begin{aligned}\theta_N=&\varepsilon^2P\alpha^{-3/2}\sqrt{\bar{\eta}}\exp\left[-\frac{\sqrt{2}}{2}\bar{\xi}\right]\left(\sin\frac{\sqrt{2}}{2}\bar{\xi}+\cos\frac{\sqrt{2}}{2}\bar{\xi}\right)\\&+\varepsilon^3\left\{\sqrt{2\eta}P(\nu+1)\exp\left[-\frac{\sqrt{2}}{2}\xi\right]\sin\frac{\sqrt{2}}{2}\xi\right.\\&\left.-\sqrt{2\bar{\eta}}\;P(\nu-\alpha^2)\alpha^{-\frac{1}{2}}\exp\left[-\frac{\sqrt{2}}{2}\bar{\xi}\right]\cos\frac{\sqrt{2}}{2}\bar{\xi}\right\}+O(\varepsilon^4)\end{aligned}\tag{3.56}$$

3. 余项估计

我们以R_N，Z_N分别表示边值问题(2.3)和(2.4)的真解θ_ε，S_ε与形式渐近解θ_N，S_N的余项，即

$$R_N=\theta_\varepsilon-\theta_N,\quad Z_N=S_\varepsilon-S_N$$

且记

$$R_N=\varepsilon^{N+1}R^N,\quad Z_N=\varepsilon^{N+1}Z^N$$

$$a(\rho)=\frac{\varepsilon^2}{\rho},\quad b(\rho)=\varepsilon^2\rho^{-1}$$

$$f(\rho,\theta,S)=\varepsilon^2\left(\frac{\theta}{\rho^2}-P\rho^{-1}+\rho^{-1}S\theta\right)-S$$

$$g(\rho,\theta,S)=\varepsilon^2\rho^{-2}S-\varepsilon^2\theta^2(2\rho)^{-1}+\theta$$

将$\theta_\varepsilon=\theta_N+\varepsilon^{N+1}R^N$，$S_\varepsilon=S_N+\varepsilon^{N+1}Z^N$代入边值问题(2.3)和(2.4)得到$R^N$，$Z^N$满足下列

边值问题

$$\left.\begin{aligned}&\varepsilon^2\frac{d^2R^N}{d\rho^2}+a(\rho)\frac{dR^N}{d\rho}=F(R^N,Z^N)+p(\rho,\varepsilon)\\&\varepsilon^2\frac{dZ^N}{d\rho^2}+b(\rho)\frac{dZ^N}{d\rho}=G(R^N,Z^N)+q(\rho,\varepsilon)\\&R^N|_{\rho=1}=O(1),\ \left(\rho\frac{dZ^N}{d\rho}-\nu Z^N\right)\Big|_{\rho=1}=O(1)\\&Z^N|_{\rho=a}=O(1),\ \left(a\frac{dR^N}{d\rho}+\nu R^N\right)\Big|_{\rho=a}=O(1)\end{aligned}\right\}\tag{3.57}$$

其中 $F(R^N,Z^N)=\frac{1}{\varepsilon^{N+1}}[f(\rho,\theta_N+\varepsilon^{N+1}R^N,S_N+\varepsilon^{N+1}Z^N)-f(\rho,\theta_N,Z_N)]$

$$G(R^N,Z^N)=\frac{1}{\varepsilon^{N+1}}[g(\rho,\theta_N+\varepsilon^{N+1}R^N,S_N+\varepsilon^{N+1}Z^N)-g(\rho,\theta_N,Z_N)]$$

$$p(\rho,\varepsilon)=C\left(1+\frac{1}{\varepsilon}\exp[-k(1-\rho)/\varepsilon]\right)\quad(\text{当}k>0)$$

$$q(\rho;\varepsilon)=O\left(1+\frac{1}{\varepsilon}\exp[-k(\rho-a)/\varepsilon]\right)\quad(\text{当}k>0)$$

为了估计余项R^N和 Z^N，我们把边值问题(3.57)化为以下积分方程组．下面为简便起见省去了R^N和Z^N的上角标．

$$\left.\begin{aligned}R(\rho,\varepsilon)=&R_0(\rho,\varepsilon)+\frac{1}{\varepsilon(1-\varepsilon)}\int_\rho^1(u^\varepsilon-u)F(R(u),Z(u))du\\&+\int_a^1B(\rho,u,\varepsilon)F(R(u),Z(u))du\\Z(\rho,\varepsilon)=&Z_0(\rho,\varepsilon)+\frac{1}{\varepsilon(1-\varepsilon)}\int_\rho^1(u^\varepsilon-u)G(R(u),Z(u))du\\&+\int_a^1B(\rho,u,\varepsilon)G(R(u),Z(u))du\end{aligned}\right\}\tag{3.58}$$

其中

$$R_0(\rho,\varepsilon)=(R(1,\varepsilon)+\varepsilon R_\rho(1,\varepsilon))-\varepsilon R_\rho(a,\varepsilon)\left\{a^\varepsilon+\frac{1}{\varepsilon(1-\varepsilon)}a^\varepsilon(1-\rho^{1-\varepsilon})\right\}$$
$$-\int_a^1u^\varepsilon p(u,\varepsilon)du-\frac{1}{\varepsilon}\int_\rho^1\int_a^v\left(\frac{u}{v}\right)^\varepsilon p(u,\varepsilon)dudv$$

$$Z_0(\rho,\varepsilon)=(S(1,\varepsilon)+\varepsilon S_\rho(1,\varepsilon))-\varepsilon S_\rho(a,\varepsilon)\left\{a^\varepsilon+\frac{1}{\varepsilon(1-\varepsilon)}a^\varepsilon(1-\rho^{1-\varepsilon})\right\}$$
$$-\int_a^1u^\varepsilon q(u,\varepsilon)du-\frac{1}{\varepsilon}\int_\rho^1\int_a^v\left(\frac{u}{v}\right)^\varepsilon q(u,\varepsilon)dudv$$

$$B(\rho,u,\varepsilon)=-\exp\left[-\frac{1}{\varepsilon}\int_u^1a(t)dt\right]-\frac{1}{\varepsilon}\eta(\rho-u)\int_\rho^1\exp\left[-\frac{1}{\varepsilon}\int_u^v a(t)dt\right]dv$$

$$\eta(\lambda)=\begin{cases}0, & (\lambda<0)\\1, & (\lambda\geqslant0)\end{cases}$$

显然 $\int_0^1 B(\rho,u,\varepsilon)du=Q(\varepsilon)$

现在，我们把(3.58)式第二项积分中的F，G线性化得到

$$\left.\begin{aligned}R(\rho)&=R_0(\rho,\varepsilon)+\int_\rho^1 K_1(u,\varepsilon)R(u)du+\int_\rho^1 K_2(u,\varepsilon)Z(u)du\\&\quad+\int_0^1 B(\rho,u,\varepsilon)F(R(u),Z(u))du+\varepsilon^{N+1}H(\rho,R(\rho),Z(\rho))\\Z(\rho)&=Z_0(\rho,\varepsilon)+\int_\rho^1 K_3(u,\varepsilon)R(u)du+\int_\rho^1 K_4(u,\varepsilon)Z(u)du\\&\quad+\int_0^1 B(\rho,u,\varepsilon)G(R(u),Z(u))du+\varepsilon^{N+1}M(\rho,R(\rho),Z(\rho))\end{aligned}\right\}\tag{3.59}$$

其中 $(K_1(u,\varepsilon),\ K_2(u,\ \varepsilon))=-(f_\theta(u,R_N,Z_N),f_S(u,R_N,Z_N))\cdot\dfrac{u^\varepsilon-u}{\varepsilon(1-\varepsilon)}$

$$(K_3(u,\varepsilon),K_4(u,\varepsilon))=-(g_\theta(u,R_N,Z_N),g_S(u,R_N,Z_N))\cdot\frac{u^\varepsilon-u}{\varepsilon(1-\varepsilon)}$$

当R，Z有界时，H，M是有界函数．

我们把(3.59)式写成下列向量形式

$$R^*=R_0^*+J_1R^*+J_2R^*\tag{3.60}$$

其中 $R^*=\begin{pmatrix}R\\Z\end{pmatrix}$，$R_0^*=\begin{pmatrix}R_0\\Z_0\end{pmatrix}$

$$J_1R^*=\int_\rho^1 K^*(u,\varepsilon)R^*(u,\ \varepsilon)du,\ \ J_2R^*=\int_0^1 M^*(R^*,\rho,u,\varepsilon)du$$

而

$$K^*=\begin{pmatrix}K_1 & K_2\\K_3 & K_4\end{pmatrix},\ \ M^*=\begin{pmatrix}B & F\\B & G\end{pmatrix}+\varepsilon^{N+1}\begin{pmatrix}H\\M\end{pmatrix}$$

因为核K^*是有界的，所以向量积分算子J_1是可逆的，即$(I-J_1)^{-1}$存在．从而(3.60)可化为

$$R^*=(I-J_1)^{-1}R_0^*+(I-J_1)^{-1}J_2R^*\tag{3.61}$$

其中 $(I-J_1)^{-1}\phi=\phi+\int_0^1 W^*(\rho,u,\varepsilon)\phi(u)du$

是对任何ϕ的K^*的预解核．当ε充分小时，它在$\alpha\leqslant\rho$，$u\leqslant1$上是有界的（参见文献[7]）．

引进范数 $\|R^*\|=\sup(|R|,|Z|;\ \alpha\leqslant\rho\leqslant1,\ 0<\varepsilon\leqslant\varepsilon_0)$

我们有

$$\left|\int_0^1 B(\rho,u,\varepsilon)(F(R_1(u),Z_1(u))-F(R_2(u),Z_2(u)))du\right|$$

$$\leqslant O(1)\left(\int_0^1 B(\rho,u,\varepsilon)du\right)\|R_1^*-R_2^*\|=O(\varepsilon)\|R_1^*-R_2^*\|$$

同理 $\left|\int_0^1 B(\rho,u,\varepsilon)(G(R_1(u),Z_1(u))-G(R_2(u),Z_2(u)))du\right|\leqslant O(\varepsilon)\|R_1^*R_2^*\|$

又因为

$$(I-J_1)^{-1}J_2R^*=\int_0^1 M^*(R^*,\rho,u,\varepsilon)du$$

$$+\int_\rho^1 W^*(\rho,u,\varepsilon)\int_\alpha^1 M^*(R^*,u,v,\varepsilon)dvdu$$

由M^*的定义，则有

$$\|(I-J_1)^{-1}(J_2R_1^*-J_2R_2^*)\|\leqslant O(\varepsilon_0)\|R_1^*-R_2^*\|$$

利用Banach不动点定理，可知在$C[\alpha,1]\times C[\alpha,1]$上存在唯一的不动点，即对充分小的$\varepsilon>0$，在$\alpha\leqslant\rho\leqslant1$上存在唯一的连续函数组$(R,Z)$满足积分方程(3.58)。

综合上述讨论，我们有下面的定理。

定理1 当ε充分小时，边值问题(2.3)和(2.4)在$\alpha\leqslant\rho\leqslant1$上存在唯一的解$(S(\rho,\varepsilon),\theta(\rho,\varepsilon))$，且对每个整数$N\geqslant0$解可表为

$$S(\rho,\varepsilon)=S_N(\rho,\varepsilon)+\varepsilon^{N+1}Z^N(\rho,\varepsilon)$$

$$\theta(\rho,\varepsilon)=\theta_N(\rho,\varepsilon)+\varepsilon^{N+1}R^N(\rho,\varepsilon)$$

其中，S_N和θ_N由(3.38)式给出，R^N和Z^N在$\alpha\leqslant\rho\leqslant1$上一致有界。即问题(2.3)和(2.4)在$\alpha\leqslant\rho\leqslant1$上的一致有效渐近解为

$$S=\varepsilon^2\Big\{P\sqrt{\rho}\,\alpha^{-3/2}\exp\Big[-\frac{\sqrt{2}(\rho-\alpha)}{2\varepsilon}\Big]\Big(\cos\frac{\sqrt{2}(\rho-\alpha)}{2\varepsilon}-\sin\frac{\sqrt{2}(\rho-\alpha)}{2\varepsilon}\Big)-P\rho^{-1}\Big\}+\varepsilon^3\sqrt{2\rho}\,P\Big\{-(\nu+1)\cdot\exp\Big[-\frac{\sqrt{2}(1-\rho)}{2\varepsilon}\Big]\cos\frac{\sqrt{2}(1-\rho)}{2\varepsilon}+\alpha^{-\frac{1}{2}}(\nu-\alpha^2)\exp\Big[-\frac{\sqrt{2}(\rho-\alpha)}{2\varepsilon}\Big]\cdot\sin\frac{\sqrt{2}(\rho-\alpha)}{2\varepsilon}\Big\}+O(\varepsilon^4)$$

$$\theta=P\varepsilon^2\sqrt{\rho}\,\alpha^{-3/2}\exp\Big[-\frac{\sqrt{2}(\rho-\alpha)}{2\varepsilon}\Big]\Big(\sin\frac{\sqrt{2}(\rho-\alpha)}{2\varepsilon}+\cos\frac{\sqrt{2}(\rho-\alpha)}{2\varepsilon}\Big)+\sqrt{2\rho}\,P\varepsilon^3\Big\{(\nu+1)\exp\Big[-\frac{\sqrt{2}(1-\rho)}{2\varepsilon}\Big]\sin\frac{\sqrt{2}(1-\rho)}{2\varepsilon}-(\nu-\alpha^2)\alpha^{-\frac{1}{2}}\exp\Big[-\frac{\sqrt{2}(\rho-\alpha)}{2\varepsilon}\Big]\cos\frac{\sqrt{2}(\rho-\alpha)}{2\varepsilon}\Big\}+O(\varepsilon^4)$$

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Singular Perturbation Solutions of the Nonlinear Stability of a Truncated Shallow Spherical Shell under Linear Distributed Loads along the Interior Edge

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Abstract

Using a singular pertubation method, the nonlinear stability of a truncated shallow spherical shell without a nondeformable rigid body at the center under linear distributed loads along the interior edge is investigated in this paper. When the geometrical parameter k is large, the uniformly valid asymptotic solutions are obtained.

Key words shallow shell, nonlinear-stability, singular perturbation solutions