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应力函数一般解的补充

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摘　要

本文指出平面问题极坐标形式应力函数一般解并不完备，不能处理曲杆受任意边界分布力的问题．为此，提出两个新的应力函数，将一般解作若干补充之后，能解曲杆 $r=a,b$ 上受任意分布力的问题．这是包含区域边界几何参数的新的应力函数．

一、应力函数一般解的局限性

由Michell提出的平面问题极坐标形式的应力函数一般式为[1]：

$$\begin{aligned}\Phi_0=&A_0\ln r+B_0r^2+C_0r^2\ln r+D_0r^2\phi+E_0\phi\\&+A_1r\phi\sin\phi+\left[B_1r^3+C_1\frac{1}{r}+D_1r\ln r\right]\cos\phi\\&-\bar{A}_1r\phi\cos\phi+\left[\bar{B}_1r^3+\bar{C}_1\frac{1}{r}+\bar{D}_1r\ln r\right]\sin\phi\\&+\sum_{n=2}^{\infty}[A_nr^n+B_nr^{n+2}+C_nr^{-n}+D_nr^{-n+2}]\cos n\phi\\&+\sum_{n=2}^{\infty}[\bar{A}_nr^n+\bar{B}_nr^{n+2}+\bar{C}_nr^{-n}+\bar{D}_nr^{-n+2}]\sin n\phi\end{aligned}\tag{1.1}$$

A_0，B_0，$C_0\cdots\bar{A}_n$，$\bar{B}_n$，$\bar{C}_n$，$\bar{D}_n$为待定系数．用它可以分析整体圆环内外表面受任意分布力（图1）的情况和许多复杂的平面问题．在著作[1]中详细论述了与各项相对应的应力状态，以及多余待定常数的物理意义和处理方法．

但是当用函数Φ_0求解曲杆表面受均布剪力或 $\cos\phi$，$\sin\phi$分布的正应力、剪应力时（图2），就得不到解．即

当 (i)
$$\left.\begin{aligned}r=a:&\quad \sigma_r=0,\ \tau_{r\varphi}=\tau_a\\r=b:&\quad \sigma_r=0,\ \tau_{r\varphi}=\tau_b\end{aligned}\right\}\tag{1.2}$$

(ii)
$$\left.\begin{aligned}r=a:&\quad \sigma_r=\sigma_a\cos\phi,\quad \tau_{r\varphi}=\tau_a\sin\phi\\r=b:&\quad \sigma_r=\sigma_b\cos\phi,\quad \tau_{r\varphi}=\tau_b\sin\phi\end{aligned}\right\}\tag{1.3}$$

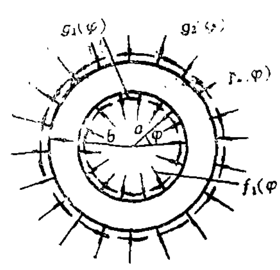

图 1

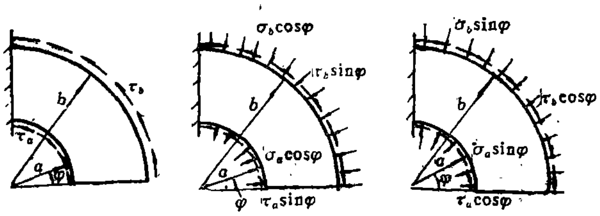

图 2

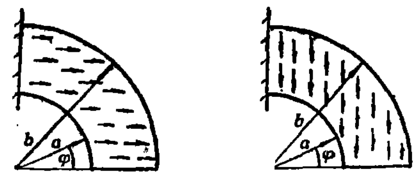

图 3

(iii)

$$\left.\begin{array}{lll} r=a: & \sigma_r=\sigma_a\sin\phi, & \tau_{r\varphi}=\tau_a\cos\phi \\ r=b: & \sigma_r=\sigma_b\sin\phi, & \tau_{r\varphi}=\tau_b\cos\phi \end{array}\right\} \tag{1.4}$$

等几种边界受力条件时，无法求解。

因此，Φ_0具有明显的局限性，它没有足够的项表达均布剪力和$\cos\phi$，$\sin\phi$分布力的作用，而这些项都是低阶项，在任意分布的面力中占主要成分。因此必须对Φ_0进行补充。

二、改进后应力函数形式

考虑到上述缺陷，我们构造了几个新的应力函数，将其补充进Φ_0中，使它更完备，它能充分表达曲杆及其它问题的各种受力状态。改进后的应力函数Φ为：

$$\begin{aligned} \Phi= & A_0\ln r+B_0r^2+C_0r^2\ln r+D_0r^2\phi+E_0\phi+\underline{F_0\phi\ln r+G_0\phi r^2\ln r} \\ & +A_1r\phi\sin\phi+\left[B_1r^3+C_1\frac{1}{r}+D_1r\ln r\right]\cos\phi \\ & +\underline{E_1\left\{\left[\frac{a^2b^2}{r}\ln r+r^3\ln r\right]\cos\phi+\left[\frac{a^2b^2}{r}+2(a^2+b^2)r\ln r-r^3\right]\phi\sin\phi\right\}} \\ & -\bar{A}_1r\phi\cos\phi+\left[\bar{B}_1r^3+\bar{C}_1\frac{1}{r}+\bar{D}_1r\ln r\right]\sin\phi \\ & +\underline{\bar{E}_1\left\{\left[\frac{a^2b^2}{r}\ln r+r^3\ln r\right]\sin\phi-\left[\frac{a^2b^2}{r}+2(a^2+b^2)r\ln r-r^3\right]\phi\cos\phi\right\}} \\ & +\sum_{n=2}^{\infty}[A_nr^n+B_nr^{n+2}+C_nr^{-n}+D_nr^{-n+2}]\cos n\phi \\ & +\sum_{n=2}^{\infty}[\bar{A}_nr^n+\bar{B}_nr^{n+2}+\bar{C}_nr^{-n}+\bar{D}_nr^{-n+2}]\sin n\phi \end{aligned} \tag{2,1}$$

式中A_0，B_0，C_0，…，$\bar{A}_n$，$\bar{B}_n$，$\bar{C}_n$，$\bar{D}_n$等为待定常数。a，b为曲杆或圆环的内、外径。下面划有横线的项为新补充的项。

根据求应力的关系式

$$\left.\begin{aligned} \sigma_r &=\frac{1}{r}\frac{\partial\Phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\Phi}{\partial\phi^2} \\ \sigma_\varphi &=\frac{\partial^2\Phi}{\partial r^2} \\ \tau_{r\varphi} &=-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\Phi}{\partial\phi}\right) \end{aligned}\right\} \tag{2.2}$$

求得相应的应力为：

$$\sigma_r=A_0\frac{1}{r^2}+B_0[2\ln r+1]+2C_0+2D_0\phi+F_0\frac{1}{r^3}\phi+G_0(2\ln r+1)\phi$$

$$+\left[A_1\frac{2}{r}+2B_1r-2C_1\frac{1}{r^3}+D_1\frac{1}{r}\right]\cos\phi$$

$$+E_1\left\{\left[-\frac{2a^2b^2}{r^3}\ln r+\frac{3a^2b^2}{r^3}+2r\ln r-r+\frac{4(a^2+b^2)}{r}\ln r\right]\cos\phi\right.$$

$$\left.+\left[-\frac{2a^2b^2}{r^3}+\frac{2(a^2+b^2)}{r}-2r\right]\phi\sin\phi\right\}$$

$$+\left[\bar{A}_1\frac{2}{r}+2\bar{B}_1r-2\bar{C}_1\frac{1}{r^3}+\bar{D}_1\frac{1}{r}\right]\sin\phi$$

$$+\bar{E}_1\left\{\left[\frac{3a^2b^2}{r^3}-\frac{2a^2b^2}{r^3}\ln r+2r\ln r-r+\frac{4(a^2+b^2)}{r}\ln r\right]\sin\phi\right.$$

$$\left.+\left[\frac{2a^2b^2}{r^3}-\frac{2(a^2+b^2)}{r}+2r\right]\phi\cos\phi\right\}$$

$$+\sum_{n=2}^{\infty}[A_n(n-n^2)r^{n-2}+B_n(n+2-n^2)r^n+C_n(-n-n^2)r^{-n-2}$$

$$+D_n((-n+2-n^2)r^{-n}]\cos n\phi$$

$$+\sum_{n=2}^{\infty}[\bar{A}_n(n-n^2)r^{n-2}+\bar{B}_n(n+2-n^2)r^n+\bar{C}_n(-n-n^2)r^{-n-2}$$

$$+\bar{D}_n(-n+2-n^2)r^{-n}]\sin n\phi \tag{2.3}$$

$$\sigma_\varphi=-A_0\frac{1}{r^2}+B_0[2\ln r+3]+2C_0+2D_0\phi-F_0\frac{1}{r^2}\phi+G_0(2\ln r+3)\phi$$

$$+\left[6B_1r+C_1\frac{2}{r^3}+D_1\frac{1}{r}\right]\cos\phi$$

$$+E_1\left\{\left[\frac{2a^2b^2}{r^3}\ln r-\frac{3a^2b^2}{r^3}+6r\ln r+5r\right]\cos\phi\right.$$

$$\left.+\left[\frac{2a^2b^2}{r^3}+2(a^2+b^2)\frac{1}{r}-6r\right]\phi\sin\phi\right\}$$

$$+\left[6\bar{B}_1r+\bar{C}_1\frac{2}{r^3}+\bar{D}_1\frac{1}{r}\right]\sin\phi$$

$$+\bar{E}_1\left\{\left[\frac{2a^2b^2}{r^3}\ln r-\frac{3a^2b^2}{r^3}+6r\ln r+5r\right]\sin\phi\right.$$

$$\left.+\left[-\frac{2a^2b^2}{r^3}-2(a^2+b^2)\frac{1}{r}+6r\right]\phi\cos\phi\right\}$$

$$+\sum_{n=2}^{\infty}[A_n n(n-1)r^{n-2}+B_n(n+1)(n+2)r^n$$

$$+C_n n(n+1)r^{-n-2}+D_n(n-1)(n-2)r^{-n}]\cos n\phi$$

$$+\sum_{n=2}^{\infty}[\bar{A}_n n(n-1)r^{n-2}+\bar{B}_n(n+1)(n+2)r^n$$

$$+\bar{C}_n n(n+1)r^{-n-2}+\bar{D}_n(n-1)(n-2)r^{-n}]\sin n\phi \tag{2.4}$$

$$\tau_{r\varphi}=-D_0+E_0\frac{1}{r^2}+F_0\frac{1}{r^2}(\ln r-1)-G_0(\ln r+1)$$

$$+\left[2B_1 r-C_1\frac{2}{r^3}+D_1\frac{1}{r}\right]\sin\phi$$

$$+E_1\left\{\left[-\frac{2a^2b^2}{r^3}\ln r+\frac{3a^2b^2}{r^3}+2r\ln r+3r-\frac{2(a^2+b^2)}{r}\right]\sin\phi\right.$$

$$\left.+\left[\frac{2a^2b^2}{r^3}-\frac{2(a^2+b^2)}{r}+2r\right]\phi\cos\phi\right\}$$

$$+\left[-2\bar{B}_1 r+\bar{C}_1\frac{2}{r^3}-\bar{D}_1\frac{1}{r}\right]\cos\phi$$

$$+\bar{E}_1\left\{\left[\frac{2a^2b_2}{r^3}\ln r-\frac{3a^2b^2}{r^3}-2r\ln r-3r+\frac{2(a^2+b^2)}{r}\right]\cos\phi\right.$$

$$\left.+\left[\frac{2a^2b^2}{r^3}-\frac{2(a^2+b^2)}{r}+2r\right]\phi\sin\phi\right\}+\sum_{n=2}^{\infty}[A_n n(n-1)r^{n-2}+B_n n(n+1)r^n$$

$$+C_n n(n-1)r^{-n+2}+D_n n(-n+1)r^{-n}]\sin n\phi+\sum_{n=2}^{\infty}[-\bar{A}_n n(n-1)r^{n-2}$$

$$-\bar{B}_n n(n+1)r^n+\bar{C}_n n(n+1)r^{-n+2}+\bar{D}_n n(-n+1)r^{-n}]\cos n\phi \tag{2.5}$$

三、曲杆在r＝a，b处受任意分布力之解

设曲杆在 $r=a$，b 边受有任意分布的面力作用，则可以将此面力表达成$(0,2\pi)$区间上的Fourier 级数形式，即

$$\left.\begin{aligned}\sigma_{r(r=a)}&=e_0+\sum_{n=1}^{\infty}e_n\cos n\phi+\sum_{n=1}^{\infty}f_n\sin n\phi\\ \sigma_{r(r=b)}&=\bar{e}_0+\sum_{n=1}^{\infty}\bar{e}_n\cos\phi+\sum_{n=1}^{\infty}\bar{f}_n\sin n\phi\end{aligned}\right\}$$

$$\left.\begin{aligned}\tau_{r\varphi(r=a)}&=g_0+\sum_{n=1}^{\infty}g_n\cos n\phi+\sum_{n=1}^{\infty}h_n\sin n\phi\\ \tau_{r\varphi(r=b)}&=\bar{g}_0+\sum_{n=1}^{\infty}\bar{g}_n\cos n\phi+\sum_{n=1}^{\infty}\bar{h}_n\sin n\phi\end{aligned}\right\} \tag{3.1}$$

式中 e_0，e_n，f_n，…，$\bar{g}_n$，$\bar{h}_n$ 为根据实际载荷求得的各项Fourier 系数。

边界条件为(图2)：

$$\left.\begin{aligned}&r=a: && \sigma_r=\sigma_{r(r=a)}, && \tau_{r\varphi}=\tau_{r\varphi(r=a)}\\ &r=b: && \sigma_r=\sigma_{r(r=b)}, && \tau_{r\varphi}=\tau_{r\varphi(r=b)}\\ &\phi=0: && \int_a^b\sigma_\varphi dr=P_0,\quad \int_a^b\sigma_\varphi r dr=M_0, && \int_a^b\tau_{r\varphi}dr=Q_0\\ &\phi=\alpha: && \int_a^b\sigma_\varphi dr=P_\alpha,\quad \int_a^b\sigma_\varphi r dr=M_\alpha, && \int_a^b\tau_{r\varphi}dr=Q_\alpha\end{aligned}\right\} \tag{3.2}$$

考虑到曲杆的整体平衡，外力的合力和合力矩为零，即$\sum M=0$，$\sum X=0$，$\sum Y=0$。因此，对$\phi=0$，$\phi=\alpha$ 的条件，实际只有一个是独立的。作为一个例子，我们取$\phi=0$ 处的三个条件来求解。

采用应力函数Φ来求解。将(2.3)，(2.4)，(2.5)，(3.1)式代入(3.2)，比较各项系数，可以解得：

$$A_0=-\frac{4a^2b^2}{N_1}\ln\frac{a}{b}\Big\{M_0-\frac{a^2b^2}{b^2-a^2}[e_0-\bar{e}_0]\ln\frac{a}{b}-\frac{1}{2}[\bar{e}_0b^2-e_0a^2]$$

$$-b^2\bar{h}_1+a^2h_1-\sum_{n=2}^{\infty}\frac{1}{n}(b^2\bar{h}_n-a^2h_n)\Big\}+\frac{a^2b^2}{b^2-a^2}(e_0-\bar{e}_0)$$

$$B_0=\frac{2(b^2-a^2)}{N_1}\Big\{M_0-\frac{a^2b^2}{b^2-a^2}[e_0-\bar{e}_0]\ln\frac{a}{b}-\frac{1}{2}[\bar{e}_0b^2-e_0a^2]$$

$$-b^2\bar{h}_1+a^2h_1-\sum_{n=2}^{\infty}\frac{1}{n}(b^2\bar{h}_n-a^2h_n)\Big\}$$

$$C_0=\frac{-1}{N_1}[2b^2\ln b-2a^2\ln a+b^2-a^2]\Big\{M_0-\frac{a^2b^2}{b^2-a^2}[e_0-\bar{e}_0]\ln\frac{a}{b}$$

$$-\frac{1}{2}[\bar{e}_0b^2-e_0a^2]-b^2\bar{h}_1+a^2h_1-\sum_{n=2}^{\infty}\frac{1}{n}(b^2\bar{h}_n-a^2h_n)\Big\}$$

$$+\frac{1}{2(b^2-a^2)}[\bar{e}_0b^2-e_0\bar{a}^2]$$

$$D_0=\frac{-g_0a^2+\bar{g}_0b^2}{N_1}[b^2-a^2+2b^2\ln b-2a^2\ln]$$

$$E_0=\frac{1}{2}(g_0a^2+\bar{g}_0b^2)+\frac{1}{2N_1}(g_0a^2-\bar{g}_0b^2)\Big[b^4-a^4+4a^2b^2(\ln ab-1)\ln\frac{a}{b}\Big]$$

$$F_0=\frac{-4a^2b^2}{N_1}(g_0a^2-\bar{g}_0b^2)\ln\frac{a}{b}$$

$$G_0=\frac{2(b^2-a^2)}{N_1}(g_0a^2-\bar{g}_0b^2)$$

$$C_1=\frac{a^2b^2}{2N_2}\Big\{-P_0+b\bar{e}_0-e_0+\frac{3}{2}[\bar{h}_1b-h_1a]-\frac{ab}{b^2-a^2}[h_1b-\bar{h}_1a]\ln\frac{a}{b}$$

$$-E_1\Big[2(a^2\ln b+b^2\ln a)\ln\frac{a}{b}+(b^2-a^2)\ln ab\Big]$$

$$+\sum_{n=2}^{\infty}\Big\{\frac{1}{1+n}[\bar{h}_nb-h_na]-\frac{1}{1-n^2}[\bar{h}_nb-\bar{e}_nb-h_na+e_na]\Big\}\Big\}$$

$$A_1=-(a^2+b^2)E_1\ln ab+\frac{1}{4}[e_1a+\bar{e}_1b-h_1a-\bar{h}_1b]$$

$$B_1=-\frac{1}{a^2b^2}C_1-E_1\ln ab+\frac{\bar{h}_1b-h_1a}{2(b^2-a^2)}$$

$$D_1=\frac{2(a^2+b^2)}{a^2b^2}C_1+E_1[-a^2-b^2+2a^2\ln b+2b^2\ln a]+\frac{ab}{b^2-a^2}(h_1b-\bar{h}_1a)$$

$$\bar{C}_1=-\frac{a^2b^2}{2N_2}\Big\{Q_0-g_0a+\bar{g}_0b+\frac{1}{2}(g_1a-\bar{g}_1b)+\frac{ab}{b^2-a^2}[bg_1-a\bar{g}_1]\ln\frac{a}{b}$$

$$+2\bar{E}_1\Big[(a^2+b^2-a^2\ln b-b^2\ln a)\ln\frac{a}{b}+(b^2-a^2)\Big(1-\frac{1}{2}\ln ab\Big)\Big]$$

$$+\sum_{n=2}^{\infty}\Big\{\frac{1}{1+n}(\bar{g}_nb-g_na)+\frac{n}{1-n^2}[\bar{g}_nb+\bar{f}_nb-g_na-f_na]\Big\}\Big\}$$

$$\bar{A}_1=-(a^2+b^2)\bar{E}_1\ln ab+\frac{1}{4}[f_1a+\bar{f}_1b+g_1a+\bar{g}_1b]$$

$$\bar{B}_1=-\frac{1}{a^2b^2}\bar{C}_1-\bar{E}_1\ln ab+\frac{1}{2(b^2-a^2)}(g_1a-\bar{g}_1b)$$

$$\bar{D}_1=\frac{2(a^2+b^2)}{ab}\bar{C}_1+\bar{E}_1[-a^2-b^2+2a^2\ln b+2b^2\ln a]+\frac{ab}{b^2-a^2}(\bar{g}_1a-g_1b)$$

其中

$$E_1=\frac{1}{4N_2}[e_1a-\bar{e}_1b-h_1a+\bar{h}_1b]$$

$$\bar{E}_1=\frac{1}{4N_2}[f_1a-\bar{f}_1b+g_1a-\bar{g}_1b]$$

$$N_1=(b^2-a^2)^2-4a^2b^2\Big(\ln\frac{a}{b}\Big)^2$$

$$N_2=b^2-a^2+(a^2+b^2)\ln\frac{a}{b}$$

对系数A_n，B_n，C_n，D_n，$\bar{A}_n$，$\bar{B}_n$，$\bar{C}_n$，$\bar{D}_n$有

$$A_n=\frac{1}{\Delta}\Big\{\frac{1}{2}(b^{2n}-a^{2n})(\bar{e}_nb^{n+2}-e_na^{n+2})-\frac{n+2}{2n}(b^{2n}-a^{2n})(\bar{h}_nb^{n+2}-h_na^{n+2})$$

$$+\frac{nb^n}{2(1-n)}(b^{2n+2}-a^{2n+2})(\bar{e}_n-\bar{h}_n)-\frac{na^n}{2(1-n)}(b^{2n+2}-a^{2n+2})(e_n-h_n)\Big\}$$

$$B_n=\frac{1}{\Delta}\Big\{\frac{b^n}{2}(b^{2n}-a^{2n})(\bar{e}_n-\bar{h}_n)-\frac{a^n}{2}(b^{2n}-a^{2n})(e_n-h_n)$$

$$-\frac{n}{2(n+1)}(b^{2n-2}-a^{2n-2})(\bar{e}_nb^{n+2}-e_na^{n+2})$$

$$+\frac{n+2}{2(n+1)}(b^{2n-2}-a^{2n-2})(\bar{h}_nb^{n+2}-h_na^{n+2})$$

$$D_n=\frac{b}{4(1+n)}(\bar{e}_n-\bar{h}_n)+\frac{a^n}{4(1-n)}(e_n-h_n)-A_n\frac{n}{2}(b^{2n-2}+a^{2n-2})-B_n\frac{n+1}{2}(b^{2n}+a^{2n})$$

$$C_n=\frac{1}{b^{-n-2}+a^{-n-2}}\Big\{\frac{-1}{n(1+n)}(\bar{h}_n+h_n)+\frac{1-n}{1+n}D_n(b^{-n}+a^{-n})$$

$$+B_n(b^n+a^n)+A_n\frac{n-1}{n+1}(b^{n-2}+a^{n-2})\Big\}$$

$$\bar{A}_n=\frac{1}{\Delta}\Big\{\frac{1}{2}(b^{2n}-a^{2n})(\bar{f}_nb^{n+2}-f_na^{n+2})-\frac{n+2}{2n}(b^{2n}-a^{2n})(g_na^{n+2}-\bar{g}_nb^{n+2})$$

$$+\frac{nb^n}{2(1-n)}(b^{2n+2}-a^{2n+2})(\bar{f}_n+\bar{g}_n)-\frac{na^n}{2(1-n)}(b^{2n+2}-a^{2n+2})(f_n+g_n)\Big\}$$

$$\bar{B}_n=\frac{1}{\Delta}\Big\{\frac{b^n}{2}(b^{2n}-a^{2n})(\bar{f}_n+\bar{g}_n)-\frac{a^n}{2}(b^{2n}-a^{2n})(f_n+g_n)$$

$$-\frac{n}{2(1+n)}(b^{2n-2}-a^{2n-2})(\bar{f}_nb^{n+2}-f_na^{n+2})$$

$$+\frac{n+2}{2(1+n)}(b^{2n-2}-a^{2n-2})(g_na^{n+2}-\bar{g}_nb^{n+2})\Big\}$$

$$\bar{D}_n=\frac{b^n}{4(1-n)}(\bar{f}_n+\bar{g}_n)+\frac{a^n}{4(1-n)}(f_n+g_n)$$

$$-\bar{A}_n\frac{n}{2}(b^{2n-2}+a^{2n-2})-\bar{B}_n\frac{n+1}{2}(b^{2n}+a^{2n})$$

$$\bar{C}_n=\frac{1}{b^{-n-2}+a^{-n-2}}\Big\{\frac{1}{n(1+n)}(g_n+\bar{g}_n)+\frac{1-n}{1+n}\bar{D}_n(b^{-n}+a^{-n})$$

$$+\bar{B}_n(b^n+a^n)+\bar{A}_n\frac{n-1}{n+1}(b^{n-2}+a^{n-2})\Big\}$$

其中

$$\Delta=(1-n^2)(b^{2n}-a^{2n})^2+n^2(b^{2n-2}-a^{2n-2})(b^{2n+2}-a^{2n+2})$$

由上可以看出Φ函的前三项（含A_0，B_0，C_0系数）是反映杆的纯弯应力状态。它由$r=a$，b边的均布径向力e_0，$\bar{e}_0$和弯矩M_0决定，同时还包含$r=a$，b边上分布剪力$h_n\sin\phi$，$\bar{h}_n\sin\phi$的影响。

Φ的第4～7项（含系数D_0，E_0，F_0，G_0），反映边界$r=a$，b上受均布剪力g_0，$\bar{g}_0$的纯剪应力状态。

Φ中带$\cos\phi$，$\phi\sin\phi$的项（系数为A_1，B_1，C_1，D_1，E_1），反映边界$r=a$，b上受按$\cos\phi$

分布的径向力、按$\sin\phi$分布的剪力的作用，并与轴向力P_0平衡。

Φ中带$\sin\phi$，$\phi\cos\phi$的项（系数为$\bar{A}_1$，$\bar{B}_1$，$\bar{C}_1$，$\bar{D}_1$，$\bar{E}_1$），反映边界$r=a$，b上受$\sin\phi$分布的径向力和$\cos\phi$分布的剪力的作用，并与横力Q_0相平衡。

系数A_n，B_n，…，$\bar{C}_n$，$\bar{D}_n$完全可以由$r=a$，b边上σ_r，τ的条件决定，含有这些系数的项，反映了边界受$\sin n\phi$，$\cos n\phi$规律分布面力的作用。

利用Φ中$\cos\phi$，$\sin\phi$以前的各项，还可以解决曲杆受水平方向及垂直方向均布体积力的问题[2](图3)。利用Φ，包含更多的级数项，还可以解决沿ϕ方向体积力按复杂规律变化的问题。Φ是比Φ_0应用范围更广的一个应力函数一般式。

值得指出的是：在Φ中新增加的应力函数项，是含有边界尺寸因素a，b的一类应力函数。实际上，在弹性力学的应力函数中，还存在着一批这种含有边界尺寸因素的应力函数，这些函数的存在将大大扩大我们获得解析解的范围。本文只是扩大极坐标一般解的一个特例。

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The Improvement of the General Expression for the Stress Function Φ of the Two-Dimensional Problem

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Abstract

In this paper, it is pointed that the general expression for the stress function Φ_0 of the plane problem in polar coordinates is incomplete. The problems of the curved bar with an arbitrary distributive load at the boundaries can't be solved by this stress function. For this reason, we suggest two new stress functions and put them in the general expression. Then, the problems of the curved bar applied with an arbitrary distributive load at $r=a$, b boundaries can be solved. This is a new stress function including geometric boundary constants.