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一类反应扩散方程组初值-边值问题的奇摄动*

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摘 要

本文研究了一类反应扩散方程组的奇摄动初值一边值问题．利用微分不等式理论证明了该问题存在一个解并得到了其解的渐近展开式．

关键词 反应扩散方程组 奇摄动 比较定理 渐近展开式

今考虑反应扩散方程组的一个模型：

$$\partial u_i/\partial t-(\varepsilon L+L_1)u_i=f_i(t,x,u_1,u_2,\varepsilon)\qquad(i=1,2)$$

$$(x\equiv(x_1,x_2,\cdots,x_n)\in\Omega,\ t\in(0,T])\tag{1}$$

$$B_iu_i\equiv\alpha_i(x)\partial u_i/\partial\nu+\beta_i(x)u_i=g_i(x,\varepsilon)\qquad(i=1,2)\quad x\in\partial\Omega\tag{2}$$

$$u_i(0,x,\varepsilon)=h_i(x,\varepsilon),\qquad(i=1,2)\tag{3}$$

其中ε为正的小参数，Ω为n维欧氏空间R^n的有界区域，$\partial\Omega$为Ω的光滑边界，$\partial/\partial\nu$为$\partial\Omega$上的内法向导数，$\alpha_i(x)\leqslant\alpha_0<0$，$\beta_i(x)\geqslant\beta_0>0$，$L$为二阶强椭圆型算子：

$$\left.\begin{aligned}&L\equiv\sum_{i,j=1}^{n}a_{ij}(x)\frac{\partial^2}{\partial x_i\partial x_j}+\sum_{i=1}^{n}a_i(x)\frac{\partial}{\partial x_i}+a(x)\\&\sum_{i,j=1}^{n}a_{ij}(x)\xi_i\xi_j\geqslant\delta_1>0,\qquad x\in\Omega\end{aligned}\right\}\tag{4}$$

其中$\forall\xi_i\in R(i=1,2,\cdots,n)$为任一组实数，且$a_{ij}(x)=a_{ji}(x)$，而$L_1$为一阶微分算子：

$$\left.\begin{aligned}&L_1\equiv\sum_{i=1}^{n}b_i(x)\frac{\partial}{\partial x_i}-b(x)\qquad b(x)\geqslant b_0>0\\&\Big[\sum_{i=1}^{n}b_i(x)\nu_i\Big]_{\partial\Omega}\geqslant\delta_2>0\end{aligned}\right\}\tag{5}$$

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并且假设f_i，g_i，h_i，a_{ij}，a_i，a，b_i，b，α_i，β_i均为其变元在所决定的范围内为充分光滑的函数，α_0，β_0，b_0，δ_1，δ_2为正常数，ν_i为$\partial\Omega$上内法线的方向素，且存在一个正常数l，满足：

$$f_{iu_1}+f_{iu_2}\leqslant -l<0 \qquad (i=1,\ 2) \tag{6}$$

反应扩散方程在生物物理，生物数学，物理化学等领域中广泛地被应用，许多近代文献中都研究了这一类问题，例如[1]～[5]．问题(1)～(3)是研究较广泛的一类问题的模型．本文是采用多重尺度法[6]求得了问题(1)～(3)的形式渐近解，并用微分不等式理论[7],[10]~[13]证明了相应解的一致有效性．

首先构造问题(1)～(3)的形式渐近解．本问题当$\varepsilon=0$时的退化情形为：

$$\begin{cases}\partial u_i/\partial t-L_1u_i=f_i(t,x,u_1,u_2,0) & (7)\\ \qquad\qquad\qquad\qquad (i=1,2) & \\ u_i(0,x,0)=h_i^{\prime}(x,0) & (8)\end{cases}$$

其中$h_i^{\prime}(x,0)$为$h_i(x,0)$从Ω上延拓到整个R^n上的充分光滑函数．(7)为对称双曲型方程组．设Cauchy问题(7)，(8)在Ω内的一组充分光滑解为

$$u_i=U_{i0}(t,x) \qquad (i=1,\ 2) \tag{9}$$

记(1)～(3)的外部解为

$$U_i(t,x,\varepsilon)\sim\sum_{j=0}^{\infty}U_{ij}(t,x)\varepsilon^j \qquad (i=1,2) \tag{10}$$

将f_i，$h_i^{\prime}$按ε Taylor展开，把(10)代入(1)，(3)．合併ε的同次幂项并令各次幂的系数等于零．考虑到U_{i0}满足(7)，(8)，我们便有：

$$\begin{cases}\partial U_{ij}/\partial t-L_1U_{ij}=F_{ij}+LU_{i(j-1)} & (11)\\ \qquad\qquad\qquad\qquad (j=1,2,\cdots;\ i=1,2) & \\ U_{ij}(0,x)=h_{ij} & (12)\end{cases}$$

其中F_{ij}，h_{ij}分别为f_i，$h_i^{\prime}$的展开系数：

$$f_i(t,x,u_1,u_2,\varepsilon)\equiv F_i(\varepsilon)\sim\sum_{j=0}^{\infty}F_{ij}\varepsilon^j \qquad (i=1,2)$$

$$h_i^{\prime}(x,\varepsilon)\sim\sum_{j=0}^{\infty}h_{ij}\varepsilon^j \qquad (i=1,2)$$

$$\begin{aligned}F_{ij}=&\frac{1}{j!}\frac{d^jF_i(\varepsilon)}{d\varepsilon^j}\bigg|_{\varepsilon=0}\\ &=f_{iu_1}(t,x,U_{10},U_{20},0)U_{1j}+f_{iu_2}(t,x,U_{10},U_{20},0)U_{2j}\\ &\quad+C_{ij}(t,x,U_{10},U_{11},\cdots,U_{1(j-1)},U_{20},U_{21},\cdots,U_{2(j-1)})\\ &\qquad\qquad (j=0,1,2,\cdots)\end{aligned}$$

$$h_{ij}=\frac{1}{j!}\frac{\partial^jh_i^{\prime}}{\partial\varepsilon^j}\bigg|_{\varepsilon=0} \qquad (j=0,1,2,\cdots)$$

这里C_{ij}为其变元关于j的逐次已知函数．其构造从略．

由线性问题(11)，(12)可以关于j依次地求出$U_{ij}(i=1,2)$．将它们及(9)代入(10)，便得到原问题的外部解．显然它未必满足边界条件(2)，还需进一步构造边界层校正项．

为此，先在$\partial\Omega$的邻域建立局部坐标$(\rho,\ \varphi)$，其中$\varphi\equiv(\varphi_1,\varphi_2,\cdots,\varphi_{n-1})$为$n-1$维流形$\partial\Omega$上点的非奇坐标。规定在$\partial\Omega$的邻域内每一点$Q$的坐标按如下方式选取：取$Q$沿$\partial\Omega$的内法线到$\partial\Omega$的距离为$\rho$的坐标，取$\rho_0$足够小，当$\rho\leqslant\rho_0$时使点$Q$的$\rho$坐标唯一地被确定。同时把过$Q$沿内法线在$\partial\Omega$上的交点的$\varphi$的坐标取为点$Q$的$\varphi$的坐标。

在上述的局部坐标系下，于$\partial\Omega$的邻域$\rho\leqslant\rho_0'$内原问题(1)～(3)可表示为：

$$\partial u_i/\partial t-(\varepsilon\bar{L}+\bar{L}_1)u_i=\bar{f}_i(t,\rho,\varphi,u_1,u_2,\varepsilon)\qquad(i=1,2)\tag{13}$$

$$\bar{B}_iu_i\equiv[\bar{\alpha}_i(\varphi)\partial u_i/\partial\rho+\bar{\beta}_i(\varphi)u_i]_{\partial\Omega}=\bar{g}_i(\varphi,\varepsilon)|_{\partial\Omega}\qquad(i=1,2)\tag{14}$$

$$u_i(0,\rho,\varphi,\varepsilon)=\bar{h}_i(\rho,\varphi,\varepsilon)\qquad(i=1,2)\tag{15}$$

其中

$$\bar{L}\equiv\bar{a}_{nn}(\rho,\varphi)\frac{\partial^2}{\partial\rho^2}+\sum_{i=1}^{n-1}\bar{a}_{in}(\rho,\varphi)\frac{\partial^2}{\partial\rho\partial\varphi_i}+\sum_{i,j=1}^{n-1}\bar{a}_{ij}(\rho,\varphi)\frac{\partial^2}{\partial\varphi_i\partial\varphi_j}$$

$$+\bar{a}_n(\rho,\varphi)\frac{\partial}{\partial\rho}+\sum_{i=1}^{n-1}\bar{a}_i(\rho,\varphi)\frac{\partial}{\partial\varphi_i}+\bar{a}(\rho,\varphi)$$

$$\bar{L}_1\equiv\bar{b}_n(\rho,\varphi)\frac{\partial}{\partial\rho}+\sum_{i=1}^{n-1}\bar{b}_i(\rho,\varphi)\frac{\partial}{\partial\varphi_i}-\bar{b}(\rho,\varphi)$$

$$\bar{f}_i(t,\rho,\varphi,u_1,u_2,\varepsilon)\equiv f_i(t,x,u_1,u_2,\varepsilon)$$

$$\bar{g}_i(\varphi,\varepsilon)\equiv g_i(x,\varepsilon)|_{\partial\Omega},\quad \bar{h}_i(\rho,\varphi,\varepsilon)\equiv h_i'(x,\varepsilon)$$

$$\bar{\alpha}_i(\varphi)\equiv\alpha_i(x)|_{\partial\Omega},\quad \bar{\beta}_i(\varphi)\equiv\beta_i(x)|_{\partial\Omega}$$

下面引入多重尺度变量[6]：

$$\tilde{\rho}=k(\rho,\varphi)/\varepsilon,\quad \tilde{\boldsymbol{\rho}}=\rho,\quad \varphi=\varphi$$

其中$k(\rho,\varphi)$是待定函数，它将在下面选定。于是有：

$$\frac{\partial}{\partial\rho}=\frac{k_\rho}{\varepsilon}\frac{\partial}{\partial\tilde{\rho}}+\frac{\partial}{\partial\tilde{\boldsymbol{\rho}}}$$

$$\frac{\partial^2}{\partial\rho^2}=\frac{k_\rho^2}{\varepsilon^2}\frac{\partial^2}{\partial\tilde{\rho}^2}+2\frac{k_\rho}{\varepsilon}\frac{\partial^2}{\partial\tilde{\rho}\partial\tilde{\boldsymbol{\rho}}}+\frac{k_{\rho\rho}}{\varepsilon}\frac{\partial}{\partial\tilde{\rho}}+\frac{\partial^2}{\partial\tilde{\boldsymbol{\rho}}^2}$$

为了书写方便起见，以下仍用变量ρ表示$\tilde{\rho}$，于是$\partial/\partial t-(\varepsilon\bar{L}+\bar{L}_1)$便为：

$$\partial/\partial t-(\varepsilon\bar{L}+\bar{L}_1)=\varepsilon^{-1}K_0+K_1+\varepsilon K_2$$

其中

$$K_0\equiv-\left[\bar{a}_{nn}k_\rho^2\frac{\partial^2}{\partial\tilde{\rho}^2}+\bar{b}_nk_\rho\frac{\partial}{\partial\tilde{\rho}}\right]\tag{16}$$

$$K_1\equiv\frac{\partial}{\partial t}-\left[2\bar{a}_{nn}k_\rho\frac{\partial^2}{\partial\tilde{\rho}\partial\rho}+\bar{a}_{nn}k_{\rho\rho}\frac{\partial}{\partial\tilde{\rho}}+\sum_{i=1}^{n-1}\bar{a}_{in}k_\rho\frac{\partial^2}{\partial\rho\partial\varphi_i}+\bar{a}_nk_\rho\frac{\partial}{\partial\tilde{\rho}}\right.$$

$$\left.+\bar{b}_n\frac{\partial}{\partial\rho}+\sum_{i=1}^{n-1}\bar{b}_i\frac{\partial}{\partial\varphi_i}-\bar{b}\right]\tag{17}$$

$$K_2\equiv-\left[\bar{a}_{nn}\frac{\partial^2}{\partial\rho^2}+\sum_{i=1}^{n-1}\bar{a}_{in}\frac{\partial^2}{\partial\rho\partial\varphi_i}+\sum_{i,j=1}^{n-1}\bar{a}_{ij}\frac{\partial^2}{\partial\varphi_i\partial\varphi_j}+\bar{a}_n\frac{\partial}{\partial\rho}\right.$$

$$+\sum_{i=1}^{n-1}\bar{a}_i\frac{\partial}{\partial\varphi_i}+\bar{a}\Big] \tag{18}$$

现设原问题(1)~(3)解的一组边界层校正项$\bar{U}_i$为：

$$\bar{U}_i(t,\tilde{\rho},\rho,\varphi,\varepsilon)\sim\sum_{j=1}^{\infty}\bar{U}_{ij}(t,\tilde{\rho},\rho,\varphi)\varepsilon^j \qquad (i=1,2) \tag{19}$$

并令

$$u_i=U_i+\bar{U}_i \qquad (i=1,2) \tag{20}$$

将(20)代入(13)得到：

$$\partial\bar{U}_i/\partial t-(\varepsilon\bar{L}+\bar{L}_1)\bar{U}_i=\bar{f}_i(t,\rho,\varphi,U_1+\bar{U}_1,U_2+\bar{U}_2,\varepsilon)-\bar{f}_i(t,\rho,\varphi,U_1,U_2,\varepsilon) \qquad (i=1,2)$$

再把(16)~(19)代入上式得：

$$\varepsilon^{-1}\sum_{j=1}^{\infty}(K_0[\bar{U}_{ij}]+K_1[\bar{U}_{i(j-1)}]+K_2[\bar{U}_{i(j-2)}])\varepsilon^j$$

$$=\bar{f}_i\Big(t,\rho,\varphi,\sum_{j=0}^{\infty}U_{1j}\varepsilon^j+\sum_{j=1}^{\infty}\bar{U}_{1j}\varepsilon^j,\ \sum_{j=0}^{\infty}U_{2j}\varepsilon^j+\sum_{j=1}^{\infty}\bar{U}_{2j}\varepsilon^j,\varepsilon\Big)$$

$$-\bar{f}_i\Big(t,\rho,\varphi,\sum_{j=0}^{\infty}U_{1j}\varepsilon^j,\sum_{j=0}^{\infty}U_{2j}\varepsilon^j,\varepsilon\Big)\equiv\sum_{j=1}^{\infty}\bar{F}_{ij}\varepsilon^j \qquad (i=1,2) \tag{21}$$

上式和下面$\bar{U}_{ij}$等的非正下标的函数值为零，而

$$\bar{F}_{ij}=\frac{1}{j!}\frac{\partial^j\bar{F}_i}{\partial\varepsilon^j}\Big|_{\varepsilon=0} \qquad (j=1,2,\cdots;\ i=1,2)$$

$$\bar{F}_i\equiv\bar{f}_i\Big(t,\rho,\varphi,\sum_{j=0}^{\infty}U_{1j}\varepsilon^j+\sum_{j=1}^{\infty}\bar{U}_{1j}\varepsilon^j,\ \sum_{j=0}^{\infty}U_{2j}\varepsilon^j+\sum_{j=1}^{\infty}\bar{U}_{2j}\varepsilon^j,\varepsilon\Big)$$

$$-f_i\Big(t,\rho,\varphi,\sum_{j=0}^{\infty}U_{1j}\varepsilon^j,\sum_{j=0}^{\infty}U_{2j}\varepsilon^j,\varepsilon\Big) \qquad (i=1,2)$$

比较(21)等式两端同次幂的系数，得

$$K_0[\bar{U}_{i1}]\equiv-\Big[\bar{a}_{nn}k_\rho^2\frac{\partial^2\bar{U}_{i1}}{\partial\tilde{\rho}^2}+\bar{b}_nk_\rho\frac{\partial\bar{U}_{i1}}{\partial\tilde{\rho}}\Big]=0 \qquad (i=1,2) \tag{22$_1$}$$

$$K_0[\bar{U}_{ij}]=-K_1[\bar{U}_{i(j-1)}]-K_2[\bar{U}_{i(j-2)}]+\bar{F}_{i(j-1)} \qquad (j=2,3,\cdots;\ i=1,2) \tag{22$_j$}$$

令

$$k_\rho=\bar{b}_n(\rho,\varphi)/\bar{a}_{nn}(\rho,\varphi)$$

即

$$k(\rho,\varphi)=\int_0^\rho\frac{\bar{b}_n(s,\varphi)}{\bar{a}_{nn}(s,\varphi)}ds \tag{23}$$

因此由$(22)_1$有解：

$$\bar{U}_{i1}=\beta_{i1}(t,\rho,\varphi)\exp[-\tilde{\rho}]=\beta_{i1}(t,\rho,\varphi)\exp[-k(\rho,\varphi)/\varepsilon] \qquad (i=1,2) \tag{24}$$

其中 $\beta_{i1}(t,\rho,\varphi)$ 为待定函数，将在下面决定．由(4)，(5)，不难看出当 $0<\rho\leqslant\rho_0\leqslant\rho_0'$ 时(23)决定的 $k(\rho,\varphi)>0$，故形如(24)的 $\bar{U}_{i1}(i=1,2)$ 是具有边界层性质的函数．

令 $(22)_2$ 的右端等于零：

$$-K_1[\bar{U}_{i1}]+\bar{F}_{i1}=0 \tag{25}$$

将(24)代入(25)，可得：

$$-\frac{\partial\beta_{i1}}{\partial t}+\Big[-2\bar{a}_{nn}k_\rho\frac{\partial\beta_{i1}}{\partial\rho}-\bar{a}_{nn}k_{\rho\rho}\beta_{i1}+\sum_{j=1}^{n-1}\bar{a}_{jn}k_\rho\frac{\partial^2\beta_{i1}}{\partial\rho\partial\varphi_j}-\bar{a}_nk_\rho\beta_{i1}$$

$$+\bar{b}_n\frac{\partial\beta_{i1}}{\partial\rho}+\sum_{j=1}^{n-1}\bar{b}_j\frac{\partial\beta_{i1}}{\partial\varphi_j}-\bar{b}\beta_{i1}\Big]$$

$$=-[\bar{f}_{iu_1}(t,\rho,\varphi,U_{10},U_{20},0)\beta_{11}+\bar{f}_{iu_2}(t,\rho,\varphi,U_{10},U_{20},0)\beta_{21}]\qquad(i=1,2) \tag{26}$$

把(20)代入(14)，(15)，并令其 ε 的同次幂的系数相等得：

$$\Big[\bar{a}_ik_\rho\frac{\partial\bar{U}_{i1}}{\partial\bar{\rho}}\Big]_{\partial\Omega}=[\bar{g}_{i0}-B_iU_{i0}]_{\partial\Omega}\qquad(i=1,2) \tag{27$_1$}$$

$$\Big[\bar{a}_ik_\rho\frac{\partial\bar{U}_{ij}}{\partial\bar{\rho}}\Big]_{\partial\Omega}=\Big[\bar{g}_{i(j-1)}-B_iU_{i(j-1)}-\bar{a}_i\frac{\partial\bar{U}_{i(j-1)}}{\partial\rho}\bar{\beta}_i-\bar{U}_{i(j-1)}\Big]_{\partial\Omega}$$

$$(j=2,3,\cdots;\ i=1,2) \tag{27$_j$}$$

$$\bar{U}_{ij}|_{t=0}=0\qquad(j=1,2,\cdots;\ i=1,2) \tag{28$_j$}$$

其中 $\bar{g}_{ij}$ 为：

$$\bar{g}_{ij}=\frac{1}{j!}\frac{\partial^j\bar{g}_i}{\partial\varepsilon^j}\Big|_{\varepsilon=0}\qquad(j=0,1,2,\cdots;\ i=1,2)$$

再将(2.4)代入 $(27)_1$, $(28)_1$，得：

$$\beta_{i1}|_{\partial\Omega}=\Big[-\frac{1}{\bar{a}_ik_\rho}(\bar{g}_{i0}-B_iU_{i0})\Big]_{\partial\Omega}\qquad(i=1,2) \tag{29$_1$}$$

$$\beta_{i1}(0,\rho,\varphi)=0 \tag{30}$$

由线性问题(26)，$(29)_1$，$(30)_1$ 求得 $\beta_{i1}(t,\rho,\varphi)$ 并代入(24)便得到 $\bar{U}_{i1}(i=1,2)$．再由 $(22)_2$ 并考虑到(25)，可得一组解：

$$\bar{U}_{i2}=\beta_{i2}(t,\rho,\varphi)\exp[-\bar{\rho}]=\beta_{i2}(t,\rho,\varphi)\exp[-k(\rho,\varphi)/\varepsilon]$$

$$(i=1,2) \tag{31}$$

其中 β_{i2} 是由令 $(22)_3$ 的右端为零所确定的线性偏微分方程组及 $(27)_2$，$(28)_2$ 所确定的边界条件和初始条件来决定．依此方法类推，我们可以逐次地定出

$$\bar{U}_{ij}=\beta_{ij}(t,\rho,\varphi)\exp[-k(\rho,\varphi)/\varepsilon]\qquad(j=3,4,\cdots;\ i=1,2) \tag{32}$$

显然由(31)，(32)所确定的 $\bar{U}_{ij}$ 也均为具有边界层性质的函数．

令

$$\tilde{U}_{ij}=\psi\bar{U}_{ij}\qquad(j=1,2,\cdots;\ i=1,2) \tag{33}$$

其中 ψ 为 $\Omega+\partial\Omega$ 上的充分光滑的函数且满足：

$$\psi=\begin{cases}1, & 0\leqslant\rho\leqslant\rho_0/3\\ 0, & 2\rho_0/3\leqslant\rho\text{及除}\rho\leqslant\rho_0\text{外}\Omega\text{中的其它点}\end{cases}$$

最后，我们便可构造原问题(1)～(3)的如下形式渐近解：

$$u_i \sim \sum_{j=0}^{\infty} U_{ij}\varepsilon^j + \sum_{j=1}^{\infty} \tilde{U}_{ij}\varepsilon^j \tag{34}$$

下面我们来证明原问题(1)～(3)的解具有形如(34)的一致有效展开式。

现构造函数$m_i(t,x,\varepsilon)$，$M_i(t,x,\varepsilon)$ $(i=1,2)$:

$$m_i(t,x,\varepsilon)=Z_{im}(t,x,\varepsilon)-(\gamma/l)\varepsilon^m \tag{35}$$

$$M_i(t,x,\varepsilon)=Z_{im}(t,x,\varepsilon)+(\gamma/l)\varepsilon^m \tag{36}$$

其中

$$Z_{im}(t,x,\varepsilon)=\sum_{j=0}^{m-1} U_{ij}\varepsilon^j + \sum_{j=1}^{m} \tilde{U}_{ij}\varepsilon^j \qquad (i=1,2) \tag{37}$$

而γ为适当大的待定正常数。

显然由(35)，(36)和(37)不难看出m_i，M_i关于变量t,x为足够光滑的函数且满足：

$$m_i(t,x,\varepsilon)<M_i(t,x,\varepsilon) \qquad (i=1,2) \tag{38}$$

又因

$$\bar{g}_i(\varphi,\varepsilon)=\sum_{j=0}^{m-1}\bar{g}_{ij}\varepsilon^j+O(\varepsilon^m) \qquad (i=1,2) \quad (0<\varepsilon\ll 1)$$

故对足够小的ε'，当$0<\varepsilon\leqslant\varepsilon'$时恒存在$d'>0$，使得：

$$\left| g_i(\varphi,\varepsilon)-\sum_{j=0}^{m-1}\bar{g}_{ij}\varepsilon^j \right| \leqslant d'\varepsilon^m \qquad (i=1,2)$$

由此，

$$B[m_i(t,x,\varepsilon)]|_{\partial\Omega}=B[Z_{im}(t,x,\varepsilon)]|_{\partial\Omega}\sim\beta_i(x)\frac{\gamma}{l}\varepsilon^m\Big|_{\partial\Omega}$$

$$\leqslant\bar{g}_i(\varphi,\varepsilon)|_{\partial\Omega}+\left(d'-\frac{\beta_0\gamma}{l}\right)\varepsilon^m \qquad (i=1,2)$$

所以取足够大的$\gamma\geqslant\gamma'=ld'/\beta_0$时，有

$$B[m_i(t,x,\varepsilon)]|_{\partial\Omega}\leqslant\bar{g}_i(\varphi,\varepsilon)|_{\partial\Omega}=g_i(x,\varepsilon)|_{\partial\Omega} \qquad (i=1,2) \tag{39}$$

同理可得：

$$B[M_i(t,x,\varepsilon)]|_{\partial\Omega}\geqslant g_i(x,\varepsilon)|_{\partial\Omega} \qquad (i=1,2) \tag{40}$$

下面分三种情形再来建立几个微分不等式：

(i) 当$x\in\Omega$但除$\rho\leqslant 2\rho_0/3$时，由(33)，$\tilde{U}_{ij}=0$ $(j=1,2,\cdots;\ i=1,2)$，故

$$m_i(t,x,\varepsilon)=\sum_{j=0}^{m-1} U_{ij}\varepsilon^j-\frac{\gamma}{l}\varepsilon^m \qquad (i=1,2)$$

$$\frac{\partial m_i}{\partial t}-(\varepsilon L+L_1)m_i=\sum_{j=0}^{m-1}\left[\frac{\partial U_{ij}}{\partial t}-(\varepsilon L+L_1)U_{ij}\right]\varepsilon^j+(\varepsilon a(x)-b(x))\frac{\gamma}{l}\varepsilon^m$$

$$\leqslant\sum_{j=0}^{m-1}\left[\frac{\partial U_{ij}}{\partial t}-(\varepsilon L+L_1)U_{ij}\right]\varepsilon^j+\frac{\gamma}{l}a(x)\varepsilon^{m+1} \qquad (i=1,2)$$

又因

$$f_i\Big(t,x,\sum_{j=0}^{m-1}U_{1j}\varepsilon^j,\sum_{j=0}^{m-1}U_{2j}\varepsilon^j,\varepsilon\Big)=\sum_{i=0}^{m-1}F_{ij}\varepsilon^j+O(\varepsilon^m)$$

$$(i=1,2)\quad 0<\varepsilon\ll 1$$

故对足够小的$\varepsilon_1>0$，当$0<\varepsilon\leqslant\varepsilon_1$时，恒有$d_1>0$，$M>0$，使得：

$$\Big|f_i\Big(t,x,\sum_{j=0}^{m-1}U_{1j}\varepsilon^j,\ \sum_{j=0}^{m-1}U_{2j}\varepsilon^j,\varepsilon\Big)-\sum_{j=0}^{m-1}F_{ij}\varepsilon^j\Big|\leqslant d_1\varepsilon^m\qquad(i=1,2)$$

$$|L[U_{i(m-1)}]|\leqslant M\qquad(i=1,2)$$

$$|a(x)|\leqslant M$$

再由(6)及中值定理，存在 $0<\theta_1,\theta_2<1$，使得：

$$\begin{aligned}f_i(t,x,m_1,m_2,\varepsilon)=&f_i\Big(t,x,\sum_{j=0}^{m-1}U_{1j}\varepsilon^j,\sum_{j=0}^{m-1}U_{2j}\varepsilon^j,\varepsilon\Big)+\Big[\frac{\partial}{\partial u_1}f_i\Big(t,x,\sum_{j=0}^{m-1}U_{1j}\varepsilon^j\\&+\theta_1\Big(m_1-\sum_{j=0}^{m-1}U_{1j}\varepsilon^j\Big)m_2,\varepsilon\Big)\Big]\Big(m_1-\sum_{j=0}^{m-1}U_{1j}\varepsilon^j\Big)+\Big[\frac{\partial}{\partial u_2}f_i\Big(t,x,\sum_{j=0}^{m-1}U_{1j}\varepsilon^j,\sum_{j=0}^{m-1}U_{2j}\varepsilon^j\\&+\theta_2\Big(m_2-\sum_{j=0}^{m-1}U_{2j}\varepsilon^j\Big),\ \varepsilon\Big)\Big]\Big(m_2-\sum_{j=0}^{m-1}U_{2j}\varepsilon^j\Big)\\&\geqslant f_i\Big(t,x,\sum_{j=0}^{m-1}U_{1j}\varepsilon^j,\ \sum_{j=0}^{m-1}U_{2j}\varepsilon^j,\varepsilon\Big)+\gamma\varepsilon^m\qquad(i=1,2)\end{aligned}$$

所以当$0<\varepsilon\leqslant\varepsilon_1'=\min(\varepsilon_1,l/2M)$时，

$$\begin{aligned}&\partial m_i/\partial t-(\varepsilon L+L_1)m_i-f_i(t,x,m_1,m_2,\varepsilon)\\&\leqslant\sum_{j=0}^{m-1}\Big[\frac{\partial U_{ij}}{\partial t}(\varepsilon L+L_1)U_{ij}\Big]\varepsilon^j+\frac{\gamma}{l}a(x)\varepsilon^{m+1}-f_i(t,x,m_1,m_2,\varepsilon)\\&\leqslant\Big[\frac{\partial U_{i0}}{\partial t}-L_1[U_{i0}]-f_i(t,x,U_{10},U_{20},0)\Big]\\&\quad+\sum_{j=1}^{m-1}\Big(\frac{\partial U_{ij}}{\partial t}-L_1[U_{ij}]-L[U_{i(j-1)}]-F_{ij}\Big)\varepsilon^j\\&\quad-L[U_{i(m-1)}]\varepsilon^m+(d_1-\gamma/2)\varepsilon^m\\&\leqslant(M+d_1-\gamma/2)\varepsilon^m\qquad(i=1,2)\end{aligned}$$

(ii) 当$\rho_0/3\leqslant\rho\leqslant2\rho_0/3$时，因为$\tilde{U}_{ij}(j=1,2,\cdots,m;\ i=1,2)$及其各阶偏导数均渐近地趋于零，且具有$o(\varepsilon^m)$的量级。所以可以仿照(i)类似地进行估计，在取足够小的 $\varepsilon_2'>0$ 时，对于$0<\varepsilon\leqslant\varepsilon_2'$也有相应的结果。

(iii) 当$0<\rho\leqslant\rho_0/3$时，$\tilde{U}_{ij}=\bar{U}_{ij}(j=1,2,\cdots,m;\ i=1,2)$，

故

$$m_i(t,x,\varepsilon)=\sum_{j=0}^{m-1}U_{ij}\varepsilon^j+\sum_{j=1}^{m}\bar{U}_{ij}\varepsilon^j-\frac{\gamma}{l}\varepsilon^m$$

$$\frac{\partial m_i}{\partial t}-(\varepsilon L+L_1)m_i=\sum_{j=0}^{m-1}\left[\frac{\partial U_{ij}}{\partial t}-(\varepsilon L+L_1)U_{ij}\right]\varepsilon^j$$

$$+\sum_{j=1}^{m}\left[\frac{\partial \bar{U}_{ij}}{\partial t}-(\varepsilon L+L_1)\bar{U}_{ij}\right]\varepsilon^j+(\varepsilon a(x)-b(x))\frac{\gamma}{l}\varepsilon^m$$

$$\leqslant\sum_{j=0}^{m-1}\left[\frac{\partial U_{ij}}{\partial t}-(\varepsilon L+L_1)U_{ij}\right]\varepsilon^j+\varepsilon^{-1}\sum_{j=1}^{m}(K_0[\bar{U}_{ij}]$$

$$+K_1[\bar{U}_{i(j-1)}]+K_2[\bar{U}_{i(j-2)}])\varepsilon^j+\frac{\gamma}{l}a(x)\varepsilon^{m+1}$$

$$-(K_1[\bar{U}_{im}]+K_2[\bar{U}_{i(m-1)}]+\varepsilon K_2[\bar{U}_{im}])\varepsilon^m$$

又因

$$f_i(t,x,Z_{1m},Z_{2m},\varepsilon)=\sum_{j=0}^{m-1}F_{ij}\varepsilon^j+\sum_{j=0}^{m-1}\bar{F}_{ij}\varepsilon^j+O(\varepsilon^m)$$

$$(i=1,2)\qquad 0<\varepsilon\ll 1$$

故对足够小的$\varepsilon_3>0$，当$0<\varepsilon\leqslant\varepsilon_3$时，恒有$d_3>0$，$M'>0$，使得：

$$\left|f_i(t,x,Z_{1m},Z_{2m},\varepsilon)-\sum_{j=0}^{m-1}F_{ij}\varepsilon^j-\sum_{j=0}^{m-1}\bar{F}_{ij}\varepsilon^j\right|\leqslant d_3\varepsilon^m\qquad(i=1,2)$$

$$|L[U_{i(m-1)}]|\leqslant M'\qquad(i=1,2)$$

$$|K_1[\bar{U}_{im}]+K_2[\bar{U}_{i(m-1)}]+\varepsilon K_2[\bar{U}_{im}]|\leqslant M'\qquad(i=1,2)$$

$$|a(x)|\leqslant M'$$

再由中值定理可得：

$$f_i(t,x,m_1,m_2,\varepsilon)\geqslant f_i(t,x,Z_{1m},Z_{2m},\varepsilon)+\gamma\varepsilon^m\qquad(i=1,2)$$

所以当$0<\varepsilon\leqslant\varepsilon_3'=\min(\varepsilon_3,\ l/2M')$时，

$$\partial m_i/\partial t-(\varepsilon L+L_1)m_i-f_i(t,x,m_1,m_2,\varepsilon)$$

$$\leqslant\left[\frac{\partial U_{i0}}{\partial t}-L_1[U_{i0}]-f_i(t,x,U_{10},U_{20},0)\right]+\sum_{j=1}^{m-1}\left[\frac{\partial U_{ij}}{\partial t}-L_1[U_{ij}]\right.$$

$$\left.-L[U_{i(j-1)}]-F_{ij}\right]\varepsilon^j+\varepsilon^{-1}\sum_{j=1}^{m}(K_0[\bar{U}_{ij}]+K_1[\bar{U}_{i(j-1)}]$$

$$+K_2[\bar{U}_{i(j-2)}]-\bar{F}_{i(j-1)})\varepsilon^j+\frac{\gamma}{l}a(x)\varepsilon^m-L[U_{i(m-1)}]\varepsilon^m$$

$$-(K_1[\bar{U}_{im}]+K_2[\bar{U}_{i(m-1)}]+\varepsilon K_2[\bar{U}_{im}])\varepsilon^m+(d_3-\gamma)\varepsilon^m$$

$$\leqslant(2M'+d_3-\gamma/2)\varepsilon^m\qquad(i=1,2)$$

综合(i)～(iii)的讨论，取足够小的$\varepsilon_0=\min(\varepsilon_1',\varepsilon_2',\varepsilon_3')$及足够大的$\gamma_0$，当$0<\varepsilon\leqslant\varepsilon_0,\gamma\geqslant\gamma_0$时，成立微分不等式：

$$\partial m_i/\partial t-(\varepsilon L+L_1)m_i\leqslant f_i(t,x,m_1,m_2,\varepsilon)\qquad(i=1,2)$$

$$0<\varepsilon\leqslant\varepsilon_0,\ (t,x)\in(0,T]\times\Omega\qquad(41$$

同理有：

$$\partial M_i/\partial t-(\varepsilon L+L_1)M_i\geqslant f_i(t,x,M_1,M_2,\varepsilon) \quad (i=1,2)$$

$$0<\varepsilon\leqslant\varepsilon_0,\ (t,x)\in(0,T]\times\Omega \qquad (42)$$

最后，利用(9)，$(28)_j$，不难得到

$$m_i(0,x,\varepsilon)\leqslant h_i(x,\varepsilon)\leqslant M_i(0,x,\varepsilon) \quad (i=1,2)\quad (0<\varepsilon\ll 1) \qquad (43)$$

由(38)～(43)，利用比较定理[8],[9]知，对足够小的$\varepsilon>0$，初值一边值问题 (1)～(3) 存在一组解$u_i(t,x,\varepsilon)$ $(i=1,2)$，满足不等式：

$$m_i(t,x,\varepsilon)\leqslant u_i(t,x,\varepsilon)\leqslant M_i(t,x,\varepsilon) \quad (i=1,2)\quad (t,x)\in[0,T]\times(\Omega+\partial\Omega)$$

由此便得到在$(t,x)\in[0,T]\times(\Omega+\partial\Omega)$上成立一致有效的展开式：

$$u_i(t,x,\varepsilon)=\sum_{j=0}^{m-1}U_{ij}\varepsilon^j+\sum_{j=1}^{m}\tilde{U}_{ij}\varepsilon^j+O(\varepsilon^m) \quad (i=1,2)\quad (0<\varepsilon\ll 1)$$

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Singular Perturbation of Initial-Boundary Value Problems for a Class of Reaction Diffusion Systems

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Abstract

In this paper, a class of singularly perturbed initial-boundary value problems for the reaction diffusion systems is considered. Using the theory of differential inequality, we prove that the initial-boundary value problems have a solution and obtain their asymptotic expansion.

Key words reaction diffusion system, singular perturbation, comparison theorem, asymptotic expansion