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变质量一般非完整力学系统的Routh方程

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摘　要

本文不使用任何变分原理，而直接从牛顿力学的动力学方程导出受有任意阶非线性非完整理想约束的变质量力学系统相对于一般非惯性参考系的 Routh 方程，并由它给出其它类型的运动微分方程．

关键词　Routh方程　变质量系统　非完整约束　非惯性参考系

一、引　言

在分析力学中，导出或推广基本的运动微分方程通常借助于各种变分原理，这充分表明了变分原理极其重要的地位和作用．但是，无论是微分型的变分原理还是积分型的变分原理，当对其本身作严格的理论探讨时，往往需要引入比其实际使用范围狭窄得多的限制条件[1,2]，因此不利用任何变分原理导出运动微分方程的方法在分析力学的发展中具有意义．吴大猷[3]曾由牛顿第二定律出发导出完整系的拉格朗日方程，Kane[4]和钟奉俄[5]由牛顿第二定律导出 Kane 方程．本文则从非惯性参考系下的变质量质点的动力学方程出发，仅利用理想约束条件，导出了受有任意阶非线性非完整约束的变质量力学系统的 Routh 方程，并由它给出不带乘子的拉格朗日型方程，Nielsen型方程和Appell型方程．

二、Routh 方程

1．导出Routh方程

设某一变质量力学系统由N个质点组成，受有g个完整约束和s个m阶非完整约束．现在，我们相对于任一非惯性参考系$O'x'y'z'$研究该力学系统的运动，如果$O'x'y'z'$相对惯性系$Oxyz$的运动为已知，即$O'x'y'z'$的原点位矢$\mathbf{r}_{O'}=\mathbf{r}_{O'}(t)$和绕$O'$点的转动角速度$\boldsymbol{\omega}=\boldsymbol{\omega}(t)$均为时间$t$的已知函数，系统相对$O'x'y'z'$的位形可引入广义坐标$q_\alpha$（$\alpha=1,2,\cdots,n$；$n=3N-g$）来确定．根据牛顿力学的有关动力学理论，变质量力学系统相对于非惯性参考系的动力学方程应为

$$m_i\overset{**}{\mathbf{r}}_i'=\mathbf{F}_i+\mathbf{R}_i^H+\mathbf{R}_i^N+\mathbf{F}_i^R-m_i\ddot{\mathbf{r}}_{O'}-m_i\dot{\boldsymbol{\omega}}\times\mathbf{r}_i'$$
$$-m_i\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}_i')-2m_i\boldsymbol{\omega}\times\overset{*}{\mathbf{r}}_i' \qquad (i=1,2,\cdots,N) \tag{2.1}$$

式中，$m_i=m_i(q_a,\ \dot{q}_a,\ \cdots,\ \overset{(m)}{q}_a,\ t)$为第$i$个质点的质量，$\mathbf{r}_i'$为$i$质点相对于$O'x'y'z'$的位矢，$\overset{**}{\mathbf{r}}_i'$就是它的相对加速度；$\mathbf{F}_i$为作用于$i$质点上的主动力，$\mathbf{R}_i^H$、$\mathbf{R}_i^N$分别表示完整约束和非完整约束的约束反力，$\mathbf{F}_i^R=\dot{m}_i\mathbf{u}_i$为反冲力，其中$\mathbf{u}_i$是微粒对$i$质点的相对分离或并入的速度。将(2.1)中第$i$式点乘$\frac{\tilde{\partial}\mathbf{r}_i'}{\partial q_a}$ $(i=1,2,\cdots,N)$，然后把它们相加可得

$$\sum_{i=1}^N m_i\overset{**}{\mathbf{r}}_i'\cdot\frac{\tilde{\partial}\mathbf{r}_i'}{\partial q_a}=\sum_{i=1}^N\mathbf{F}_i\cdot\frac{\tilde{\partial}\mathbf{r}_i'}{\partial q_a}+\sum_{i=1}^N\mathbf{R}_i^H\cdot\frac{\tilde{\partial}\mathbf{r}_i'}{\partial q_a}+\sum_{i=1}^N\mathbf{R}_i^N\cdot\frac{\tilde{\partial}\mathbf{r}_i'}{\partial q_a}$$
$$+\sum_{i=1}^N\mathbf{F}_i^R\cdot\frac{\tilde{\partial}\mathbf{r}_i'}{\partial q_a}-\sum_{i=1}^N m_i\ddot{\mathbf{r}}_{O'}\cdot\frac{\tilde{\partial}\mathbf{r}_i'}{\partial q_a}-\sum_{i=1}^N m_i(\dot{\boldsymbol{\omega}}\times\mathbf{r}_i')\cdot\frac{\tilde{\partial}\mathbf{r}_i'}{\partial q_a}$$
$$-\sum_{i=1}^N m_i[\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}_i')]\cdot\frac{\tilde{\partial}\mathbf{r}_i'}{\partial q_a}-\sum_{i=1}^N 2m_i(\boldsymbol{\omega}\times\overset{*}{\mathbf{r}}_i')\cdot\frac{\tilde{\partial}\mathbf{r}_i'}{\partial q_a}$$
$$(a=1,2,\cdots,n) \tag{2.2}$$

本文采用Kane[4]的观点，即认为求矢量的各种导数均依赖于参考系．设$\frac{d}{dt}$，$\frac{\partial}{\partial q_a}$，$\frac{\partial}{\partial t}$表示在惯性系$Oxyz$中对矢量作导数运算；$\frac{\bar{d}}{dt}$，$\frac{\bar{\partial}}{\partial q_a}$，$\frac{\bar{\partial}}{\partial t}$为平动系$O'xyz$中对矢量作导数运算；$\frac{\tilde{d}}{dt}$，$\frac{\tilde{\partial}}{\partial q_a}$，$\frac{\tilde{\partial}}{\partial t}$为非惯性系$O'x'y'z'$中对矢量作导数运算．由于对标量的导数不依赖于参考系，故计算标量导数时，三类导数记号具有相同的含义．因为

$$\mathbf{r}_i(q_a,\ t)=\mathbf{r}_{O'}(t)+\mathbf{r}_i'(q_a,\ t) \tag{2.3}$$

故有

$$\left.\begin{aligned}&\frac{\partial\mathbf{r}_i'}{\partial q_a}=\frac{\tilde{\partial}\mathbf{r}_i'}{\partial q_a} &&\frac{\partial\mathbf{r}_i}{\partial q_a}=\frac{\tilde{\partial}\mathbf{r}_i'}{\partial q_a}\\ &\frac{d\mathbf{r}_i'}{dt}=\frac{\tilde{d}}{dt}\mathbf{r}_i'+\boldsymbol{\omega}\times\mathbf{r}_i' &&\frac{d\mathbf{r}_i}{dt}=\frac{\tilde{d}}{dt}\mathbf{r}_i+\boldsymbol{\omega}\times\mathbf{r}_i\end{aligned}\right\} \tag{2.4}$$

其中$\frac{\tilde{d}\mathbf{r}_i'}{dt}=\overset{*}{\mathbf{r}}_i'$就是$i$质点的相对速度．若引入凝固导数和凝固偏导数记号，令$\frac{\varDelta}{\varDelta\dot{q}_a}$，$\frac{\varDelta}{\varDelta q_a}$分别表示把质量当常数时在$Oxyz$中对$\dot{q}_a$、$q_a$的偏导数，$\frac{D}{Dt}$为将质量当常数时在$Oxyz$中对时间的导数，余类推．这样一来，(2.2)式左边

$$\sum m_i\overset{**}{\mathbf{r}}_i'\cdot\frac{\partial\mathbf{r}_i'}{\partial q_a}=\frac{\tilde{D}}{Dt}\frac{\tilde{\varDelta}T_r}{\varDelta\dot{q}_a}-\frac{\tilde{\varDelta}T_r}{\varDelta q_a}=\frac{D}{Dt}\frac{\varDelta T_r}{\varDelta\dot{q}_a}-\frac{\varDelta T_r}{\varDelta q_a}$$

其中$T_r=\sum_{i=1}^N\frac{1}{2}m_i\overset{*}{\mathbf{r}}_i'\cdot\overset{*}{\mathbf{r}}_i'$为系统的总相对动能．而(2.2)式右边各项为

广义主动力 $Q_\alpha \triangleq \sum_{i=1}^{N} \mathbf{F}_i \cdot \frac{\widetilde{\partial} \mathbf{r}_i'}{\partial q_\alpha} = \sum_{i=1}^{N} \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_\alpha}$

广义反冲力 $Q_\alpha^R \triangleq \sum_{i=1}^{N} \mathbf{F}_i^R \cdot \frac{\widetilde{\partial} \mathbf{r}_i'}{\partial q_\alpha} = \sum_{i=1}^{N} \mathbf{F}_i^R \cdot \frac{\partial \mathbf{r}_i}{\partial q_\alpha}$

广义牵连转动惯性力

$$Q_\alpha^{\dot{\omega}} - \frac{\Pi}{\Pi q_\alpha} V^\omega \triangleq -\sum_{i=1}^{N} m_i (\dot{\omega} \times \mathbf{r}_i') \cdot \frac{\widetilde{\partial} \mathbf{r}_i'}{\partial q_\alpha} - \sum_{i=1}^{N} m_i [\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_i')] \cdot \frac{\widetilde{\partial} \mathbf{r}_i'}{\partial q_\alpha}$$

$$= -\sum_{i=1}^{N} m_i (\dot{\omega} \times \mathbf{r}_i') \cdot \frac{\widetilde{\partial} \mathbf{r}_i'}{\partial q_\alpha} - \frac{\widetilde{\Pi}}{\Pi q_\alpha} \left[-\frac{1}{2} \boldsymbol{\omega} \cdot \boldsymbol{I}_{O'} \cdot \boldsymbol{\omega} \right]$$

其中$\boldsymbol{I}_{O'}$表示力学系统对O'点的惯量张量。

广义牵连平动惯性力

$$-\frac{\Pi}{\Pi q_\alpha} V^{O'} \triangleq -\sum_{i=1}^{N} m_i \ddot{\mathbf{r}}_{O'} \cdot \frac{\widetilde{\partial} \mathbf{r}_i'}{\partial q_\alpha} = -\frac{\widetilde{\Pi}}{\Pi q_\alpha} [M(\overset{*}{v}_{O'} + \boldsymbol{\omega} \times \mathbf{v}_{O'}) \cdot \mathbf{r}_c']$$

广义科里奥利惯性力

$$\Gamma_\alpha \triangleq -\sum_{i=1}^{N} 2m_i (\boldsymbol{\omega} \times \overset{*}{\mathbf{r}}_i') \cdot \frac{\widetilde{\partial} \mathbf{r}_i'}{\partial q_\alpha}$$

$$= -2\boldsymbol{\omega} \cdot \left\{ \sum_{k=1}^{n} \left(\sum_{i=1}^{N} m_i \frac{\widetilde{\partial} \mathbf{r}_i'}{\partial q_k} \times \frac{\widetilde{\partial} \mathbf{r}_i'}{\partial q_\alpha} \right) \dot{q}_k + \sum_{i=1}^{N} m_i \frac{\widetilde{\partial} \mathbf{r}_i'}{\partial t} \times \frac{\widetilde{\partial} \mathbf{r}_i'}{\partial q_\alpha} \right\}$$

如果约束都是理想的，则对于完整的理想约束应有

$$\sum_{i=1}^{N} \mathbf{R}_i^H \cdot \frac{\widetilde{\partial} \mathbf{r}_i'}{\partial q_\alpha} = 0 \qquad (\alpha = 1, 2, \cdots, n)$$

而对m阶非完整的理想约束

$$\varphi_\rho(q_\alpha, \dot{q}_\alpha, \cdots, \overset{(m)}{q}_\alpha, t) = 0 \qquad (\rho = 1, 2, \cdots, s) \tag{2.5}$$

其广义约束反力[6]

$$N_\alpha \triangleq \sum_{i=1}^{N} \mathbf{R}_i^N \cdot \frac{\widetilde{\partial} \mathbf{r}_i'}{\partial q_\alpha} = \sum_{\rho=1}^{s} \lambda_\rho \frac{\partial \varphi_\rho}{\partial \overset{(m)}{q}_\alpha} \qquad (\alpha = 1, 2, \cdots, n) \tag{2.6}$$

将以上结果代入(2.2)式，我们得到

$$\frac{D}{Dt} \frac{\Pi T_r}{\Pi \dot{q}_\alpha} - \frac{\Pi T_r}{\Pi q_\alpha} = Q_\alpha + Q_\alpha^R + Q_\alpha^{\dot{\omega}} - \frac{\Pi}{\Pi q_\alpha}(V^{O'} + V^\omega)$$

$$+ \Gamma_\alpha + \sum_{\rho=1}^{s} \lambda_\rho \frac{\partial \varphi_\rho}{\partial \overset{(m)}{q}_\alpha} \qquad (\alpha = 1, 2, \cdots, n) \tag{2.7}$$

上式就是既受有完整约束，又受有一般非完整约束的变质量力学系统相对于非惯性参考系的动力学方程——Routh方程，其中λ_ρ为拉格朗日乘子。

2. 充分性讨论

我们论证由方程(2.7)可导出方程(2.1)．在力学系统初刻条件给定，诸作用力、惯性力和约束方程给定的条件下，力学系统的运动将由方程(2.7)唯一确定．根据前面的推导，由(2.7)可得出(2.2)，即有

$$\sum_{i=1}^{N}\{\mathbf{F}_i+\mathbf{F}_i^R+\mathbf{R}_i^H+\mathbf{R}_i^N-m_i\ddot{\mathbf{r}}_{O'}-m_i\dot{\boldsymbol{\omega}}\times\mathbf{r}_i'-m_i\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}_i')$$

$$-2m_i(\boldsymbol{\omega}\times\overset{*}{\mathbf{r}}_i')-m_i\overset{**}{\mathbf{r}}_i'\}\cdot\frac{\widetilde{\partial}\mathbf{r}_i'}{\partial q_a}=0\qquad(a=1,2,\cdots,n)\tag{2.8}$$

在一般情况下，N项中的大括弧因子可能分别等于零，也可能全部或部分非零．不失普遍性可设

$$\mathbf{F}_i+\mathbf{F}_i^R+\mathbf{R}_i^H+\mathbf{R}_i^N-m_i\ddot{\mathbf{r}}_{O'}-m_i\dot{\boldsymbol{\omega}}\times\mathbf{r}_i'-m_i\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}_i')$$
$$-2m_i(\boldsymbol{\omega}\times\overset{*}{\mathbf{r}}_i')-m_i\overset{**}{\mathbf{r}}_i'\triangleq\mathbf{f}_i\qquad(i=1,2,\cdots,N)\tag{2.9}$$

则(2.8)式化为

$$\sum_{i=1}^{N}\mathbf{f}_i\cdot\frac{\widetilde{\partial}\mathbf{r}_i'}{\partial q_a}=0$$

即可得到

$$\sum_{a=1}^{n}\sum_{i=1}^{N}\mathbf{f}_i\cdot\frac{\widetilde{\partial}\mathbf{r}_i'}{\partial q_a}\,\delta\overset{(m)}{q_a}=0\tag{2.10}$$

将(2.9)中的 $\mathbf{f}_i$ 移到等号左边和 $\mathbf{R}_i^N$ 合并，限令 $\mathbf{R}_i^N-\mathbf{f}_i\triangleq\mathbf{R}_i'$，注意到 m 阶非完整理想约束条件为[6]

$$\sum_{i=1}^{N}\mathbf{R}_i^N\cdot\delta\overset{(m)}{\mathbf{r}_i'}=\sum_{a=1}^{n}\sum_{i=1}^{N}\mathbf{R}_i^N\cdot\frac{\widetilde{\partial}\overset{(m)}{\mathbf{r}_i'}}{\partial\overset{(m)}{q_a}}\,\delta\overset{(m)}{q_a}$$

$$=\sum_{a=1}^{n}\sum_{i=1}^{N}\mathbf{R}_i^N\cdot\frac{\widetilde{\partial}\mathbf{r}_i'}{\partial q_a}\,\delta\overset{(m)}{q_a}=0\tag{2.11}$$

则由(2.10)、(2.11)有

$$\sum_{i=1}^{N}\mathbf{R}_i'\cdot\delta\overset{(m)}{\mathbf{r}_i'}=0\tag{2.12}$$

上式表明，$\mathbf{R}_i'$ $(i=1,2,\cdots,N)$ 仍为满足理想约束条件的m阶非完整约束反力，因此我们得到

$$\mathbf{F}_i+\mathbf{F}_i^R+\mathbf{R}_i^H+\mathbf{R}_i'-m_i\ddot{\mathbf{r}}_{O'}-m_i\dot{\boldsymbol{\omega}}\times\mathbf{r}_i'-m_i\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}_i')$$
$$-2m_i\boldsymbol{\omega}\times\overset{*}{\mathbf{r}}_i'=m_i\overset{**}{\mathbf{r}}_i'\qquad(i=1,2,\cdots,N)\tag{2.13}$$

从式中各量的含义看，这就是牛顿力学中变质量质点在非惯性参考系下的动力学方程．由于在初刻条件给定，作用力、惯性力和约束方程给定的条件下，力学系统的运动是唯一确定的，因此不难得出非完整约束的约束反力均是唯一的，即有 $\mathbf{R}_i^N=\mathbf{R}_i'$ $(i=1,2,\cdots,N)$，则(2.13)就是前面的(2.1)．

三、由Routh方程导出其它类型的方程

1. 导出拉格朗日型方程

令方程(2.7)中的

$$\frac{D}{Dt}\frac{\Delta T_r}{\Delta \dot{q}_\alpha}-\frac{\Delta T_r}{\Delta q_\alpha}-Q_\alpha-Q_\alpha^R-Q_\alpha^{\dot{\omega}}+\frac{\Delta}{\Delta q_\alpha}(V^{o'}+V^{\omega})-\Gamma_\alpha\triangleq a_\alpha$$

$$(\alpha=1,2,\cdots,n) \tag{3.1}$$

则(2.7)可以表为

$$\left.\begin{aligned}
&\lambda_1\frac{\partial\varphi_1}{\partial \overset{(m)}{q}_1}+\lambda_2\frac{\partial\varphi_2}{\partial \overset{(m)}{q}_1}+\cdots+\lambda_s\frac{\partial\varphi_s}{\partial \overset{(m)}{q}_1}=a_1\\
&\lambda_1\frac{\partial\varphi_1}{\partial \overset{(m)}{q}_2}+\lambda_2\frac{\partial\varphi_2}{\partial \overset{(m)}{q}_2}+\cdots+\lambda_s\frac{\partial\varphi_s}{\partial \overset{(m)}{q}_2}=a_2\\
&\qquad\vdots\qquad\qquad\qquad\qquad\quad\vdots\\
&\lambda_1\frac{\partial\varphi_1}{\partial \overset{(m)}{q}_n}+\lambda_2\frac{\partial\varphi_2}{\partial \overset{(m)}{q}_n}+\cdots+\lambda_s\frac{\partial\varphi_s}{\partial \overset{(m)}{q}_n}=a_n
\end{aligned}\right\} \tag{3.2}$$

由于力学系统受有s个m阶非完整约束(2.5)，则函数φ_1，φ_2，…，φ_s应互相独立，故至少存在一个s阶雅可比行列式不等于零[7]，不失普遍性，可设(3.2)中最末s个方程式系数构成的雅可比行列式非零(通过变换方程式的顺序和更换变量的编号，必可达到)，即

$$\begin{vmatrix}
\frac{\partial\varphi_1}{\partial \overset{(m)}{q}_{n-s+1}} & \frac{\partial\varphi_2}{\partial \overset{(m)}{q}_{n-s+1}} & \cdots & \frac{\partial\varphi_s}{\partial \overset{(m)}{q}_{n-s+1}}\\
\frac{\partial\varphi_1}{\partial \overset{(m)}{q}_{n-s+2}} & \frac{\partial\varphi_2}{\partial \overset{(m)}{q}_{n-s+2}} & \cdots & \frac{\partial\varphi_s}{\partial \overset{(m)}{q}_{n-s+2}}\\
\vdots & & & \vdots\\
\frac{\partial\varphi_1}{\partial \overset{(m)}{q}_n} & \frac{\partial\varphi_2}{\partial \overset{(m)}{q}_n} & \cdots & \frac{\partial\varphi_s}{\partial \overset{(m)}{q}_n}
\end{vmatrix}\neq 0 \tag{3.3}$$

那末，由(3.2)最末的s个方程必可解出s个λ_ρ来．为此目的，可将(2.3)的前$n-s$个方程和余下的后s个方程分别表为矩阵形式

$$\Phi_1\Lambda=A_1 \tag{3.4}$$

$$\Phi_2\Lambda=A_2 \tag{3.5}$$

其中

$$\Phi_1=\begin{bmatrix}
\frac{\partial\varphi_1}{\partial \overset{(m)}{q}_1} & \frac{\partial\varphi_2}{\partial \overset{(m)}{q}_1} & \cdots & \frac{\partial\varphi_s}{\partial \overset{(m)}{q}_1}\\
\vdots & & & \vdots\\
\frac{\partial\varphi_1}{\partial \overset{(m)}{q}_{n-s}} & \frac{\partial\varphi_2}{\partial \overset{(m)}{q}_{n-s}} & \cdots & \frac{\partial\varphi_s}{\partial \overset{(m)}{q}_{n-s}}
\end{bmatrix},$$

$$\Phi_2=\begin{bmatrix} \dfrac{\partial\varphi_1}{\partial \overset{(m)}{q}_{n-s+1}} & \dfrac{\partial\varphi_2}{\partial \overset{(m)}{q}_{n-s+1}} & \cdots & \dfrac{\partial\varphi_s}{\partial \overset{(m)}{q}_{n-s+1}} \\ \vdots & & & \vdots \\ \dfrac{\partial\varphi_1}{\partial \overset{(m)}{q}_{n}} & \dfrac{\partial\varphi_2}{\partial \overset{(m)}{q}_{n}} & \cdots & \dfrac{\partial\varphi_s}{\partial \overset{(m)}{q}_{n}} \end{bmatrix}$$

$$A_1=\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-s} \end{bmatrix},\quad A_2=\begin{bmatrix} a_{n-s+1} \\ a_{n-s+2} \\ \vdots \\ a_n \end{bmatrix},\quad \Lambda=\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix}$$

因有条件(3.3)，Φ_2必存在逆阵Φ_2^{-1}，由(3.4)和(3.5)可得

$$\Phi_1\Phi_2^{-1}A_2=A_1 \tag{3.6}$$

引入 $C^T\triangleq-\Phi_1\Phi_2^{-1}$，并将(3.6)还原为$n-s$个不含乘子$\lambda_\rho$的方程，即有

$$a_\nu=-\sum_{\alpha=1}^{s}C_{\alpha\nu}a_{n-s+\alpha}\qquad(\nu=1,2,\cdots,n-s)$$

也就是

$$\frac{D}{Dt}\frac{\varDelta T_r}{\varDelta\dot{q}_\nu}-\frac{\varDelta T_r}{\varDelta q_\nu}-Q_\nu-Q_\nu^R-Q_\nu^{\dot{\omega}}+\frac{\varDelta}{\varDelta q_\nu}(V^{o'}+V^{\omega})-\Gamma_\nu$$

$$=-\sum_{\alpha=1}^{s}\left\{\frac{D}{Dt}\frac{\varDelta T_r}{\varDelta\dot{q}_{n-s+\alpha}}-\frac{\varDelta T_r}{\varDelta q_{n-s+\alpha}}-Q_{n-s+\alpha}-Q_{n-s+\alpha}^R-Q_{n-s+\alpha}^{\dot{\omega}}\right.$$

$$\left.+\frac{\varDelta}{\varDelta q_{n-s+\alpha}}(V^{o'}+V^{\omega})-\Gamma_{n-s+\alpha}\right\}C_{\alpha\nu}\qquad(\nu=1,2,\cdots,n-s) \tag{3.7}$$

上式就是受有完整约束和任意阶非完整约束的变质量力学系统相对于非惯性参考系所满足的拉格朗日方程，它是文献[5]中所得(28)式的推广。

当m阶非完整约束方程设为

$$\overset{(m)}{q}_{n-s+\beta}=f_\beta(q_\sigma,\ \dot{q}_\sigma,\ \cdots,\ \overset{(m-1)}{q}_\sigma,\ \overset{(m)}{q}_\sigma,\ t)$$

$$(\beta=1,2,\cdots,s;\ \sigma=1,2,\cdots,n-s) \tag{3.8}$$

时，可将上式看作

$$\varphi_\beta=f_\beta(q_\sigma,\ \dot{q}_\sigma,\ \cdots,\ \overset{(m-1)}{q}_\sigma,\ \overset{(m)}{q}_\sigma,\ t)-\overset{(m)}{q}_{n-s+\beta}=0$$

因此可以求得 $C=\Phi_1^T$，即有

$$C_{\alpha\nu}=\frac{\partial f_\alpha}{\partial\overset{(m)}{q}_\nu}\qquad(\alpha=1,2,\cdots,s;\ \nu=1,2,\cdots,n-s)$$

则(3.7)化为Maggi型方程[8]

$$\frac{D}{Dt}\frac{\varDelta T_r}{\varDelta\dot{q}_\nu}-\frac{\varDelta T_r}{\varDelta q_\nu}-Q_\nu-Q_\nu^R-Q_\nu^{\dot{\omega}}+\frac{\varDelta}{\varDelta q_\nu}(V^{o'}+V^{\omega})-\Gamma_\nu$$

$$=-\sum_{\alpha=1}^{s}\left\{\frac{D}{Dt}\frac{\varDelta T_r}{\varDelta\dot{q}_{n-s+\alpha}}-\frac{\varDelta T_r}{\varDelta q_{n-s+\alpha}}-Q_{n-s+\alpha}-Q_{n-s+\alpha}^R-Q_{n-s+\alpha}^{\dot{\omega}}\right.$$

$$+\frac{Д}{Дq_{n-s+a}}(V^{0'}+V^{\omega})-\Gamma_{n-s+a}\Big\}\frac{\partial f_a}{\partial \overset{(m)}{q}_\nu}\qquad(\nu=1,2,\cdots,n-s)\tag{3.9}$$

2. **导出Nielsen型方程**

通过对T_r直接计算凝固导数和凝固偏导数可以得到

$$\frac{D}{Dt}\frac{ДT_r}{Д\dot{q}_a}-\frac{ДT_r}{Дq_a}=m\frac{D}{Dt}\frac{Д}{Д\overset{(m)}{q}_a}\frac{D^{(m-1)}T_r}{Dt^{(m-1)}}-\frac{Д}{Д\overset{(m-1)}{q}_a}\frac{D^{(m-1)}T_r}{Dt^{(m-1)}}\tag{3.10}$$

由(3.10)可将(3.9)式变为

$$m\frac{D}{Dt}\frac{Д}{Д\overset{(m)}{q}_\nu}\frac{D^{(m-1)}T_r}{Dt^{(m-1)}}-\frac{Д}{Д\overset{(m-1)}{q}_\nu}\frac{D^{(m-1)}T_r}{Dt^{(m-1)}}-Q_\nu-Q_\nu^R-Q_\nu^{\dot{\omega}}+\frac{Д}{Дq_\nu}(V^{0'}+V^{\omega})-\Gamma_\nu$$
$$+\sum_{a=1}^{s}\Big\{m\frac{D}{Dt}\frac{Д}{Д\overset{(m)}{q}_{n-s+a}}\frac{D^{(m-1)}T_r}{Dt^{(m-1)}}-\frac{Д}{Д\overset{(m-1)}{q}_{n-s+a}}\frac{D^{(m-1)}T_r}{Dt^{(m-1)}}-Q_{n-s+a}-Q^R_{n-s+a}-Q^{\dot{\omega}}_{n-s+a}$$
$$+\frac{Д}{Дq_{n-s+a}}(V^{0'}+V^{\omega})-\Gamma_{n-s+a}\Big\}\frac{\partial f_a}{\partial \overset{(m)}{q}_\nu}=0\qquad(\nu=1,2,\cdots,n-s)\tag{3.11}$$

令$\frac{D^{(m-1)}\tilde{T}_r}{Dt^{(m-1)}}$代表$\frac{D^{(m-1)}T_r}{Dt^{(m-1)}}$中已用(3.8)消去不独立的$\overset{(m)}{q}_{n-s+\beta}$而得到的表达式，则有

$$\left.\begin{aligned}&\frac{D}{Dt}\frac{Д}{Д\overset{(m)}{q}_\nu}\frac{D^{(m-1)}\tilde{T}_r}{Dt^{(m-1)}}=\frac{D}{Dt}\frac{Д}{Д\overset{(m)}{q}_\nu}\frac{D^{(m-1)}T_r}{Dt^{(m-1)}}\\&\qquad+\sum_{a=1}^{s}\frac{D}{Dt}\frac{Д}{Д\overset{(m)}{q}_{n-s+a}}\frac{D^{(m-1)}T_r}{Dt^{(m-1)}}\cdot\frac{f_a}{\overset{(m)}{q}_\nu}+\sum_{a=1}^{s}\frac{Д}{Д\overset{(m)}{q}_{n-s+a}}\frac{D^{(m-1)}T_r}{Dt^{(m-1)}}\frac{d}{dt}\frac{f_a}{\overset{(m)}{q}_\nu}\\&\frac{Д}{Д\overset{(m-1)}{q}_\nu}\frac{D^{(m-1)}\tilde{T}_r}{Dt^{(m-1)}}=\frac{Д}{Д\overset{(m-1)}{q}_\nu}\frac{D^{(m-1)}T_r}{Dt^{(m-1)}}+\sum_{a=1}^{s}\frac{Д}{Д\overset{(m)}{q}_{n-s+a}}\frac{D^{(m-1)}T_r}{Dt^{(m-1)}}\frac{\partial f_a}{\partial\overset{(m)}{q}_\nu}\end{aligned}\right\}\tag{3.12}$$

将(3.12)代入(3.11)，我们得到

$$m\frac{D}{Dt}\frac{Д}{Д\overset{(m)}{q}_\nu}\frac{D^{(m-1)}\tilde{T}_r}{Dt^{(m-1)}}-\frac{Д}{Д\overset{(m-1)}{q}_\nu}\frac{D^{(m-1)}\tilde{T}_r}{Dt^{(m-1)}}-\sum_{a=1}^{s}\frac{Д}{Д\overset{(m)}{q}_{n-s+a}}\frac{D^{(m-1)}T_r}{Dt^{(m-1)}}\Big(m\frac{d}{dt}\frac{f_a}{\overset{(m)}{q}_\nu}$$
$$-\frac{f_a}{\partial\overset{(m-1)}{q}_\nu}\Big)-\sum_{a=1}^{s}\frac{Д}{Д\overset{(m-1)}{q}_{n-s+a}}\frac{D^{(m-1)}T_r}{Dt^{(m-1)}}\frac{\partial f_a}{\partial\overset{(m)}{q}_\nu}=Q_\nu+Q_\nu^R+Q_\nu^{\dot{\omega}}-\frac{Д}{Дq_\nu}(V^{0'}+V^{\omega})+\Gamma_\nu$$
$$+\sum_{a=1}^{s}\Big\{Q_{n-s+a}+Q^R_{n-s+a}+Q^{\dot{\omega}}_{n-s+a}-\frac{Д}{Дq_{n-s+a}}(V^{0'}+V^{\omega})+\Gamma_{n-s+a}\Big\}\frac{f_a}{\partial\overset{(m)}{q}_\nu}$$
$$(\nu=1,2,\cdots,n-s)\tag{3.13}$$

上式即为广义Чаплыгин型方程。

注意到还有下述关系式成立：

$$\frac{Д}{Д\overset{(m)}{q}_\nu}\frac{D^{(m)}\tilde{T}_r}{Dt^{(m)}}=\frac{D}{Dt}\frac{Д}{Д\overset{(m)}{q}_\nu}\frac{D^{(m-1)}\tilde{T}_r}{Dt^{(m-1)}}+\frac{Д}{Д\overset{(m-1)}{q}_\nu}\frac{D^{(m-1)}\tilde{T}_r}{Dt^{(m-1)}}$$

$$\left.\begin{aligned}&+\sum_{\alpha=1}^{s}\frac{\varDelta}{\varDelta\overset{(m-1)}{q}_{n-s+\alpha}}\frac{\tilde{D}^{(m-1)}\tilde{T}_r}{Dt^{(m-1)}}\frac{\partial f_\alpha}{\partial\overset{(m)}{q}_\nu}\\ &\frac{\partial\dot{f}_\alpha}{\partial\overset{(m)}{q}_\nu}=\frac{d}{dt}\frac{\partial f_\alpha}{\partial\overset{(m)}{q}_\nu}+\frac{\partial f_\alpha}{\partial\overset{(m-1)}{q}_\nu}+\sum_{\beta=1}^{s}\frac{\partial f_\alpha}{\partial\overset{(m-1)}{q}_{n-s+\beta}}\frac{\partial f_\beta}{\partial\overset{(m)}{q}_\nu}\end{aligned}\right\}\tag{3.14}$$

将上式代入方程(3.13)，就得到广义Nielsen型方程

$$\begin{aligned}&m\frac{\varDelta}{\varDelta\overset{(m)}{q}_\nu}\frac{D^{(m)}\tilde{T}_r}{Dt^{(m)}}-(m+1)\frac{\varDelta}{\varDelta\overset{(m-1)}{q}_\nu}\frac{D^{(m-1)}\tilde{T}_r}{Dt^{(m-1)}}\\ &-(m+1)\sum_{\alpha=1}^{s}\frac{\varDelta}{\varDelta\overset{(m-1)}{q}_{n-s+\alpha}}\frac{D^{(m-1)}T_r}{Dt^{(m-1)}}\frac{\partial f_\alpha}{\partial\overset{(m)}{q}_\nu}-\sum_{\alpha=1}^{s}\frac{\varDelta}{\varDelta\overset{(m)}{q}_{n-s+\alpha}}\frac{D^{(m-1)}T_r}{Dt^{(m-1)}}\Bigg[m\frac{\partial f_\alpha}{\partial\overset{(m)}{q}_\nu}\\ &-(m+1)\frac{\partial f_\alpha}{\partial\overset{(m-1)}{q}_\nu}\Bigg]=Q_\nu+Q_\nu^R+Q_\nu^{\dot\omega}-\frac{\varDelta}{\varDelta q_\nu}(V^{O'}+V^{\omega})+\Gamma_\nu+\sum_{\alpha=1}^{s}\Big\{Q_{n-s+\alpha}\\ &+Q^R_{n-s+\alpha}+Q^{\dot\omega}_{n-s+\alpha}-\frac{\varDelta}{\varDelta q_{n-s+\alpha}}(V^{O'}+V^{\omega})+\Gamma_{n-s+\alpha}\Big\}\frac{\partial f_\alpha}{\partial\overset{(m)}{q}_\nu}\qquad(\nu=1,2,\cdots,n-s)\end{aligned}\tag{3.15}$$

3. 导出Appell型方程

现在，我们仍由方程(2.7)导出Appell型方程．先对 T_r 直接计算凝固导数和凝固偏导数，则有

$$\left.\begin{aligned}&\frac{\tilde{D}}{Dt}\frac{\tilde{\varDelta}T_r}{\varDelta\dot{q}_a}=\sum_{i=1}^{N}m_i\overset{**}{r}_i'\cdot\frac{\tilde{\partial}\overset{*}{r}_i'}{\partial\dot{q}_a}+\sum_{i=1}^{N}m_i\overset{*}{r}_i'\cdot\frac{\tilde{d}}{dt}\frac{\tilde{\partial}\overset{*}{r}_i'}{\partial\dot{q}_a}\\ &\frac{\tilde{\varDelta}T_r}{\varDelta q_a}=\sum_{i=1}^{N}m_i\overset{*}{r}_i'\cdot\frac{\tilde{\partial}\overset{*}{r}_i'}{\partial q_a}\end{aligned}\right\}\tag{3.16}$$

不难证明还有以下关系式成立：

$$\frac{\tilde{d}}{dt}\frac{\tilde{\partial}\overset{(k)}{r}_i'}{\partial\overset{(k)}{q}_a}=\frac{1}{(k+1)}\frac{\tilde{\partial}\overset{(k+1)}{r}_i'}{\partial\overset{(k)}{q}_a},\quad\frac{\tilde{\partial}\overset{(k)}{r}_i'}{\partial\overset{(k)}{q}_a}=\frac{\tilde{\partial}r_i'}{\partial q_a}\qquad(k=0,1,\cdots)\tag{3.17}$$

对力学系统的总相对加速度能S_r直接计算凝固偏导数并利用(3.17)，得

$$\frac{\varDelta S_r}{\varDelta\ddot{q}_a}=\frac{\tilde{\varDelta}S_r}{\varDelta\ddot{q}_a}=\sum_{i=1}^{N}m_i\overset{**}{r}_i'\cdot\frac{\tilde{\partial}\overset{**}{r}_i'}{\partial\ddot{q}_a}$$

因此有

$$\frac{D}{Dt}\frac{\varDelta T_r}{\varDelta\dot{q}_a}-\frac{\varDelta T_r}{\varDelta q_a}=\frac{\tilde{D}}{Dt}\frac{\tilde{\varDelta}T_r}{\varDelta\dot{q}_a}-\frac{\tilde{\varDelta}T_r}{\varDelta q_a}=\frac{\varDelta S_r}{\varDelta\ddot{q}_a}\tag{3.18}$$

将(3.18)代入(2.7)，就得

$$\frac{\varPi S_r}{\varPi \ddot{q}_a}=Q_a+Q_a^R+Q_a^{\dot{\omega}}-\frac{\varPi}{\varPi q_a}(V^{O'}+V^{\omega})+\Gamma_a$$

$$+\sum_{\rho=1}^{s}\lambda_\rho\frac{\partial\varphi_\rho}{\partial \overset{(m)}{q_a}}\qquad(a=1,2,\cdots,n)\qquad(3.18)'$$

这就是带有拉格朗日乘子的Appell型方程。

按照和先前类似的方法从(3.18)中消去乘子λ_ρ，就得到不带乘子的Appell方程

$$\frac{\varPi S_r}{\varPi \ddot{q}_\nu}+\sum_{a=1}^{s}\frac{\varPi S_r}{\varPi \ddot{q}_{n-s+a}}C_{a\nu}=Q_\nu+Q_\nu^R+Q_\nu^{\dot{\omega}}-\frac{\varPi}{\varPi q_\nu}(V^{O'}+V^{\omega})+\Gamma_\nu$$

$$+\sum_{a=1}^{s}\left\{Q_{n-s+a}+Q_{n-s+a}^R+Q_{n-s+a}^{\dot{\omega}}-\frac{\varPi}{\varPi q_{n-s+a}}(V^{O'}+V^{\omega})+\Gamma_{n-s+a}\right\}C_{a\nu}$$

$$(\nu=1,2,\cdots,n-s)\qquad(3.19)$$

特别地，当m阶非完整约束方程为(3.8)式时，$C_{a\nu}=\frac{\partial f_a}{\partial \overset{(m)}{q_\nu}}$。如果令$\tilde{S}_r$表示从$S_r$中借助(3.8)消去不独立的$\overset{(m)}{q}_{n-s+a}$而得到的表达式，则有

$$\frac{\varPi \tilde{S}_r}{\varPi \ddot{q}_\nu}=\frac{\varPi S_r}{\varPi \ddot{q}_\nu}+\sum_{a=1}^{s}\frac{\varPi S_r}{\varPi \overset{(m)}{q}_{n-s+a}}\frac{\partial f_a}{\partial \overset{(m)}{q_\nu}}\qquad(3.20)$$

于是(3.19)可以表为

$$\frac{\varPi \tilde{S}_r}{\varPi \ddot{q}_\nu}=Q_\nu+Q_\nu^R+Q_\nu^{\dot{\omega}}-\frac{\varPi}{\varPi q_\nu}(V^{O'}+V^{\omega})+\Gamma_\nu$$

$$+\sum_{a=1}^{s}\left\{Q_{n-s+a}+Q_{n-s+a}^R+Q_{n-s+a}^{\dot{\omega}}-\frac{\varPi}{\varPi q_{n-s+a}}(V^{O'}+V^{\omega})+\Gamma_{n-s+a}\right\}\frac{\partial f_a}{\partial \overset{(m)}{q_\nu}}$$

$$(\nu=1,2,\cdots,n-s)\qquad(3.21)$$

这是Appell型方程的又一种形式。

四、例 子

例1 变质量质点的质量为$m=m(t)$，在以O为力心的牛顿引力场中运动。相对于原点在O、以匀角速$\boldsymbol{\omega}$绕定轴转动的非惯性参考系，质点的运动受有速度为常数的非线性非完整约束，试建立质点相对运动的运动微分方程，并给出广义约束反力的表达式。

解 本题利用Routh方程(2.7)求解。设动系$Ox'y'z'$和惯性系$Oxyz$的原点重合于O，转动轴Oz'和Oz轴重合，取广义坐标为相对动系的球坐标r，ψ，φ，则约束方程为

$$f=\dot{r}^2+r^2\dot{\psi}^2+r^2c^2\psi\dot{\varphi}^2-c^2=0\qquad(c\text{为常数})\qquad(4.1)$$

力学系统的总相对动能

$$T_r=\frac{1}{2}m(t)[\dot{r}^2+r^2\dot{\psi}^2+r^2c^2\psi\dot{\varphi}^2]\qquad(4.2)$$

对T_r计算凝固导数和凝固偏导数得到

$$\frac{D}{Dt}\frac{\varDelta T_r}{\varDelta \dot r}=m(t)\ddot r \qquad \frac{\varDelta T_r}{\varDelta r}=m(t)r(\dot\psi^2+c^2\psi\dot\varphi^2)$$

$$\frac{D}{Dt}\frac{\varDelta T_r}{\varDelta \dot\psi}=m(t)\frac{d}{dt}(r^2\dot\psi) \qquad \frac{\varDelta T_r}{\varDelta \psi}=-m(t)r^2\dot\varphi^2 s\psi c\psi$$

$$\frac{D}{Dt}\frac{\varDelta T_r}{\varDelta \dot\varphi}=m(t)\frac{d}{dt}(r^2c^2\psi\dot\varphi) \qquad \frac{\varDelta T_r}{\varDelta \varphi}=0$$

由约束方程(4.1)有

$$\frac{\partial f}{\partial \dot r}=2\dot r \qquad \frac{\partial f}{\partial\dot\psi}=2r^2\dot\psi \qquad \frac{\partial f}{\partial\dot\varphi}=2r^2\dot\varphi c^2\psi$$

根据诸广义力的计算公式可以求得

$$Q_r=-\frac{k^2}{r^2}m(t) \qquad Q_r^R=\dot m(t)u_r' \qquad Q_r^{\dot\omega}=0$$

$$Q_\psi=0 \qquad Q_\psi^R=\dot m(t)ru_\psi' \qquad Q_\psi^{\dot\omega}=0$$

$$Q_\varphi=0 \qquad Q_\varphi^R=\dot m(t)rc\psi u_\varphi' \qquad Q_\varphi^{\dot\omega}=0$$

$$-\frac{\varDelta}{\varDelta r}(V^O+V^\omega)=m(t)\omega^2rc^2\psi$$

$$-\frac{\varDelta}{\varDelta\psi}(V^O+V^\omega)=-m(t)\omega^2r^2s\psi c\psi$$

$$-\frac{\varDelta}{\varDelta\varphi}(V^O+V^\omega)=0$$

$$\varGamma_r=2m(t)\omega r\dot\varphi c^2\psi$$

$$\varGamma_\psi=-2m(t)\omega r\dot\varphi^2 s\psi c\psi$$

$$\varGamma_\varphi=-2m(t)\omega rc\psi[\dot rc\psi-r\dot\psi s\psi]$$

将以上结果代入(2.7)，经过整理可得

$$\left.\begin{aligned}
&m(t)\ddot r-m(t)r[\dot\psi^2+\dot\varphi^2c^2\psi]=-\frac{k^2}{r^2}m(t)+\dot m(t)u_r'\\
&\qquad+2m(t)\omega r\dot\varphi c^2\psi+m(t)\omega^2rc^2\psi+2\dot r\lambda\\
&m(t)\frac{d}{dt}(r^2\dot\psi)+m(t)r^2\dot\varphi^2s\psi c\psi=\dot m(t)ru_\psi'+2r\dot\psi^2\lambda\\
&\qquad-2m(t)\omega r^2\dot\varphi s\psi c\psi-m(t)r^2\omega^2s\psi c\psi\\
&m(t)\frac{d}{dt}(r^2\dot\varphi c^2\psi)=\dot m(t)ru_\varphi'c\psi+2m(t)\omega r^2\dot\psi s\psi c\psi\\
&\qquad-2m(t)\omega r\dot rc^2\psi+2r^2\dot\varphi c^2\psi\lambda
\end{aligned}\right\}\tag{4.3}$$

根据(2.6)，广义约束反力可表示为

$$\left.\begin{aligned}
&N_r=m(t)\ddot r-m(t)r[\dot\psi^2+\dot\varphi^2c^2\psi]+\frac{k^2}{r^2}m(t)-\dot m(t)u_r'\\
&\qquad-2m(t)\omega r\dot\varphi c^2\psi-m(t)\omega^2rc^2\psi\\
&N_\psi=m(t)\frac{d}{dt}(r\dot\psi^2)+m(t)r^2\dot\varphi^2s\psi c\psi-\dot m(t)ru_\psi'\\
&\qquad+2m(t)\omega r^2\dot\varphi s\psi c\psi+m(t)r^2\omega^2s\psi c\psi\\
&N_\varphi=m(t)\frac{d}{dt}(r^2\dot\varphi c^2\psi)-2m(t)\omega r^2\dot\psi s\psi c\psi\\
&\qquad-\dot m(t)ru_\varphi'c\psi+2m(t)\omega r\dot rc^2\psi
\end{aligned}\right\}\tag{4.4}$$

例2 在Чаплыгин-Caratheodory问题中，物体对称轴PC上另有一点B，其质量为$m=m(t)$，分离微粒的相对速度为$\mathbf{u}'$，方向与PC成α角．若物体质量为M，它对于自身质心C的惯量矩为J_C．物体-变质量质点系统的位置可由接触点相对于冰面坐标系$O'x'y'$的坐标(x', y')以及PC与$O'x'$轴间的夹角θ来确定．假如$O'x'y'$也在水平面内运动，其O'点的速度$\mathbf{v}_{O'}$在水平面内，而转动角速度$\boldsymbol{\omega}$垂直于水平面，且$\boldsymbol{\omega}$和$\mathbf{v}_{O'}$都是时间t的已知函数．试建立问题的相对运动微分方程和导出广义约束反力的表达式．

解 我们仍由方程(2.7)求解．取物体-变质量质点系统为研究对象，选取x'，y'，θ为广义坐标，则该问题的非完整约束方程为

$$\varphi=\dot{y}'-\dot{x}'\operatorname{tg}\theta=0 \tag{4.5}$$

由此可得

$$\frac{\partial\varphi}{\partial\dot{x}'}=-\operatorname{tg}\theta \qquad \frac{\partial\varphi}{\partial\dot{y}'}=1 \qquad \frac{\partial\varphi}{\partial\theta}=0$$

由于系统所受的主动力仅有重力，故

$$Q_{x'}=Q_{y'}=Q_\theta=0$$

根据诸广义力的计算公式，可以求得

$$Q^R_{x'}=-\dot{m}(t)u'c(\alpha+\theta)$$

$$Q^R_{y'}=-\dot{m}(t)u's(\alpha+\theta)$$

$$Q^R_\theta=-\dot{m}(t)u'bs\alpha$$

$$Q^{\dot{\omega}}_{x'}=\dot{\omega}y'[M+m(t)]+\dot{\omega}s\theta[aM+bm(t)]$$

$$Q^{\dot{\omega}}_{y'}=-\dot{\omega}x'[M+m(t)]-\dot{\omega}c\theta[aM+bm(t)]$$

$$Q^{\dot{\omega}}_\theta=-\dot{\omega}(x'c\theta+y's\theta)[aM+bm(t)]-\dot{\omega}[J_C+a^2M+b^2m(t)]$$

$$-\frac{\varPi}{\varPi x'}V^{O'}=-(\dot{v}_{O'x}c\varphi+\dot{v}_{O'y}s\varphi)[M+m(t)]$$

$$-\frac{\varPi}{\varPi y'}V^{O'}=(\dot{v}_{O'x}s\varphi-\dot{v}_{O'y}c\varphi)[M+m(t)]$$

$$-\frac{\varPi}{\varPi\theta}V^{O'}=\{\dot{v}_{O'x}s(\theta+\varphi)-\dot{v}_{O'y}c(\theta+\varphi)\}[aM+bm(t)]$$

$$-\frac{\varPi}{\varPi x'}V^{\omega}=\omega^2x'[M+m(t)]+\omega^2c\theta[aM+bm(t)]$$

$$-\frac{\varPi}{\varPi y'}V^{\omega}=\omega^2y'[M+m(t)]+\omega^2s\theta[aM+bm(t)]$$

$$-\frac{\varPi}{\varPi\theta}V^{\omega}=-\omega^2(x's\theta-y'c\theta)[aM+bm(t)]$$

$$\Gamma_{x'}=2\omega\dot{y}'[M+m(t)]+2\omega\dot{\theta}c\theta[aM+bm(t)]$$

$$\Gamma_{y'}=-2\omega\dot{x}'[M+m(t)]+2\omega\dot{\theta}s\theta[aM+bm(t)]$$

$$\Gamma_\theta=-2\omega(\dot{x}'c\theta+\dot{y}'s\theta)[aM+bm(t)]$$

力学系统的总相对动能应为

$$T_r=\frac{1}{2}M\{(\dot{x}'-a\dot{\theta}s\theta)^2+(\dot{y}'+a\dot{\theta}c\theta)^2\}+\frac{1}{2}J_C\dot{\theta}^2$$
$$+\frac{1}{2}m(t)\{(\dot{x}'-b\dot{\theta}s\theta)^2+(\dot{y}'+b\dot{\theta}c\theta)^2\} \tag{4.6}$$

对T_r作凝固导数和凝固偏导数计算，可得

$$\frac{D}{Dt}\frac{\varPi T_r}{\varPi\dot{x}'}=[M+m(t)]\ddot{x}'-[aM+bm(t)](\ddot{\theta}s\theta+\dot{\theta}^2c\theta)$$

$$\frac{D}{Dt}\frac{\varPi T_r}{\varPi\dot{y}'}=[M+m(t)]\ddot{y}'-[aM+bm(t)](\ddot{\theta}c\theta-\dot{\theta}^2s\theta)$$

$$\frac{D}{Dt}\frac{\varPi T_r}{\varPi\dot{\theta}}=[J_C+a^2M+b^2m(t)]\ddot{\theta}+[aM+bm(t)]$$
$$\cdot(\ddot{y}'c\theta-\ddot{x}'s\theta-\dot{y}'\dot{\theta}s\theta-\dot{x}'\dot{\theta}c\theta)$$

$$\frac{\varPi T_r}{\varPi x'}=0 \qquad \frac{\varPi T_r}{\varPi y'}=0$$

$$\frac{\varPi T_r}{\varPi\theta}=-\dot{\theta}[aM+bm(t)](\dot{x}'c\theta+\dot{y}'s\theta)$$

将以上结果代入(2.7)并整理，可得

$$[M+m(t)]\ddot{x}'-[aM+bm(t)](\ddot{\theta}s\theta+\dot{\theta}^2c\theta)$$
$$=-\dot{m}(t)u'c(\alpha+\theta)+[aM+bm(t)](\omega^2c\theta+2\omega\dot{\theta}c\theta+\dot{\omega}s\theta)$$
$$+[M+m(t)](\dot{\omega}y'+2\omega\dot{y}'+\omega^2x'-\dot{v}_{O'x}c\varphi-\dot{v}_{O'y}s\varphi)-\lambda\operatorname{tg}\theta \tag{4.7}$$

$$[M+m(t)]\ddot{y}'+[aM+bm(t)](\ddot{\theta}c\theta-\dot{\theta}^2s\theta)$$
$$=-\dot{m}(t)u's(\alpha+\theta)+[aM+bm(t)](\omega^2s\theta+2\omega\dot{\theta}s\theta-\dot{\omega}c\theta)$$
$$+[M+m(t)](-\dot{\omega}x'-2\omega\dot{x}'+\omega^2y'+\dot{v}_{O'x}s\varphi-\dot{v}_{O'y}c\varphi)+\lambda \tag{4.8}$$

$$[J_C+a^2M+b^2m(t)]\ddot{\theta}+[aM+bm(t)](\ddot{y}'c\theta-\ddot{x}'s\theta)$$
$$=-\dot{m}(t)u'bs\alpha-[aM+bm(t)]\{\dot{\omega}(x'c\theta+y's\theta)$$
$$+2\omega(\dot{x}'c\theta+\dot{y}'s\theta)+\omega^2(x's\theta-y'c\theta)-\dot{v}_{O'x}s(\theta+\varphi)$$
$$+\dot{v}_{O'y}c(\theta+\varphi)\}-[J_C+a^2M+b^2m(t)]\dot{\omega} \tag{4.9}$$

从(4.7)和(4.8)中消去乘子λ，即有

$$[M+m(t)](\ddot{x}'c\theta+\ddot{y}'s\theta)-[aM+bm(t)](\dot{\theta}+\omega)^2$$
$$=-\dot{m}(t)u'c\alpha+[M+m(t)]\{\dot{\omega}(y'c\theta-x's\theta)-2\omega(\dot{x}'s\theta-\dot{y}'c\theta)$$
$$+\omega^2(x'c\theta+y's\theta)-\dot{v}_{O'x}c(\varphi+\theta)-\dot{v}_{O'y}s(\varphi+\theta)\} \tag{4.10}$$

(4.9)和(4.10)即为系统满足的运动微分方程，结合约束方程(4.5)，原则上可求出x', y'和θ. 最后，再根据(2.6)式，系统所受的广义约束反力可以表为

$$N_{x'}=[M+m(t)](\ddot{x}'+\dot{v}_{O'x}c\varphi+\dot{v}_{O'y}s\varphi-\dot{\omega}y'-2\omega\dot{y}'-\omega^2x')$$
$$-[aM+bm(t)](\ddot{\theta}s\theta+\dot{\theta}^2c\theta+\dot{\omega}s\theta+\omega^2c\theta$$
$$+2\omega\dot{\theta}c\theta)+\dot{m}(t)u'c(\alpha+\theta) \tag{4.11}$$

$$N_{y'}=[M+m(t)](\ddot{y}'-\dot{v}_{O'x}s\varphi+\dot{v}_{O'y}c\varphi+\dot{\omega}x'+2\omega\dot{x}'-\omega^2y')$$
$$+[aM+bm(t)](\ddot{\theta}c\theta-\dot{\theta}^2s\theta+\dot{\omega}c\theta-\omega^2s\theta$$
$$-2\omega\dot{\theta}s\theta)+\dot{m}(t)u's(\alpha+\theta) \tag{4.12}$$

$$N_\theta=0 \tag{4.13}$$

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Routh's Equations for General Nonholonomic Mechanical Systems of Variable Mass

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Abstract

In this paper, Routh's equations for the mechanical systems of the variable mass with nonlinear nonholonomic constraints of arbitrary orders in a noninertial reference system have been deduced not from any variation principles, but from the dynamical equations of Newtonian mechanics. And then again the other forms of equations for nonholonomic systems of variable mass are obtained from Routh's equations.

Key words Routh's equations, variable mass system, nonholonomic constraint, noninertial reference system