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正交异性矩形板后屈曲摄动分析*

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摘　　要

本文从正交异性板 Kármán 型大挠度方程出发，以挠度为摄动参数，采用直接摄动法，研究了正交异性矩形板在面内压缩作用下的后屈曲性态。

本文讨论了两种面内边界条件，同时考虑了初始挠度的影响。

本文给出了多种复合材料板的计算结果。所得结果与实验结果的比较表明二者是一致的。

一、引　　言

正交异性板理论是研究复合材料板，层合板以及加筋板的基础。弄清正交异性矩形板在面内压缩作用下的后屈曲性态具有重要的实用意义。

对于各向同性矩形板屈后性态已作过广泛的研究，相比之下，人们对于正交异性矩形板后屈曲性态还缺乏更多的了解。关于在面内压缩作用下，四边简支正交异性矩形板的后屈曲问题，许多学者已用各种方法计算过，如Yusuff[1]，Prabhakara 和 Chia[2] 用双 Fourier 级数法，Chan[3] 用 Raleigh-Ritz 法，此外，Chandra 和 Raju[4] 用连续摄动法，以载荷为摄动参数求得了正交异性矩形板后屈曲平衡路径二级渐近解。这种以载荷为摄动参数的连续摄动法曾为 Stein 在文[5]中所使用。近些年来，Bank[6][7][8] 又对正交异性矩形板作过诸多理论分析和实验研究。由于各人所用方法不同，所考虑的面内边界条件也不完全相同，因此，所得结果存在或多或少的差异。

在文[9]中，作者曾以挠度为摄动参数，采用直接摄动法研究了各向同性矩形板在面内压缩作用下的后屈曲性态。这一方法将在本文中得到进一步运用。

本文将正交异性板 Kármán 型大挠度方程化为一系列线性方程求解，在求得大挠度渐近解的基础上，利用边界条件直接导得后屈曲平衡路径的四级渐近表达式。

本文讨论了两种面内边界条件，一种为纵向边缘可移简支，一种为纵向边缘不可移简支。

本文同时考虑了初挠度的影响。初挠度的形式取作和矩形板小挠度解的形式一致。

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二、基 本 方 程

假定四边简支正交异性矩形板的长为a，宽为b，厚度为t，受到对边均布压力。取坐标系如图1所示。并以W^*和W分别表示初始的和附加的挠度，以ϕ表示应力函数，那么正交异性矩形板 Kármán 型大挠度方程可表为如下形式：

$$L_1W=t\Big[\frac{\partial^2\phi}{\partial y^2}\frac{\partial^2 W}{\partial x^2}-2\frac{\partial^2\phi}{\partial x\partial y}\frac{\partial^2 W}{\partial x\partial y}+\frac{\partial^2\phi}{\partial x^2}\frac{\partial^2 W}{\partial y^2}+\frac{\partial^2\phi}{\partial y^2}\frac{\partial^2 W^*}{\partial x^2}$$
$$-2\frac{\partial^2\phi}{\partial x\partial y}\frac{\partial^2 W^*}{\partial x\partial y}+\frac{\partial^2\phi}{\partial x^2}\frac{\partial^2 W^*}{\partial y^2}\Big] \tag{2.1}$$

$$L_2\phi=\Big[\Big(\frac{\partial^2 W}{\partial x\partial y}\Big)^2-\frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial y^2}+2\frac{\partial^2 W}{\partial x\partial y}\frac{\partial^2 W^*}{\partial x\partial y}-\frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W^*}{\partial y^2}-\frac{\partial^2 W}{\partial y^2}\frac{\partial^2 W^*}{\partial x^2}\Big] \tag{2.2}$$

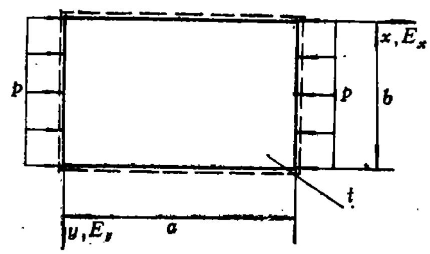


图1 正交异性矩形板

其中

$$\left.\begin{aligned}L_1&=D_x\frac{\partial^4}{\partial x^4}+2(D_x\nu_y+2D_{xy})\frac{\partial^4}{\partial x^2\partial y^2}+D_y\frac{\partial^4}{\partial y^4}\\ L_2&=\frac{1}{E_y}\frac{\partial^4}{\partial x^4}+2\Big(\frac{1}{2G}-\frac{\nu_x}{E_x}\Big)\frac{\partial^4}{\partial x^2\partial y^2}+\frac{1}{E_x}\frac{\partial^4}{\partial y^4}\end{aligned}\right\} \tag{2.3}$$

$$D_x=\frac{E_xt^3}{12(1-\nu_x\nu_y)},\quad D_y=\frac{E_yt^3}{12(1-\nu_x\nu_y)},\quad D_{xy}=\frac{Gt^3}{12} \tag{2.4}$$

D_x，D_y；D_{xy}分别为抗弯、抗扭刚度，E_x，E_y，G为弹性模数，ν_x，ν_y 为 Poisson 比，且有

$$\nu_y=\nu_x\frac{E_y}{E_x} \tag{2.5}$$

板中的内力：

$$N_x=t\frac{\partial^2\phi}{\partial y^2},\quad N_{xy}=-t\frac{\partial^2\phi}{\partial x\partial y},\quad N_y=t\frac{\partial^2\phi}{\partial x^2} \tag{2.6}$$

面内位移U，V与W，W^*及ϕ的关系为：

$$\left.\begin{aligned}&\frac{\partial U}{\partial x}+\frac{1}{2}\left(\frac{\partial W}{\partial x}\right)^2+\frac{\partial W}{\partial x}\frac{\partial W^*}{\partial x}=\frac{1}{E_x}\frac{\partial^2\phi}{\partial y^2}-\frac{\nu_y}{E_y}\frac{\partial^2\phi}{\partial x^2}\\&\frac{\partial V}{\partial y}+\frac{1}{2}\left(\frac{\partial W}{\partial y}\right)^2+\frac{\partial W}{\partial y}\frac{\partial W^*}{\partial y}=\frac{1}{E_y}\frac{\partial^2\phi}{\partial x^2}-\frac{\nu_x}{E_x}\frac{\partial^2\phi}{\partial y^2}\\&\frac{\partial U}{\partial y}+\frac{\partial V}{\partial x}+\frac{\partial W}{\partial x}\frac{\partial W}{\partial y}+\frac{\partial W}{\partial x}\frac{\partial W^*}{\partial y}+\frac{\partial W}{\partial y}\frac{\partial W^*}{\partial x}=-\frac{1}{G}\frac{\partial^2\phi}{\partial x\partial y}\end{aligned}\right\}\tag{2.7}$$

假定边界支承为四边简支的，那么边界条件为

$$x=0,\ a;\ W=W_{,xx}=0,\ N_{xy}=0 \tag{2.8a}$$

$$\int_0^b N_x dy+p=0 \tag{2.8b}$$

$$y=0,\ b;\ W=W_{,yy}=0,\ N_{xy}=0 \tag{2.9a}$$

$$\int_0^a N_y dx=0 \qquad \text{（纵向边缘可移简支）} \tag{2.9b}$$

$$V=\text{常数} \qquad \text{（纵向边缘不可移简支）} \tag{2.9c}$$

单位端部缩短：

$$\Delta_x/a=-\frac{1}{ab}\int_0^b\int_0^a\frac{\partial U}{\partial x}dxdy$$

$$=-\frac{1}{ab}\int_0^b\int_0^a\left[\left(\frac{1}{E_x}\frac{\partial^2\phi}{\partial y^2}-\frac{\nu_y}{E_y}\frac{\partial^2\phi}{\partial x^2}\right)-\frac{1}{2}\left(\frac{\partial W}{\partial x}\right)^2-\frac{\partial W}{\partial x}\frac{\partial W^*}{\partial x}\right]dxdy \tag{2.10}$$

$$\Delta_y/b=-\frac{1}{ab}\int_0^a\int_0^b\frac{\partial V}{\partial y}dydx$$

$$=-\frac{1}{ab}\int_0^a\int_0^b\left[\left(\frac{1}{E_y}\frac{\partial^2\phi}{\partial x^2}-\frac{\nu_x}{E_x}\frac{\partial^2\phi}{\partial y^2}\right)-\frac{1}{2}\left(\frac{\partial W}{\partial y}\right)^2-\frac{\partial W}{\partial y}\frac{\partial W^*}{\partial y}\right]dydx \tag{2.11}$$

引进

$$\left.\begin{aligned}&\bar{x}=\frac{\pi}{a}x,\quad \bar{y}=\frac{\pi}{b}y,\quad \beta=\frac{a}{b},\quad \bar{W}=\frac{W}{t}\sqrt{12(1-\nu_x\nu_y)}\\&\bar{W}^*=\frac{W^*}{t}\sqrt{12(1-\nu_x\nu_y)},\quad \bar{\phi}=\frac{\phi t}{\sqrt{D_xD_y}},\quad \frac{D_x\nu_y+2D_{xy}}{D_x}=\gamma_1^2\\&\sqrt{\frac{D_y}{D_x}}=\gamma_2^2,\quad E_y\left(\frac{1}{2G}-\frac{\nu_x}{E_x}\right)=\gamma_3^2,\quad \lambda_x=\frac{\sigma_x b^2 t}{4\pi^2\sqrt{D_xD_y}}\\&\delta_x=\frac{12(1-\nu_x\nu_y)b^2}{4\pi^2t^2}\frac{\Delta_x}{a},\quad \delta_y=\frac{12(1-\nu_x\nu_y)b^2}{4\pi^2t^2}\frac{\Delta_y}{b}\end{aligned}\right\}\tag{2.12}$$

那么方程（2.1）、（2.2）可化为如下无量纲形式（略去字母上的“—”号）：

$$L_1^*W=\gamma_1^2\beta^2\Big[\frac{\partial^2\phi}{\partial y^2}\frac{\partial^2W}{\partial x^2}-2\frac{\partial^2\phi}{\partial x\partial y}\frac{\partial^2W}{\partial x\partial y}+\frac{\partial^2\phi}{\partial x^2}\frac{\partial^2W}{\partial y^2}+\frac{\partial^2\phi}{\partial y^2}\frac{\partial^2W^*}{\partial x^2}-2\frac{\partial^2\phi}{\partial x\partial y}\frac{\partial^2W^*}{\partial x\partial y}+\frac{\partial^2\phi}{\partial x^2}\frac{\partial^2W^*}{\partial y^2}\Big] \tag{2.13}$$

$$L_2^*\phi=\gamma_2^2\beta^2\Big[\Big(\frac{\partial^2W}{\partial x\partial y}\Big)^2-\frac{\partial^2W}{\partial x^2}\frac{\partial^2W}{\partial y^2}+2\frac{\partial^2W}{\partial x\partial y}\frac{\partial^2W^*}{\partial x\partial y}-\frac{\partial^2W}{\partial x^2}\frac{\partial^2W^*}{\partial y^2}-\frac{\partial^2W}{\partial y^2}\frac{\partial^2W^*}{\partial x^2}\Big] \tag{2.14}$$

其中

$$\left.\begin{aligned}L_1^*&=\frac{\partial^4}{\partial x^4}+2\gamma_1^2\beta^2\frac{\partial^4}{\partial x^2\partial y^2}+\gamma_2^4\beta^4\frac{\partial^4}{\partial y^4}\\ L_2^*&=\frac{\partial^4}{\partial x^4}+2\gamma_3^2\beta^2\frac{\partial^4}{\partial x^2\partial y^2}+\gamma_2^4\beta^4\frac{\partial^4}{\partial y^4}\end{aligned}\right\} \tag{2.15}$$

边界条件化为

$$x=0,\ \pi;\quad W=W_{,xx}=0,\quad \frac{\partial^2\phi}{\partial x\partial y}=0 \tag{2.16a}$$

$$\frac{1}{\pi}\int_0^\pi\beta^2\frac{\partial^2\phi}{\partial y^2}dy+4\lambda_x\beta^2=0 \tag{2.16b}$$

$$y=0,\ \pi;\quad W=W_{,yy}=0,\quad \frac{\partial^2\phi}{\partial x\partial y}=0 \tag{2.17a}$$

$$\frac{1}{\pi}\int_0^\pi\frac{\partial^2\phi}{\partial x^2}dx=0 \quad (\text{可移简支}) \tag{2.17b}$$

$$\delta_y=0 \quad (\text{不可移简支}) \tag{2.17c}$$

单位端部缩短化为

$$\delta_x=-\frac{1}{4\pi^2\gamma_2^2\beta^2}\int_0^\pi\int_0^\pi\Big[\Big(\gamma_2^4\beta^2\frac{\partial^2\phi}{\partial y^2}-\nu_y\frac{\partial^2\phi}{\partial x^2}\Big)-\frac{1}{2}\gamma_2^2\Big(\frac{\partial W}{\partial x}\Big)^2-\gamma_2^2\frac{\partial W}{\partial x}\frac{\partial W^*}{\partial x}\Big]dxdy \tag{2.18}$$

$$\delta_y=-\frac{1}{4\pi^2\gamma_2^2\beta^2}\int_0^\pi\int_0^\pi\Big[\Big(\frac{\partial^2\phi}{\partial x^2}-\nu_y\beta^2\frac{\partial^2\phi}{\partial y^2}\Big)-\frac{1}{2}\gamma_2^2\beta^2\Big(\frac{\partial W}{\partial y}\Big)^2-\gamma_2^2\beta^2\frac{\partial W}{\partial y}\frac{\partial W^*}{\partial y}\Big]dydx \tag{2.19}$$

方程（2.13）至（2.19）为简支正交异性矩形板在面内压缩作用下后屈曲问题控制方程。本文将以摄动法来构造其渐近解。

三、大挠度渐近解

设方程(2.13)、(2.14)的解为如下渐近展开式：

$$W(x,y,\varepsilon)=\sum_{n=1}\varepsilon^nW_n(x,y),\quad \phi(x,y,\varepsilon)=\sum_{n=0}\varepsilon^n\phi_n(x,y) \tag{3.1}$$

并假定板的初挠度（或初始缺陷）与小挠度解的形式一致，即

$$W^*=\varepsilon A_{11}^{*}\sin mx\sin ny=\varepsilon\mu A_{11}^{(1)}\sin mx\sin ny \tag{3.2}$$

其中，$\mu=A_{11}^{*}/A_{11}^{(1)}$ 为缺陷参数。

将式(3.1)、(3.2)代入方程(2.13)、(2.14)便可获得各级摄动方程，采用文[9]相同的摄动步骤，计及边界条件(2.16a)、(2.17a)，我们可以得到大挠度渐近解

$$W=\varepsilon[A_{11}^{(1)}\sin mx\sin ny]+\varepsilon^3[A_{13}^{(3)}\sin mx\sin 3ny+A_{31}^{(3)}\sin 3mx\sin ny]+O(\varepsilon^5) \tag{3.3}$$

$$\begin{aligned}\phi=&-B_{00}^{(0)}\frac{y^2}{2}-b_{00}^{(0)}\frac{x^2}{2}+\varepsilon^2\Big[-B_{00}^{(2)}\frac{y^2}{2}-b_{00}^{(2)}\frac{x^2}{2}\\&+B_{20}^{(2)}\cos 2mx+B_{02}^{(2)}\cos 2ny\Big]+\varepsilon^4\Big[-B_{00}^{(4)}\frac{y^2}{2}-b_{00}^{(4)}\frac{x^2}{2}\\&+B_{20}^{(4)}\cos 2mx+B_{02}^{(4)}\cos 2ny+B_{22}^{(4)}\cos 2mx\cos 2ny\\&+B_{40}^{(4)}\cos 4mx+B_{04}^{(4)}\cos 4ny+B_{24}^{(4)}\cos 2mx\cos 4ny\\&+B_{42}^{(4)}\cos 4mx\cos 2ny\Big]+O(\varepsilon^6)\end{aligned} \tag{3.4}$$

其中，系数 $B_{00}^{(i)}$ 和 $b_{00}^{(i)}$ $(i=0,2,4,\cdots)$ 的关系为

$$\gamma_2^2(\beta^2B_{00}^{(0)}m^2+b_{00}^{(0)}n^2\beta^2)=\frac{m^4+2\gamma_1^2m^2n^2\beta^2+\gamma_2^4n^4\beta^4}{(1+\mu)} \tag{3.5}$$

$$\gamma_2^2(\beta^2B_{00}^{(2)}m^2+b_{00}^{(2)}n^2\beta^2)=\frac{1}{16}(m^4+\gamma_2^4n^4\beta^4)(1+2\mu)A_{11}^{(1)}A_{11}^{(1)} \tag{3.6}$$

$$\begin{aligned}\gamma_2^2(\beta^2B_{00}^{(4)}m^2+b_{00}^{(4)}n^2\beta^2)=&-\frac{1}{256}(1+2\mu)[2(1+\mu)^2\\&+(1+2\mu)]\Big[\frac{m^8}{(m^4+18\gamma_1^2m^2n^2\beta^2+81\gamma_2^4n^4\beta^4)-\gamma_2^2(\beta^2B_{00}^{(0)}m^2+b_{00}^{(0)}9n^2\beta^2)}\\&+\frac{\gamma_2^8n^8\beta^8}{(81m^4+18\gamma_1^2m^2n^2\beta^2+\gamma_2^4n^4\beta^4)-\gamma_2^2(\beta^2B_{00}^{(0)}9m^2+b_{00}^{(0)}n^2\beta^2)}\Big]A_{11}^{(1)}A_{11}^{(1)}A_{11}^{(1)}A_{11}^{(1)}\end{aligned} \tag{3.7}$$

其它系数皆可表为 $A_{11}^{(1)}$ 的形式，如

$$\left.\begin{aligned}
B_{20}^{(2)}&=\frac{1}{32}\frac{\gamma_2^2n^2\beta^2}{m^2}(1+2\mu)A_{11}^{(1)}A_{11}^{(1)}\\
B_{02}^{(2)}&=\frac{1}{32}\frac{m^2}{\gamma_2^2n^2\beta^2}(1+2\mu)A_{11}^{(1)}A_{11}^{(1)}\\
A_{13}^{(3)}&=\frac{1}{16}\frac{m^4}{(m^4+18\gamma_1^2m^2n^2\beta^2+81\gamma_2^4n^4\beta^4)-\gamma_2^2(\beta^2B_{00}^{(0)}m^2+b_{00}^{(0)}9n^2\beta^2)}\\
&\quad\cdot(1+\mu)(1+2\mu)A_{11}^{(1)}A_{11}^{(1)}A_{11}^{(1)}\\
A_{31}^{(3)}&=\frac{1}{16}\frac{\gamma_2^4n^4\beta^4}{(81m^4+18\gamma_1^2m^2n^2\beta^2+\gamma_2^4n^4\beta^4)-\gamma_2^2(\beta^2B_{00}^{(0)}9m^2+b_{00}^{(0)}n^2\beta^2)}\\
&\quad\cdot(1+\mu)(1+2\mu)A_{11}^{(1)}A_{11}^{(1)}A_{11}^{(1)}\\
B_{20}^{(4)}&=-\frac{1}{256}\frac{\gamma_2^2n^2\beta^2}{m^2}\\
&\quad\cdot\frac{\gamma_2^4n^4\beta^4}{(81m^4+18\gamma_1^2m^2n^2\beta^2+\gamma_2^4n^4\beta^4)-\gamma_2^2(\beta^2B_{00}^{(0)}9m^2+b_{00}^{(0)}n^2\beta^2)}\\
&\quad\cdot(1+\mu)^2(1+2\mu)A_{11}^{(1)}A_{11}^{(1)}A_{11}^{(1)}A_{11}^{(1)}\\
B_{02}^{(4)}&=-\frac{1}{256}\frac{m^2}{\gamma_2^2n^2\beta^2}\frac{m^4}{(m^4+18\gamma_1^2m^2n^2\beta^2+81\gamma_2^4n^4\beta^4)-\gamma_2^2(\beta^2B_{00}^{(0)}m^2+b_{00}^{(0)}9n^2\beta^2)}\\
&\quad\cdot(1+\mu)^2(1+2\mu)A_{11}^{(1)}A_{11}^{(1)}A_{11}^{(1)}A_{11}^{(1)}
\end{aligned}\right\}\tag{3.8}$$

四、后屈曲性态

现在考虑两种不同的面内边界条件

1.纵向边缘可移简支

将式（3.4）代入边界条件（2.16b）得

$$4\lambda_x\beta^2=\beta^2B_{00}^{(0)}+\varepsilon^2\beta^2B_{00}^{(2)}+\varepsilon^4\beta^2B_{00}^{(4)}+\cdots\tag{4.1}$$

将式（3.4）代入边界条件(2.17b)，我们有

$$b_{00}^{(i)}=0\qquad(i=0,2,4,\cdots)\tag{4.2}$$

那么由式(3.5)、(3.6)、(3.7)有

$$\gamma_2^2\beta^2B_{00}^{(0)}=\frac{m^4+2\gamma_1^2m^2n^2\beta^2+\gamma_2^4n^4\beta^4}{(1+\mu)m^2}\tag{4.3}$$

$$\gamma_2^2\beta^2B_{00}^{(2)}=\frac{1}{16}\frac{m^4+\gamma_2^4n^4\beta^4}{m^2}(1+2\mu)A_{11}^{(1)}A_{11}^{(1)}\tag{4.4}$$

$$\gamma_2^2\beta^2B_{00}^{(4)}=-\frac{1}{256m^2}(1+\mu)(1+2\mu)[2(1+\mu)^2+(1+2\mu)]$$

$$\cdot\Big[\frac{m^8}{(m^4+18\gamma_1^2m^2n^2\beta^2+81\gamma_2^4n^4\beta^4)(1+\mu)-(m^4+2\gamma_1^2m^2n^2\beta^2+\gamma_2^4n^4\beta^4)}$$

$$+\frac{\gamma_2^8 n^8\beta^8}{(81m^4+18\gamma_1^2 m^2n^2\beta^2+\gamma_2^4 n^4\beta^4)(1+\mu)-9(m^4+2\gamma_1^2 m^2n^2\beta^2+\gamma_2^4 n^4\beta^4)}\Bigg]$$

$$\cdot A_{11}^{(1)}A_{11}^{(1)}A_{11}^{(1)}A_{11}^{(1)} \tag{4.5}$$

将式(4.3)、(4.4)、(4.5)代入 (4.1) 得

$$\lambda_x=\frac{1}{4\gamma_2^2\beta^2}\Bigg\{\frac{m^4+2\gamma_1^2m^2n^2\beta^2+\gamma_2^4n^4\beta^4}{(1+\mu)m^2}+\frac{1}{16}\frac{m^4+\gamma_2^4n^4\beta^4}{m^2}(1+2\mu)A_{11}^{(1)}A_{11}^{(1)}\varepsilon^2$$

$$-\frac{1}{256m^2}(1+\mu)(1+2\mu)[2(1+\mu)^2$$

$$+(1+2\mu)]\Bigg[\frac{m^8}{(m^4+18\gamma_1^2m^2n^2\beta^2+81\gamma_2^4n^4\beta^4)(1+\mu)-(m^4+2\gamma_1^2m^2n^2\beta^2+\gamma_2^4n^4\beta^4)}$$

$$+\frac{\gamma_2^8n^8\beta^8}{(81m^4+18\gamma_1^2m^2n^2\beta^2+\gamma_2^4n^4\beta^4)(1+\mu)-9(m^4+2\gamma_1^2m^2n^2\beta^2+\gamma_2^4n^4\beta^4)}\Bigg]$$

$$\cdot A_{11}^{(1)}A_{11}^{(1)}A_{11}^{(1)}A_{11}^{(1)}\varepsilon^4+\cdots\Bigg\} \tag{4.6}$$

将式(3.3)、(3.4)及(3.2)代入 (2.18) 得

$$\delta_x=\lambda_x\gamma_2^2+\frac{1}{32}\frac{m^2}{\beta^2}(1+2\mu)A_{11}^{(1)}A_{11}^{(1)}\varepsilon^2+\cdots \tag{4.7}$$

式(4.6)、(4.7)中，摄动参数$A_{11}^{(1)}\varepsilon$具有明显的物理意义，因为由式(3.3)当$x=\pi/2m$，$y=\pi/2n$时，最大无量纲挠度

$$w_m=A_{11}^{(1)}\varepsilon-(A_{13}^{(3)}+A_{31}^{(3)})\varepsilon^3+O(\varepsilon^5) \tag{4.8}$$

反之

$$A_{11}^{(1)}\varepsilon=w_m+\frac{1}{16}(1+\mu)^2(1+2\mu)$$

$$\cdot\Bigg[\frac{m^4}{(m^4+18\gamma_1^2m^2n^2\beta^2+81\gamma_2^4n^4\beta^4)(1+\mu)-(m^4+2\gamma_1^2m^2n^2\beta^2+\gamma_2^4n^4\beta^4)}$$

$$+\frac{\gamma_2^4n^4\beta^4}{(81m^4+18\gamma_1^2m^2n^2\beta^2+\gamma_2^4n^4\beta^4)(1+\mu)-9(m^4+2\gamma_1^2m^2n^2\beta^2+\gamma_2^4n^4\beta^4)}\Bigg]$$

$$\cdot w_m^3+O(w_m^5) \tag{4.9}$$

将式 (4.9) 代入(4.6)、(4.7)我们得到以最大无量纲挠度为摄动参数的表达式

$$\lambda_x=\frac{1}{4\gamma_2^2\beta^2}\Bigg\{\frac{m^4+2\gamma_1^2m^2n^2\beta^2+\gamma_2^4n^4\beta^4}{(1+\mu)m^2}+\frac{1}{16}\frac{m^4+\gamma_2^4n^4\beta^4}{m^2}(1+2\mu)w_m^2$$

$$+\frac{1}{256m^2}\Big\langle 2(1+\mu)^2(1+2\mu)^2$$

$$\cdot\Bigg[\frac{m^4(m^4+\gamma_2^4n^4\beta^4)}{(m^4+18\gamma_1^2m^2n^2\beta^2+81\gamma_2^4n^4\beta^4)(1+\mu)-(m^4+2\gamma_1^2m^2n^2\beta^2+\gamma_2^4n^4\beta^4)}$$

$$+\frac{\gamma_2^4n^4\beta^4(m^4+\gamma_2^4n^4\beta^4)}{(81m^4+18\gamma_1^2m^2n^2\beta^2+\gamma_2^4n^4\beta^4)(1+\mu)-9(m^4+2\gamma_1^2m^2n^2\beta^2+\gamma_2^4n^4\beta^4)}\Bigg]$$

$$-(1+\mu)(1+2\mu)[2(1+\mu)^2+(1+2\mu)]$$

$$\cdot\left[\frac{m^8}{(m^4+18\gamma_1^2m^2n^2\beta^2+81\gamma_2^4n^4\beta^4)(1+\mu)-(m^4+2\gamma_1^2m^2n^2\beta^2+\gamma_2^4n^4\beta^4)}\right.$$

$$\left.+\frac{\gamma_2^8n^8\beta^8}{(81m^4+18\gamma_1^2m^2n^2\beta^2+\gamma_2^4n^4\beta^4)(1+\mu)-9(m^4+2\gamma_1^2m^2n^2\beta^2+\gamma_2^4n^4\beta^4)}\right]\Big\rangle w_m^4+\cdots\Big\} \tag{4.10}$$

$$\delta_x=\lambda_x\gamma_2^2+\frac{1}{32}\frac{m^2}{\beta^2}(1+2\mu)w_m^2+\frac{1}{256}\frac{m^2}{\beta^2}(1+\mu)^2(1+2\mu)^2$$

$$\cdot\left[\frac{m^4}{(m^4+18\gamma_1^2m^2n^2\beta^2+81\gamma_2^4n^4\beta^4)(1+\mu)-(m^4+2\gamma_1^2m^2n^2\beta^2+\gamma_2^4n^4\beta^4)}\right.$$

$$\left.+\frac{\gamma_2^4n^4\beta^4}{(81m^4+18\gamma_1^2m^2n^2\beta^2+\gamma_2^4n^4\beta^4)(1+\mu)-9(m^4+2\gamma_1^2m^2n^2\beta^2+\gamma_2^4n^4\beta^4)}\right]w_m^4+\cdots \tag{4.11}$$

2.纵向边缘不可移简支

将式(3.3)、(3.4)及 (3.2) 代入边界条件(2.17c)，我们导得

$$-(b_{00}^{(0)}+\varepsilon^2b_{00}^{(2)}+\varepsilon^4b_{00}^{(4)}+\cdots)+\nu_y(\beta^2B_{00}^{(0)}+\varepsilon^2\beta^2B_{00}^{(2)}+\varepsilon^4\beta^2B_{00}^{(4)}+\cdots)$$

$$-\frac{1}{8}(1+2\mu)\gamma_2^2n^2\beta^2A_{11}^{(1)}A_{11}^{(1)}\varepsilon^2+\cdots=0 \tag{4.12}$$

令$\varepsilon\to0$，有

$$b_{00}^{(0)}=\nu_y\beta^2B_{00}^{(0)} \tag{4.13}$$

代入 (3.5) 式得

$$\gamma_2^2\beta^2B_{00}^{(0)}=\frac{m^4+2\gamma_1^2m^2n^2\beta^2+\gamma_2^4n^4\beta^4}{(1+\mu)(m^2+\nu_yn^2\beta^2)} \tag{4.14}$$

将式(3.5)、(3.6)、(3.7)三式相加，并计及(4.13)、(4.14)，我们有

$$\gamma_2^2m^2(\beta^2B_{00}^{(0)}+\varepsilon^2\beta^2B_{00}^{(2)}+\varepsilon^4\beta^2B_{00}^{(4)}+\cdots)+\gamma_2^2n^2\beta^2(b_{00}^{(0)}+\varepsilon^2b_{00}^{(2)}+\varepsilon^4b_{00}^{(4)}+\cdots)$$

$$=\frac{m^4+2\gamma_1^2m^2n^2\beta^2+\gamma_2^4n^4\beta^4}{(1+\mu)}+\frac{1}{16}(m^4+\gamma_2^4n^4\beta^4)(1+2\mu)A_{11}^{(1)}A_{11}^{(1)}\varepsilon^2$$

$$-\frac{1}{256}(1+\mu)(1+2\mu)[2(1+\mu)^2+(1+2\mu)]$$

$$\cdot\left[\frac{m^8(m^2+\nu_yn^2\beta^2)}{(m^4+18\gamma_1^2m^2n^2\beta^2+81\gamma_2^4n^4\beta^4)(m^2+\nu_yn^2\beta^2)(1+\mu)-(m^4+2\gamma_1^2m^2n^2\beta^2+\gamma_2^4n^4\beta^4)(m^2+9\nu_yn^2\beta^2)}\right.$$

$$\left.+\frac{\gamma_2^8n^8\beta^8(m^2+\nu_yn^2\beta^2)}{(81m^4+18\gamma_1^2m^2n^2\beta^2+\gamma_2^4n^4\beta^4)(m^2+\nu_yn^2\beta^2)(1+\mu)-(m^4+2\gamma_1^2m^2n^2\beta^2+\gamma_2^4n^4\beta^4)(9m^2+\nu_yn^2\beta^2)}\right]$$

$$\cdot A_{11}^{(1)}A_{11}^{(1)}A_{11}^{(1)}A_{11}^{(1)}\varepsilon^4+\cdots \tag{4.15}$$

对于纵边不可移简支式 (4.1)、(4.8)仍然成立，此时

$$A_{11}^{(1)}\varepsilon=w_m+\frac{1}{16}(1+\mu^2)(1+2\mu)$$

$$\cdot\left[\frac{m^4(m^2+\nu_y n^2\beta^2)}{\begin{array}{c}(m^4+18\gamma_1^2 m^2n^2\beta^2+81\gamma_2^4n^4\beta^4)(m^2+\nu_y n^2\beta^2)(1+\mu)\\-(m^4+2\gamma_1^2m^2n^2\beta^2+\gamma_2^4n^4\beta^4)(m^2+9\nu_y n^2\beta^2)\end{array}}\right.$$

$$\left.+\frac{\gamma_2^4 n^4\beta^4(m^2+\nu_yn^2\beta^2)}{\begin{array}{c}(81m^4+18\gamma_1^2m^2n^2\beta^2+\gamma_2^4n^4\beta^4)(m^2+\nu_yn^2\beta^2)(1+\mu)\\-(m^4+2\gamma_1^2m^2n^2\beta^2+\gamma_2^4n^4\beta^4)(9m^2+\nu_yn^2\beta^2)\end{array}}\right]w_m^3+O(w_m^5) \tag{4.16}$$

由式(4.1)、(4.12)、(4.15)、(4.16)我们得到以最大无量纲挠度为摄动参数的表达式：

$$\lambda_x=\frac{1}{4\gamma_2^2\beta^2}\left\{\frac{m^4+2\gamma_1^2m^2n^2\beta^2+\gamma_2^4n^4\beta^4}{(1+\mu)(m^2+\nu_yn^2\beta^2)}+\frac{1}{16}\,\frac{m^4+3\gamma_2^4n^4\beta^4}{m^2+\nu_yn^2\beta^2}(1+2\mu)w_m^2\right.$$

$$+\frac{1}{256}\left\langle 2(1+\mu)^2(1+2\mu)^2\left[\frac{m^4(m^4+3\gamma_2^4n^4\beta^4)}{\begin{array}{c}(m^4+18\gamma_1^2m^2n^2\beta^2+81\gamma_2^4n^4\beta^4)(m^2+\nu_yn^2\beta^2)(1+\mu)\\-(m^4+2\gamma_1^2m^2n^2\beta^2+\gamma_2^4n^4\beta^4)(m^2+9\nu_yn^2\beta^2)\end{array}}\right.\right.$$

$$\left.+\frac{\gamma_2^4n^4\beta^4(m^4+3\gamma_2^4n^4\beta^4)}{\begin{array}{c}(81m^4+18\gamma_1^2m^2n^2\beta^2+\gamma_2^4n^4\beta^4)(m^2+\nu_yn^2\beta^2)(1+\mu)\\-(m^4+2\gamma_1^2m^2n^2\beta^2+\gamma_2^4n^4\beta^4)(9m^2+\nu_yn^2\beta^2)\end{array}}\right]$$

$$-(1+\mu)(1+2\mu)[2(1+\mu)^2+(1+2\mu)]$$

$$\cdot\left[\frac{m^8}{\begin{array}{c}(m^4+18\gamma_1^2m^2n^2\beta^2+81\gamma_2^4n^4\beta^4)(m^2+\nu_yn^2\beta^2)(1+\mu)\\-(m^4+2\gamma_1^2m^2n^2\beta^2+\gamma_2^4n^4\beta^4)(m^2+9\nu_yn^2\beta^2)\end{array}}\right.$$

$$\left.\left.\left.+\frac{\gamma_2^8n^8\beta^8}{\begin{array}{c}(81m^4+18\gamma_1^2m^2n^2\beta^2+\gamma_2^4n^4\beta^4)(m^2+\nu_yn^2\beta^2)(1+\mu)\\-(m^4+2\gamma_1^2m^2n^2\beta^2+\gamma_2^4n^4\beta^4)(9m^2+\nu_yn^2\beta^2)\end{array}}\right]\right\rangle w_m^4+\cdots\right\} \tag{4.17}$$

相应的端部缩短为：

$$\delta_x=(1-\nu_x\nu_y)\lambda_x\gamma_2^2+\frac{1}{32}\,\frac{m^2+\nu_yn^2\beta^2}{\beta^2}(1+2\mu)w_m^2+\frac{1}{256}\,\frac{m^2+\nu_yn^2\beta^2}{\beta^2}(1+\mu)^2(1+2\mu)^2$$

$$\cdot\left[\frac{m^4(m^2+\nu_yn^2\beta^2)}{\begin{array}{c}(m^4+18\gamma_1^2m^2n^2\beta^2+81\gamma_2^4n^4\beta^4)(m^2+\nu_yn^2\beta^2)(1+\mu)\\-(m^4+2\gamma_1^2m^2n^2\beta^2+\gamma_2^4n^4\beta^4)(m^2+9\nu_yn^2\beta^2)\end{array}}\right.$$

$$\left.+\frac{\gamma_2^4n^4\beta^4(m^2+\nu_yn^2\beta^2)}{\begin{array}{c}(81m^4+18\gamma_1^2m^2n^2\beta^2+\gamma_2^4n^4\beta^4)(m^2+\nu_yn^2\beta^2)(1+\mu)\\-(m^4+2\gamma_1^2m^2n^2\beta^2+\gamma_2^4n^4\beta^4)(9m^2+\nu_yn^2\beta^2)\end{array}}\right]w_m^4+\cdots \tag{4.18}$$

在式(4.10)、(4.17)中，令 $w_m=0$，即得线性临界值。而当 $\gamma_1=\gamma_2=\gamma_3=1$ 时，我们即可得到简支各向同性矩形板在单向压缩作用下的后屈曲平衡路径[9]。

五、结 果 和 讨 论

根据渐近分析导出的公式我们计算了三种纤维增强复合板对应两种面内边界条件的后屈曲载荷—端部缩短曲线，材料的弹性常数在表1中给出（单位：10^6 磅/时^2）。所得结果如图2所示。图中o号曲线对应各向同性板，即 $E_y/E_x=1$，$G/E_x=0.3846$，$\nu_x=0.3$。计算结果表明，通常情况下正交异性矩形板的后屈曲性态与各向同性板相类似。屈后强度程度上的差异

主要取决于刚度比E_y/E_x。一般说来，刚度比$E_y/E_x>1$，屈后强度高于各向同性板。反之，$E_y/E_x<1$时则低于各向同性板。在某些特殊情况下，可以获得类似曲板的后屈曲载荷一端部缩短曲线，此时，板对初始缺陷变得敏感。

图3为本文结果与文[3]、文[4]计算结果的比较，其中板的弹性常数在表2中给出（单位：10^6磅/吋^2）。可以看出，Chan的结果一般偏高，而Chandra 和 Raju 的结果，特别是9号板的结果有相当大的偏差。曾如我们在文[9]中所指出，采用载荷为摄动参数，往往使载荷一挠度曲线在挠度较大时并不收敛于真实解。

图4，图5为本文结果与文[8]实验结果的比较。板的长宽比及弹性常数在表3中给出（单位：GN/m^2）。图示表明，在理论与实验间得到了非常合理的符合。其中实验结果更接近初挠度$W^*/t=0.1$的理论曲线。

表 1

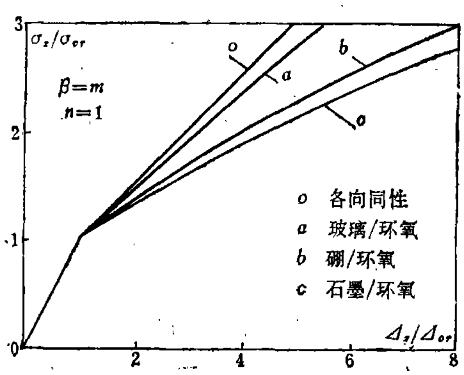
	E_x	E_y	G	ν_x	E_y/E_x	G/E_x
玻璃/环氧	7.5	2.7	1.3	0.25	0.3600	0.1733
硼/环氧	32.5	3.69	1.05	0.39	0.1135	0.0323
石墨/环氧	18.0	1.55	0.85	0.57	0.0861	0.0472

表 2

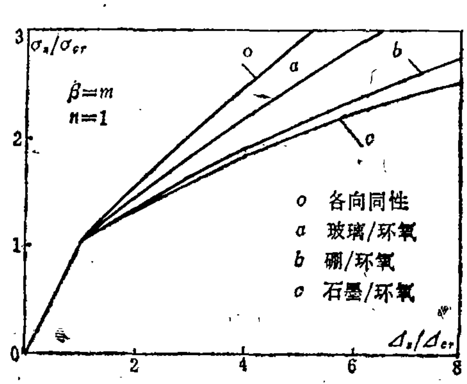
板 №	E_x	E_y	G	ν_x	E_y/E_x	G/E_x
6	2.6	7.5	1.1	0.08675	2.89	0.424
9	7.5	2.6	1.1	0.25	0.347	0.147

表 3

板 №	E_x	E_y	G	ν_x	E_y/E_x	G/E_x	a/b
1	29.80	6.30	2.06	0.33	0.2114	0.0691	1.0
2	27.38	8.06	2.71	0.33	0.2944	0.0990	1.3575
3	25.60	4.47	2.30	0.33	0.1746	0.0898	2.0
4	6.30	29.80	2.06	0.07	4.7302	0.3270	1.0
5	5.35	29.35	2.19	0.06	5.4860	0.4093	0.5



(a) 纵边可移简支



(b) 纵边不可移简支

图2 复合板 载荷—端部缩短曲线

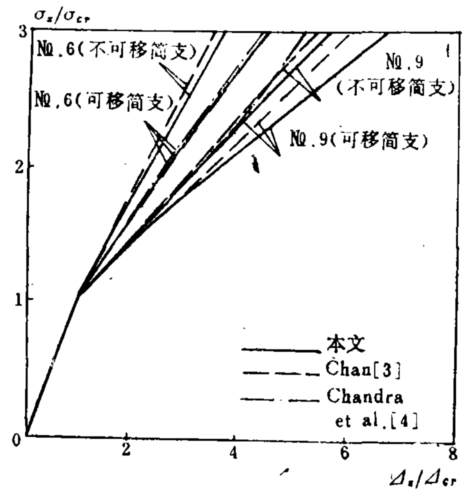


图3 载荷—端部缩短曲线，理论结果比较

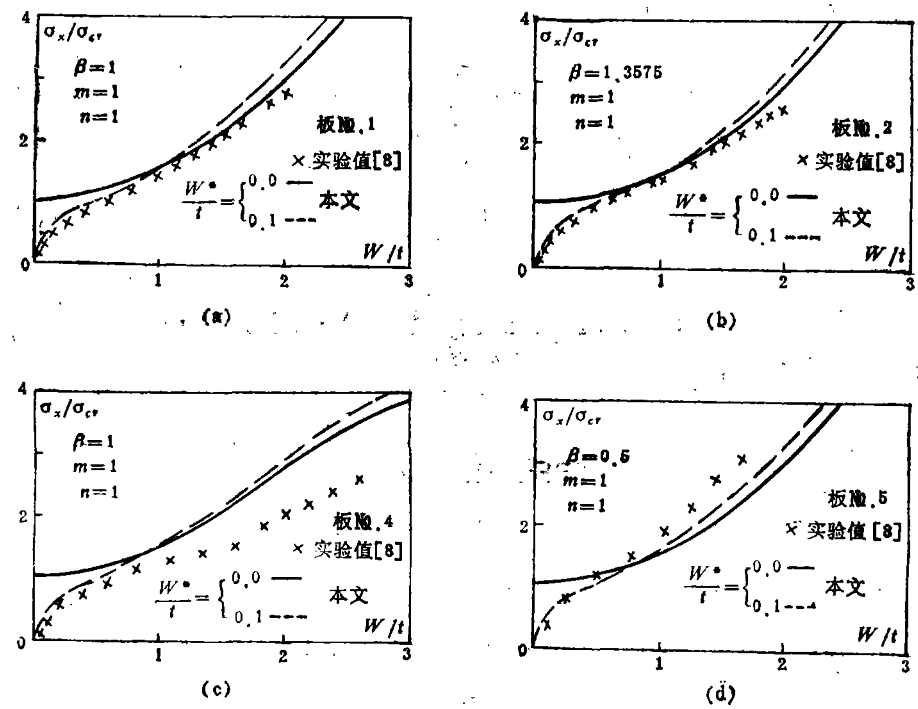


图4 载荷—挠度曲线，理论和实验结果比较

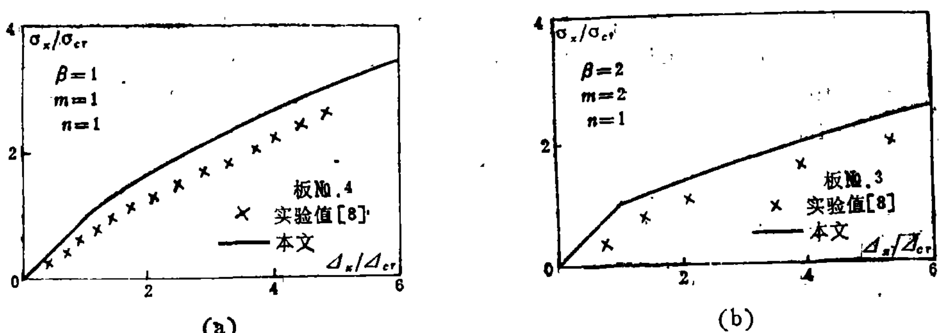


图5 载荷—端部缩短曲线，理论和实验结果比较

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Perturbation Analysis for the Postbuckling of Rectangular Orthotropic Plates

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Abstract

In this paper, applying perturbation method to von Kármán-type nonlinear large deflection equations of orthotropic plates by taking deflection as perturbation parameter, the postbuckling behavior of simply supported rectangular orthotropic plates under in-plane compression is investigated. Two types of in-plane boundary conditions are now considered and the effects of initial imperfections are also studied. Numerical results are presented for various cases of orthotropic composite plates having different elastic properties. It is found that the results obtained are in good agreement with those of experiments.