

非线性大变形问题的边界元分析法*

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摘 要

本文采用有限变形理论的拖带坐标描述法导出了瞬时位形上的速率形式非线性影响函数(近似速率型基本解), 从而导出以瞬时位形为基准的非线性大变形的边界积分方程. 由编制的 NBEM 计算程序的算例表明本文建立的非线性边界元方法是可行的.

一、前 言

自60年代以来发展的边界元法已成功应用于弹性力学. 对固体力学的非线性问题, 也求解了一些非线性材料的小变形问题. 例如: Mukhejee^[2], Telles^[3]求解了与时间相关的非弹性的平面和轴对称问题. 轴的扭转, 板的弯曲等等问题. 然而由于边界元法是一种影响函数法, 使得在几何非线性方面的应用存在很多困难. 已有文献[4]、[8]提出了大变形边界元法的算法, 但是文献的作者仍采用以小变形的影响函数(Kelvin基本解), 并以增量的 Betti 互等定理推导出边界积分方程. 我们知道, 对非线性大变形力学问题, 只有在瞬时位形上, 速率形式的 Betti 互等定理才可能近似成立. 为能把边界元法准确地应用于几何非线性力学领域, 本文采用拖带坐标描述法, 首先导出在瞬时位形上二维速率形式的影响函数(近似基本解). 然后在此基础上推导出适用于几何非线性的边界积分方程. 显然本文给出的边界积分方程是建立在拖带坐标系上以瞬时位形为基准的, 它消除了现有文献中把非线性项(物理非线性或几何非线性)转化为体积分作为初应力(或初应变)来求解的缺点.

二、应力函数法求解二维弹性力学问题的基本方程

由陈至达 S-R 分解定理^[1], 应变速率为:

$$\dot{S}_i^j = \frac{1}{2} [V^i \parallel_j + V^j \parallel_i] = \frac{1}{2} \left[\bar{F}_k^i \frac{\partial \bar{V}^k}{\partial x^j} + \bar{F}_k^j \frac{\partial \bar{V}^k}{\partial x^i} \right] \quad (2.1)$$

这里 $(\cdot) \parallel_j$ 表示在瞬时位形上对拖带坐标 x^j 的协变导数, \bar{F}_k^i 是形变梯度 F_k^i 的逆表示. V^i 是在拖带坐标系上的速度分量, \bar{V}^i 是在整体系上的速度分量.

* 陈至达推荐.

平面问题的物性方程为:

$$\left. \begin{aligned} \dot{\sigma}_i^j &= \frac{E}{1+\nu} \left[\dot{S}_i^j + \frac{\nu}{1-2\nu} \dot{S}_k^k \delta_i^j \right] \\ \text{或} \quad \dot{S}_i^j &= \frac{1+\nu}{E} [\dot{\sigma}_i^j - \nu \dot{\sigma}_k^k \delta_i^j] \end{aligned} \right\} \quad (2.2)$$

设应力速率函数 $\Phi(x^1, x^2)$, 且

$$\dot{\sigma}_i^j = \epsilon^{i\lambda} \epsilon_{j\beta} \Phi \|_{\lambda}^{\beta} \quad (2.3)$$

这里 ϵ_{ij} 是置换张量 (permutation tensor).

在瞬时位形上, 设拖带系畸变对瞬时应力分量的影响较应力本身数值变化为小, 则应力速率平衡方程近似可用:

$$\dot{\sigma}_\beta^a \|_a + \rho \dot{f}_\beta = 0 \quad (2.4)$$

(2.3) 式代入 (2.4) 式得:

$$\rho \dot{f}_\beta + \dot{\sigma}_\beta^a \|_a = \epsilon^{\alpha\lambda} \epsilon_{\beta\mu} \Phi \|_{\lambda}^{\mu} \|_a + \rho \dot{f}_\beta = 0 \quad (2.5)$$

如不考虑体力, 则

$$\dot{\sigma}_\beta^a \|_a = \epsilon^{\alpha\lambda} \epsilon_{\beta\mu} \Phi \|_{\lambda}^{\mu} \|_a = 0 \quad (2.6)$$

由 (2.1) 式得

$$2 \dot{S}_i^j \|_{\rho}^k \epsilon^{j\rho} \epsilon_{ik} = 2 V^i \|_{\rho}^k \epsilon^{j\rho} \epsilon_{ik} + V^j \|_{\rho}^k \epsilon^{i\rho} \epsilon_{jk} \quad (2.7)$$

改变上式中最后一项的哑指标记号得:

$$\dot{S}_i^j \|_{\rho}^k \epsilon^{j\rho} \epsilon_{ik} = V^i \|_{\rho}^k \epsilon^{j\rho} \epsilon_{ik} \quad (2.8)$$

因为 $V^i \|_{\rho}^k = V^i \|_{\rho}^k$, $\epsilon^{j\rho} = -\epsilon^{\rho j}$ 可推得

$$\dot{S}_i^j \|_{\rho}^k \epsilon^{j\rho} \epsilon_{ik} = 0 \quad (2.9)$$

物性方程 (2.2) 式可以改写成

$$\begin{aligned} \dot{S}_i^j &= \frac{1+\nu}{E} [\dot{\sigma}_i^j - \nu \dot{\sigma}_m^m \delta_i^j] = \frac{1+\nu}{E} [\dot{\sigma}_\beta^a \delta_a^i \delta_j^b - \nu \dot{\sigma}_\beta^a \delta_a^b \delta_j^i] \\ &= \frac{1+\nu}{E} [\delta_a^i \delta_j^b - \nu \delta_a^b \delta_j^i] \dot{\sigma}_\beta^a \end{aligned} \quad (2.10)$$

(2.10) 式代入 (2.9) 式得:

$$\dot{\sigma}_\beta^a \|_{\rho}^k [\delta_a^i \delta_j^b - \nu \delta_a^b \delta_j^i] \epsilon^{i\rho} \epsilon_{jk} = 0$$

(2.6) 式代入上式

$$\Phi \|_{\lambda}^k \epsilon^{\alpha\lambda} \epsilon_{\beta\mu} [\delta_a^i \delta_j^b - \nu \delta_a^b \delta_j^i] \epsilon^{j\rho} \epsilon_{ik} = 0 \quad (2.11)$$

其中 $\epsilon^{\alpha\lambda} \epsilon_{\beta\mu} [\delta_a^i \delta_j^b - \nu \delta_a^b \delta_j^i] \epsilon^{j\rho} \epsilon_{ik} = [\epsilon^{i\lambda} \epsilon_{j\mu} - \nu \epsilon^{\beta\lambda} \epsilon_{\beta\mu} \delta_j^i] \epsilon^{j\rho} \epsilon_{ik}$

$$\begin{aligned} &= \epsilon^{i\lambda} \epsilon_{j\mu} \epsilon^{j\rho} \epsilon_{ik} - \nu \delta_\mu^\lambda \delta_j^i \epsilon^{j\rho} \epsilon_{ik} \\ &= \delta_\mu^\rho \delta_k^i - \nu \delta_\mu^i \delta_k^\rho \end{aligned} \quad (2.12)$$

(2.12) 式代入 (2.11) 式得:

$$\Phi \|_{\lambda}^k [\delta_\mu^\rho \delta_k^i - \nu \delta_\mu^i \delta_k^\rho] = \Phi \|_{\mu}^k - \nu \Phi \|_{\mu}^k = (1-\nu) \Phi \|_{\mu}^k = 0$$

即

$$\Phi \|_{\mu}^k = 0 \quad (2.13)$$

(2.13) 式是以瞬时位形为基准的双调和方程, $\Phi(x^1, x^2)$ 为双调和函数。从而速率形式的解就可归结为求解建立在瞬时位形上的双调和方程。

三、速率形式的非线性基本解

设无限弹性体在 t 瞬时位形上, 施加单位点载荷速率 $\dot{P}_2=1$, 如图 1 所示. 为描述点载荷速率作用导致的奇异性, 人为地把全区域划分为两个区域 Ω 和 Ω^Δ , 由当地拖带坐标系 (x^1, x^2) 建立其边界条件为:

$$\left. \begin{aligned} \dot{\sigma}_2^\Delta - \dot{\sigma}_2^\Delta &= \delta(x^1, x^2) & (x^2=0) \\ \dot{\sigma}_2^1 &= \dot{\sigma}_2^1 & (x^2=0) \\ V_x &= V_x; \quad V_y = V_y & (x^2=0) \end{aligned} \right\} \quad (3.1)$$

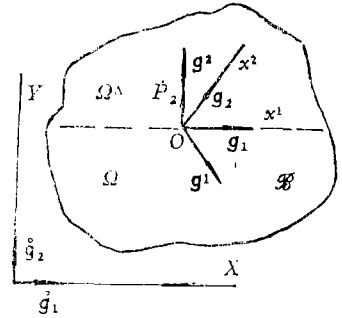


图 1 受点载荷速率的无限弹性体

其中下标 Δ 表示区域 Ω^Δ 中的量.

以该瞬时位形为基准引入傅里叶变换相空间 (ξ, x^2) $\in \mathcal{A}$, 则应力增量, 速度, 应变速率在 \mathcal{A} 中有定义, 但位形的几何尺规量在 \mathcal{A} 中无定义. 设映射 $\bar{\Phi}: \mathcal{A} \rightarrow \mathcal{A}$ 满足所需要的数学条件:

$$\left. \begin{aligned} \bar{\Phi}^\Delta &= \left(\begin{array}{c} A \\ |\xi| \end{array} + Bx^2 \right) \exp[-\xi x^2] & x^2 \geq 0 \\ \bar{\Phi} &= \left(\begin{array}{c} C \\ |\xi| \end{array} + Dx^2 \right) \exp[\xi x^2] & x^2 \leq 0 \end{aligned} \right\} \quad (3.2)$$

其中 A, B, C, D 为待定参数, 满足 (2.13) 式. 相应于相空间的边界条件为

$$\left. \begin{aligned} \bar{\sigma}_2^\Delta - \bar{\sigma}_2^\Delta &= 1 & x^2=0 \\ \bar{\sigma}_2^1 &= \bar{\sigma}_2^1 & x^2=0 \\ \bar{V}_x &= \bar{V}_x; \quad \bar{V}_y = \bar{V}_y & x^2=0 \end{aligned} \right\} \quad (3.3)$$

由 (2.1) 式和 (2.3) 式可推得:

$$\begin{aligned} \frac{d\bar{V}_y}{dx^2} &= \frac{1}{2G(i\xi) (\bar{F}_1^1 \bar{F}_2^2 - \bar{F}_2^1 \bar{F}_1^2)} \left\{ i\xi \bar{F}_1^1 [(1-\nu)\bar{\sigma}_2^2 - \nu\bar{\sigma}_1^1] + \bar{F}_1^2 [(1-\nu)\frac{d\bar{\sigma}_1^1}{dx^2} - \nu\frac{d\bar{\sigma}_2^2}{dx^2}] \right\} \\ \frac{d\bar{V}_x}{dx^2} &= \frac{1}{2G(i\xi) (\bar{F}_1^1 \bar{F}_2^2 - \bar{F}_2^1 \bar{F}_1^2)} \left\{ \bar{F}_2^2 [(1-\nu)\frac{d\bar{\sigma}_1^1}{dx^2} - \nu\frac{d\bar{\sigma}_2^2}{dx^2}] + i\xi \bar{F}_1^1 [(1-\nu)\bar{\sigma}_2^2 - \nu\bar{\sigma}_1^1] \right\} \end{aligned}$$

由此得到

$$\begin{aligned} (-i\xi)\bar{V}_x &= \frac{1}{2G(\bar{F}_1^1 \bar{F}_2^2 - \bar{F}_2^1 \bar{F}_1^2)} \left\{ -\bar{F}_2^2 \cdot 2 \cdot \bar{\sigma}_1^1 - \frac{\bar{F}_1^2}{2\xi} [(1-\nu)\frac{d\bar{\sigma}_1^1}{dx^2} - \nu\frac{d\bar{\sigma}_2^2}{dx^2}] + \bar{F}_2^2 [(1-\nu)\bar{\sigma}_1^1 - \nu\bar{\sigma}_2^2] \right\} \\ (-i\xi)\bar{V}_y &= \frac{1}{2G(\bar{F}_1^1 \bar{F}_2^2 - \bar{F}_2^1 \bar{F}_1^2)} \left\{ 2\bar{F}_1^1 \bar{\sigma}_1^1 + \frac{\bar{F}_1^1}{i\xi} [(1-\nu)\frac{d\bar{\sigma}_1^1}{dx^2} - \nu\frac{d\bar{\sigma}_2^2}{dx^2}] \right\} \end{aligned} \quad (3.4)$$

$$-\bar{F}_2^{-1}[(1-\nu)\bar{\sigma}_1 - \nu\bar{\sigma}_2] \quad (3.5)$$

全部由应力函数表示为:

$$\begin{aligned} (-i\xi)V_x = & \frac{1}{2G(\bar{F}_1^{-1}\bar{F}_2^{-1} - \bar{F}_2^{-1}\bar{F}_1^{-1})} \left\{ -2\bar{F}_2^{-1}i\xi \frac{d\bar{\Phi}}{dx^2} - \bar{F}_2^{-1}i\xi \left[(1-\nu) \frac{d^3\bar{\Phi}}{d(x^2)^3} \right. \right. \\ & \left. \left. + \nu\xi^2 \frac{d\bar{\Phi}}{dx^2} \right] + \bar{F}_2^{-1} \left[(1-\nu) \frac{d^2\bar{\Phi}}{d(x^2)^2} + \nu\xi^2\bar{\Phi} \right] \right\} \quad (3.6) \end{aligned}$$

$$\begin{aligned} (-i\xi)V_r = & \frac{1}{2G(\bar{F}_1^{-1}\bar{F}_2^{-1} - \bar{F}_2^{-1}\bar{F}_1^{-1})} \left\{ 2\bar{F}_1^{-1}i\xi \frac{d\bar{\Phi}}{dx^2} + \bar{F}_1^{-1}i\xi \left[(1-\nu) \frac{d^3\bar{\Phi}}{d(x^2)^3} \right. \right. \\ & \left. \left. + \nu \frac{d\bar{\Phi}}{dx^2} \right] - \bar{F}_1^{-1} \left[(1-\nu) \frac{d^2\bar{\Phi}}{d(x^2)^2} + \nu\xi^2\bar{\Phi} \right] \right\} \quad (3.7) \end{aligned}$$

代入相空间边界条件 (3.3) 式得到:

$$-|\xi|C + |\xi|A = 1 \quad (3.8)$$

$$C + D = B - A \quad (3.9)$$

$$\begin{aligned} 2\bar{F}_2^{-1}i\xi|\xi|A = & [(3-2\nu)i\xi|\xi|\bar{F}_2^{-1} + 2(1-\nu)\xi^2\bar{F}_2^{-1}]B \\ & + [\bar{F}_2^{-1}i\xi|\xi|(1-2\nu) - 2\bar{F}_2^{-1}\xi^2(1-\nu)]D \quad (3.10) \end{aligned}$$

$$\begin{aligned} 2\bar{F}_1^{-1}i\xi|\xi|A = & [2(1-\nu)\xi^2\bar{F}_1^{-1} + (3-2\nu)i\xi|\xi|\bar{F}_1^{-1}]B \\ & + [-2(1-\nu)\xi^2\bar{F}_1^{-1} + (1-2\nu)\bar{F}_1^{-1}i\xi|\xi|]D \quad (3.11) \end{aligned}$$

由此确定 A, B, C, D , 得到所求的应力函数为:

$$\begin{aligned} \bar{\Phi}^\Delta = & \left. \begin{aligned} & \frac{1}{4(1-\nu)} \left[2(1-\nu) \frac{1}{\xi^2} + \frac{x^2}{|\xi|} \right] \exp[-|\xi|x^2] & x^2 \geq 0 \\ & \frac{1}{4(1-\nu)} \left[-2(1-\nu) \frac{1}{\xi^2} + \frac{x^2}{|\xi|} \right] \exp[|\xi|x^2] & x^2 \leq 0 \end{aligned} \right\} \quad (3.12) \end{aligned}$$

由此可求得

$$\frac{d\bar{\Phi}}{dx^2} = \begin{cases} \frac{-|\xi|}{4(1-\nu)} \left[2(1-\nu) \frac{1}{\xi^2} + \frac{x^2}{|\xi|} \right] \exp[-|\xi|x^2] + \frac{1}{4(1-\nu)} \exp[-|\xi|x^2] & x^2 \geq 0 \\ \frac{|\xi|}{4(1-\nu)} \left[-(1-2\nu) \frac{1}{\xi^2} + \frac{x^2}{|\xi|} \right] \exp[|\xi|x^2] & x^2 \leq 0 \end{cases} \quad (3.13a)$$

$$\frac{d^2\bar{\Phi}}{d(x^2)^2} = \begin{cases} \frac{\xi^2}{4(1-\nu)} \left[-2\nu \frac{1}{\xi^2} + \frac{x^2}{|\xi|} \right] \exp[-|\xi|x^2] & x^2 \geq 0 \\ \frac{\xi^2}{4(1-\nu)} \left[2\nu \frac{1}{\xi^2} + \frac{x^2}{|\xi|} \right] \exp[|\xi|x^2] & x^2 \leq 0 \end{cases} \quad (3.13b)$$

$$\frac{d^3\bar{\Phi}}{d(x^2)^3} = \begin{cases} \frac{-\xi^2|\xi|}{4(1-\nu)} \left[-(1+2\nu) \frac{1}{\xi^2} + \frac{x^2}{|\xi|} \right] \exp[-|\xi|x^2] & x^2 \geq 0 \\ \frac{\xi^2|\xi|}{4(1-\nu)} \left[(1+2\nu) \frac{1}{\xi^2} + \frac{x^2}{|\xi|} \right] \exp[|\xi|x^2] & x^2 \leq 0 \end{cases} \quad (3.13c)$$

$$\bar{\sigma}_1^1 = \begin{cases} \frac{\xi^2}{4(1-\nu)} \left[-2\nu \frac{1}{\xi^2} + \frac{x^2}{|\xi|} \right] \exp[-|\xi|x^2] & x^2 \geq 0 \\ \frac{\xi^2}{4(1-\nu)} \left[2\nu \frac{1}{\xi^2} + \frac{x^2}{|\xi|} \right] \exp[|\xi|x^2] & x^2 \leq 0 \end{cases} \quad (3.14a)$$

$$\bar{\sigma}_2^2 = \begin{cases} \frac{-\xi^2}{4(1-\nu)} \left[2(1-\nu) \frac{1}{\xi^2} + \frac{x^2}{|\xi|} \right] \exp[-|\xi|x^2] & x^2 \geq 0 \\ \frac{-\xi^2}{4(1-\nu)} \left[-2(1-\nu) \frac{1}{\xi^2} + \frac{x^2}{|\xi|} \right] \exp[|\xi|x^2] & x^2 \leq 0 \end{cases} \quad (3.14b)$$

$$\bar{\sigma}_2^1 = \begin{cases} \frac{-i\xi|\xi|}{4(1-\nu)} \left[(1-2\nu) \frac{1}{\xi^2} + \frac{x^2}{|\xi|} \right] \exp[-|\xi|x^2] & x^2 \geq 0 \\ \frac{i\xi}{4(1-\nu)} \left[-(1-2\nu) \frac{1}{\xi^2} + \frac{x^2}{|\xi|} \right] \exp[|\xi|x^2] & x^2 \leq 0 \end{cases} \quad (3.14c)$$

$$V_x = \begin{cases} \frac{\exp[-|\xi|x^2]}{8G(1-\nu)} \left\{ 2F_2^1 |\xi| \left[(1-2\nu) \frac{1}{\xi^2} + \frac{x^2}{|\xi|} \right] - \frac{F_1^1}{\xi^2} (x^2 \xi^2 - |\xi|) - \frac{F_1^1}{i\xi} (x^2 |\xi|) \right\} & x^2 \geq 0 \\ \frac{\exp[|\xi|x^2]}{8G(1-\nu)} \left\{ -2F_2^1 |\xi| \left[-(1-2\nu) \frac{1}{\xi^2} + \frac{x^2}{|\xi|} \right] + \frac{F_1^1}{\xi^2} [x^2 \xi^2 + |\xi|] - \frac{F_1^1}{i\xi} (x^2 |\xi|) \right\} & x^2 \leq 0 \end{cases} \quad (3.15a)$$

$$V_y = \begin{cases} \frac{\exp[-|\xi|x^2]}{8G(1-\nu)} \left\{ 2F_2^2 |\xi| \left[(1-2\nu) \frac{1}{\xi^2} + \frac{x^2}{|\xi|} \right] - \frac{F_2^2}{\xi^2} [x^2 \xi^2 - |\xi|] - \frac{F_2^2}{i\xi} (x^2 |\xi|) \right\} & x^2 \geq 0 \\ \frac{\exp[-|\xi|x^2]}{8G(1-\nu)} \left\{ -2F_2^2 |\xi| \left[-(1-2\nu) \frac{1}{\xi^2} + \frac{x^2}{|\xi|} \right] + \frac{F_2^2}{\xi^2} [x^2 \xi^2 + |\xi|] - \frac{F_2^2}{i\xi} (x^2 |\xi|) \right\} & x^2 \leq 0 \end{cases} \quad (3.15b)$$

在上面的推导中我们利用了关系式

$$F_i^j \bar{F}_j^k = \delta_i^k \quad (3.16)$$

由傅里叶反变换公式^[5]

$$f(x^1, x^2) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \bar{f}(\xi, x^2) \exp[i|\xi|x^1] d\xi \quad (3.17)$$

我们可获得问题的解:

$$\left. \begin{aligned} \sigma_1^1 &= \frac{-x^2}{4\pi(1-\nu)r^2} \left[2\nu + \frac{(x^1)^2 - (x^2)^2}{r^2} \right] \\ \sigma_2^2 &= \frac{-x^2}{4\pi(1-\nu)r^2} \left[2(1-\nu) - \frac{(x^1)^2 - (x^2)^2}{r^2} \right] \\ \sigma_2^1 &= \frac{-x^1}{4\pi(1-\nu)r^2} \left[(1-2\nu) + \frac{2(x^2)^2}{r^2} \right] \end{aligned} \right\} \quad (3.18)$$

$$\left. \begin{aligned} V_x &= 8G(1-\nu)\pi \left[F_1^1 \frac{x^1 x^2}{r^2} + F_2^1 \frac{(x^2)^2}{r^2} - F_2^1 (3-4\nu) \ln r \right] \\ V_y &= 8G(1-\nu)\pi \left[F_1^2 \frac{x^1 x^2}{r^2} - F_2^2 (3-4\nu) \ln r + F_2^2 \frac{(x^2)^2}{r^2} \right] \end{aligned} \right\} \quad (3.19)$$

类似地我们可求出在 t 瞬时位形上, 速率点载荷 $\dot{P}_i=1$ 时的解答. 以下标 (i) 表示 $\dot{P}_i=1$ 时的解答, 以*表示基本解. 则:

$$\left. \begin{aligned} \dot{\sigma}_{(1)}^{*1} &= \frac{-x^1}{4\pi(1-\nu)r^2} \left[2(1-\nu) + \frac{(x^1)^2 - (x^2)^2}{r^2} \right] \\ \dot{\sigma}_{(1)}^{*2} &= \frac{-x^1}{4\pi(1-\nu)r^2} \left[2\nu - \frac{(x^1)^2 - (x^2)^2}{r^2} \right] \\ \dot{\sigma}_{(1)}^{*1} &= \frac{-x^2}{4\pi(1-\nu)r^2} \left[(1-2\nu) + \frac{2(x^1)^2}{r^2} \right] \\ \dot{\sigma}_{(2)}^{*2} &= \frac{-x^2}{4\pi(1-\nu)r^2} \left[2(1-\nu) + \frac{(x^1)^2 - (x^2)^2}{r^2} \right] \\ \dot{\sigma}_{(2)}^{*1} &= \frac{-x^2}{4\pi(1-\nu)r^2} \left[2\nu + \frac{(x^1)^2 - (x^2)^2}{r^2} \right] \\ \dot{\sigma}_{(2)}^{*2} &= \frac{-x^1}{4\pi(1-\nu)r^2} \left[(1-2\nu) + \frac{2(x^2)^2}{r^2} \right] \end{aligned} \right\} \quad (3.20)$$

$$\left. \begin{aligned} V_{(1)}^* &= 8G(1-\nu)\pi \left[F_2^1 \frac{x^1 x^2}{r^2} + F_1^1 \frac{(x^1)^2}{r^2} - F_1^1 (3-4\nu) \ln r \right] \\ V_{(1)}^* &= 8G(1-\nu)\pi \left[F_2^2 \frac{x^1 x^2}{r^2} - F_1^2 (3-4\nu) \ln r + F_1^2 \frac{(x^1)^2}{r^2} \right] \\ V_{(2)}^* &= 8G(1-\nu)\pi \left[F_1^1 \frac{x^1 x^2}{r^2} + F_2^1 \frac{(x^2)^2}{r^2} - F_2^1 (3-4\nu) \ln r \right] \\ V_{(2)}^* &= 8G(1-\nu)\pi \left[F_1^2 \frac{x^1 x^2}{r^2} - F_2^2 (3-4\nu) \ln r + F_2^2 \frac{(x^2)^2}{r^2} \right] \end{aligned} \right\} \quad (3.21)$$

上面的推导均是在当地拖带系中进行的, 如在整体拖带系中, 则在(3.20)式和(3.21)式中 $x^1 \rightarrow (x^1 - x_k^1)$, $x^2 \rightarrow (x^2 - x_k^2)$, $r \rightarrow [(x^1 - x_k^1)^2 + (x^2 - x_k^2)^2]^{\frac{1}{2}}$. x_k^i 为点载荷速率作用点的拖带坐标值.

四、边界积分方程推导和求解

对非线性大变形情况, 在瞬时位形上速率形式的 Betti 互等定理成立, 即

$$\int_{\Omega} \dot{\sigma}^*_{ij} \dot{S}^j d\Omega = \int_{\Omega} \dot{\sigma}^j_{ij} \dot{S}^* d\Omega \quad (4.1)$$

(4.1) 式左边

$$\begin{aligned} \int_{\Omega} \dot{\sigma}^*_{ij} \dot{S}^j d\Omega &= \int_{\Omega} \dot{\sigma}^*_{ij} (V^j_{||i} - L^j_{||i}) d\Omega = \int_{\Omega} \dot{\sigma}^*_{ij} V^j_{||i} d\Omega \\ &= \int_{\Omega} [\dot{\sigma}^*_{ij} V^j]_{||i} d\Omega - \int_{\Omega} \dot{\sigma}^*_{ij} V^j d\Omega \end{aligned}$$

$$\begin{aligned}
&= \int_{\Gamma} \dot{\sigma}^*_{ij} n_i V^j d\Gamma + \int_{\Omega} \dot{P}^*_i \delta(x^1 - x^1_k, x^2 - x^2_k) V^j d\Omega \\
&= \int_{\Gamma} (\dot{\sigma}^*_{ij} n_i) V^j d\Gamma + V^i
\end{aligned} \quad (4.2)$$

(4.1)式右边(类似)

$$\int_{\Omega} \dot{\sigma}^*_i \dot{S}^*_{ij} d\Omega = \int_{\Gamma} \dot{\sigma}^*_i n_i V^{*j} d\Omega + \int_{\Omega} \rho f^*_j V^{*j} d\Omega \quad (4.3)$$

在(4.2)式推导中应用了速度梯度与应变速率和角速度的协调条件: $V^i_{,j} = \dot{S}^i_j + L^i_j \dot{\vartheta}$, L^i_j 为瞬时转轴的方向余弦张量, $\dot{\vartheta}$ 为角速度。

(4.2)式和(4.3)式代入(4.1)式且应用 $\dot{P}^*_j = \dot{\sigma}^*_j n_j$

$$V^i = \int_{\Gamma} \dot{P}^*_j V^{*j} d\Gamma - \int_{\Gamma} \dot{P}^*_i V^j d\Gamma + \int_{\Omega} \rho f^*_j V^{*j} d\Omega \quad (4.4)$$

注意上式中 \dot{P}^*_i 和 V^{*j} 是在 t 瞬时的速率基本解在拖带系上的分量。即作用在“K”点沿 \mathbf{g}^i 方向的单位点载荷速率引起的引力速率和速度分量。如考虑二个方向作用点载荷速率, 上式为:

$$V^i + \int_{\Gamma} \dot{P}^*_{ni} V^m d\Gamma = \int_{\Gamma} \dot{P}^*_m V^{*n} d\Gamma + \int_{\Omega} \rho f^*_m V^{*n} d\Omega \quad (4.5)$$

(4.5)式的增量形式为

$$\Delta u^i_j + \int_{\Gamma} \dot{P}^*_{ni} (\Delta u^m) d\Gamma = \int_{\Gamma} (\Delta P^*_m) V^{*n} d\Gamma + \int_{\Omega} \Delta(\rho f^*_m) V^{*n} d\Omega \quad (4.6)$$

其中 $\Gamma = \Gamma_u + \Gamma_P$,

$$\Delta P^*_m = \sqrt{\frac{g}{g_0}} \frac{1}{\det(F^i_j)} [\Delta \tilde{P}^*_q F^q_m + \tilde{P}^*_i \Delta u^i_{,j} \tilde{F}^j_i F^i_q F^q_m] \quad (4.7)$$

这里 \tilde{P}^*_i , $\Delta \tilde{P}^*_i$ 分别为在整体系中表面力分量和其增量分量; $g = \sqrt{g_{11}g_{22} - g_{12}g_{21}}$; g_0 为初始位形下的 g 值。

(4.5)式就是大变形的边界积分方程, 形式上与小变形线弹性边界积分方程一致。但是(4.5)式整个积分方程是建立在拖带坐标系上, 以瞬时位形为基准的。并且每个积分中的积分变量均是瞬时构形的函数, 是非线性的。在具体数值计算时, 必须根据标架无差异原理及不变量原则把拖带系中的量转化为整体系中的量来计算。

关于非线性边界积分方程求解, 我们采用修正拉格朗日和迭代算法。在所编制的计算程序 NBEM 中, 采用如下迭代求解步骤:

(1) 使用 t 瞬时位形求 $t + \Delta t$ 时刻的位移增量, 此时(4.6)式为:

$$\Delta u^i_{j(t+\Delta t)} + \int_{\Gamma} \dot{P}^*_{ni} \Delta u^m_{(t+\Delta t)} d\Gamma = \int_{\Gamma} \Delta P^*_m V^{*n}_{(t)} d\Gamma + \int_{\Omega} \Delta(\rho f^*_m) V^{*n}_{(t)} d\Omega$$

是线性方程, 可求解出 $\Delta u^i_{(1)}$ 。

(2) 修正位形的尺规:

$$\left. \begin{aligned}
u^i_{(t+\Delta t)} &= u^i_{(t)} + \Delta u^i_{(1)}; & F^i_{(t+\Delta t)} &= \delta^i_j + u^i_{(t+\Delta t),j} \\
g_{ij} &= F^k_i F^k_j
\end{aligned} \right\}$$

由(4.7)式修正 $\Delta P^*_m(F^i_j)$ 及基本解 $\dot{P}^*_{ni}(F^i_j)$ 和 $V^{*n}(F^i_j)$, 然后重新求解积分方程(4.6)

式得到 $\Delta u^i_{(2)}$,

$$\Delta u_{(t+\Delta t)}^{i(2)} + \int_{\Gamma_u} \dot{P}_{m,t}^* \Delta u_{(t+\Delta t)}^{i(2)} d\Gamma = \int_{\Gamma_p} \Delta P_m V_{(t+\Delta t)}^{*n} d\Gamma + \int_{\Omega} \Delta(\rho f)_m V_{(t+\Delta t)}^{*n} d\Omega$$

(3) 类似地, 可以得到 $\Delta u_{(t+\Delta t)}^{i(3)}, \dots, \Delta u_{(t+\Delta t)}^{i(n-1)}, \Delta u_{(t+\Delta t)}^{i(n)}$. 当 $\Delta u_{(t+\Delta t)}^{i(n)} - \Delta u_{(t+\Delta t)}^{i(n-1)} \leq \varepsilon$ (给定误差) 时, $\Delta u_{(t+\Delta t)}^i = \Delta u_{(t+\Delta t)}^{i(n)}$, 否则从第(2)步开始重新求解.

五、算 例

1. 无限弹性体圆形孔洞受单向拉伸初应力作用

由于本文采用拖带坐标描述法, 以瞬时位形为基准, 在小变形下, 位形变化微小, 则 NBEM 计算结果与小变形理论解应是一致的. 作为算例, 用本文的非线性程序 NBEM 计算了无限弹性体内圆形孔洞在单向拉伸初应力下的应力状态. 计算结果示于图 2 中. 图 2 表明 NBEM 的计算值与本问题的理论解吻合很好.

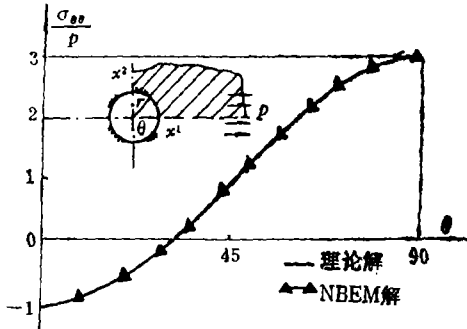


图 2 孔周边的切向应力

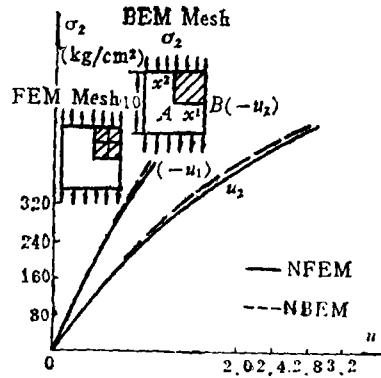


图 3 应力-位移关系曲线

2. 简单拉伸下大变形分析

选用明胶材料, 受单向拉伸载荷, 使之进行大变形. 认为材料是弹性的. 用本文的 NBEM 程序和非线性大变形有限元程序 NFEM^{[6],[7]} 进行了计算, 计算结果见图 3.

3. 厚壁圆筒受内压 p 作用

作为算例, 选厚壁圆筒受内压 p 作用, 设材料是弹性的, $E=2100.0\text{T/m}^2, \nu=0.3$. 几何尺寸如图 4 所示. 用本文的 NBEM 程序和非线性有限元程序 NFEM^{[6],[7]} 进行了计算, 结果比较接近. 如图 4、图 5 所示.

六、结 论

1. 本文作为边界元法在几何非线性力学问题中应用的初次探讨. 考虑非线性大变形的特点, 以瞬时位形为基准, 推导出建立在拖带系上的非线性大变形的边界积分方程. 消除了传统地把非线性项转化为体积分作为初应力(或初应变)来求解的缺点.

2. 本文导出了瞬时位形上速率形式的非线性影响(近似的基本解), 从而建立非线性大变形边界积分方程. 这是应用边界元法求解固体力学中的非线性力学问题的新的思想方法.

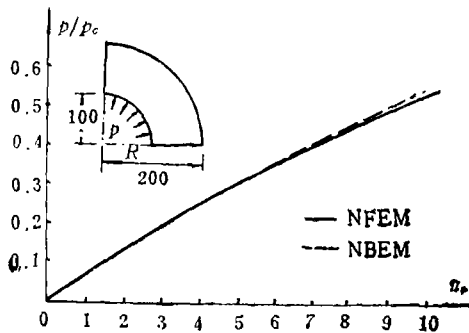


图4 内表面位移-内压关系

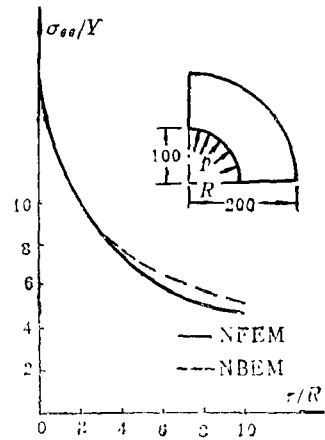


图5 沿径向应力分布

3. 对于物理非线性: 弹塑性, 非弹性材料具有流变特性的大变形力学问题的边界元法, 仅只在本文的基础上的一个简单延拓. 关于这些领域的研究将另文介绍. 最后, 作者感谢陈至达教授对本文工作给予的指导.

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Analysis of Nonlinear Large Deformation Problems by Boundary Element Method

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Abstract

In this paper, the author deduces an approximate solution of nonlinear influence function in rate for two-dimensional elastic problems on current configuration by the method of co-moving coordinate system. Here BEM formulation of large deformation based on Chen's theory⁽¹⁾ is given. The computational processes of nonlinear boundary integral equation is discussed. The author also compiles a nonlinear computing program NBEM. Numerical examples show that the results presented here is available to the solution of engineering problems.