

均布载荷作用下开顶扁球壳的 非线性稳定问题*

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摘 要

本文利用修正迭代法研究了具有硬中心的开顶扁球壳在均布载荷作用下的轴对称非线性稳定问题, 得到了临界载荷的二次近似解析公式。

一、引 言

在工程结构中, 大量使用薄壁壳体。按照设计要求, 需要核算它的稳定性。这对于选择材料以及结构的尺寸有着重要的意义。

近几十年来, 薄壳非线性稳定理论的研究一直是引人注意的课题。但是, 这项课题的研究不仅在理论上, 而且在实验上都遇到了巨大的困难, 所以解决的范围较窄、问题较少, 大都只讨论较简单的扁球壳、圆柱壳和扁锥壳, 而对于扁球壳开顶等较复杂的情况, 则研究甚少。刘人怀^[1~4]、罗祖道^[5]、Tillman^[6]等人先后对开顶扁球壳的非线性稳定问题作了研究, 获得了一些有益结果。然而, 在这一领域内, 迄今还有一些急待解决的问题尚未被讨论过, 本问题即属此范围。

我们考虑一个具有刚性中心的开顶扁球壳, 外边缘完全固定, 承受均布载荷作用。我们采用叶开沅和刘人怀于1965年所提出的修正迭代法^[1,7,8], 克服了非线性数学和壳体开顶的困难, 获得了临界载荷的解析公式。这一方法由于吸收了钱伟长^[9]的以中心挠度做摄动参数的摄动法以及逐次逼近法的优点, 因而程序简单明确, 计算量少, 而且结果精度较高。本文所获得的结果可供工程设计部门参考使用。

二、边值问题的建立

考虑一个均布载荷作用下具有硬中心的开顶扁球壳, 如图1所示。壳体厚度为 h , 中曲面

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半径为 R ，内、外边缘半径分别为 b ， a ； r 是扁球壳中曲面点至对称中心轴的距离。壳体边缘是夹紧的。

根据均布载荷 q 作用下的扁球壳非线性弯曲理论^[1,2,10]，本文所涉及的壳体的挠度 w 和径向薄膜力 N_r 所满足的基本方程是

$$\left. \begin{aligned} \frac{D}{r} \frac{d}{dr} r \frac{d}{dr} \frac{1}{r} \frac{d}{dr} (r\vartheta) - \frac{1}{r} \frac{d}{dr} [rN_r(\theta+\vartheta)] &= q \\ \frac{r}{Eh} \frac{d}{dr} \frac{1}{r} \frac{d}{dr} (r^2N_r) + \vartheta \left(\theta + \frac{1}{2} \vartheta \right) &= 0 \end{aligned} \right\} \quad (2.1a, b)$$

其中， ϑ 和 θ 分别是壳体径线方向弧的旋转角和倾斜角， E 为弹性模量， ν 为泊松比， D 是抗弯刚度

$$\vartheta = \frac{dw}{dr}, \quad \theta = \frac{r}{R}, \quad D = \frac{Eh^3}{12(1-\nu^2)} \quad (2.2)$$

求出 w 和 N_r 后，我们可由下列公式计算径向薄膜位移 u ，径向剪力 Q_r 以及径向弯矩 M_r

$$\left. \begin{aligned} u &= \frac{r}{Eh} \left[r \frac{dN_r}{dr} + (1-\nu)N_r \right] \\ Q_r &= -D \frac{d}{dr} \frac{1}{r} \frac{d}{dr} (r\vartheta) \\ M_r &= -D \left(\frac{d\vartheta}{dr} + \frac{\nu}{r} \vartheta \right) \end{aligned} \right\} \quad (2.3a, b, c)$$

为简化计算，先将方程(2.1a)积分得

$$Dr \frac{d}{dr} \frac{1}{r} \frac{d}{dr} (r\vartheta) - rN_r(\theta+\vartheta) = F(r) \quad (2.4)$$

其中 $F(r)$ 为载荷函数， C 为积分常数

$$F(r) = \int qr dr + C \quad (2.5)$$

然后，应用式(2.3b)，方程(2.4)化为

$$r[Q_r + N_r(\theta+\vartheta)] = -F(r) \quad (2.6)$$

现在，我们来确定载荷函数 $F(r)$ 。为此，用圆锥曲面从半径为 r 处切割壳体，如图2所示。将作用在这部份壳体上的所有力向对称轴投影，并注意 $\theta+\vartheta$ 的微小条件，这样就得到

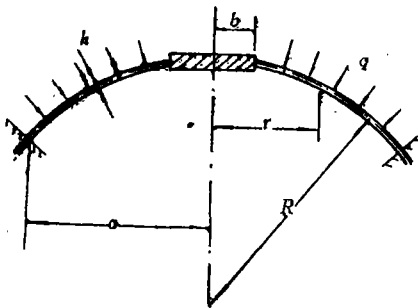


图1 均布载荷作用下的开顶扁球壳

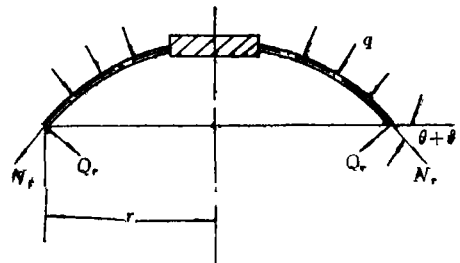


图2 载荷函数 $F(r)$ 的确定

$$2\pi r[Q_r + N_r(\theta+\vartheta)] + \pi q(r^2 - b^2) = 0 \quad (2.7)$$

将式(2.7)与(2.6)比较，就得到

$$F(r) = \frac{1}{2} q(r^2 - b^2) \quad (2.8)$$

这样, 方程(2.4)最后化为

$$Dr \frac{d}{dr} \frac{1}{r} \frac{d}{dr} (r\vartheta) - rN_r(\theta + \vartheta) = \frac{1}{2} q(r^2 - b^2) \quad (2.9)$$

因此, 方程(2.9)和(2.16)就组成具有硬中心的开顶扁球壳在均布载荷作用下的轴对称大挠度方程组.

我们在此仅讨论边缘夹紧情况.

当 $r=a$ 时, 外边缘夹紧:

$$w=0, \vartheta=0, u=0 \quad (2.10)$$

当 $r=b$ 时, 内边缘被固定在可上、下移动的无变形的一块硬中心上:

$$\vartheta=0, u=0 \quad (2.11)$$

为了计算方便, 引入下列无量纲量

$$\begin{aligned} \rho &= \frac{r}{a}, \quad \alpha = \frac{b}{a}, \quad y = \sqrt{12\lambda_1\lambda_2} \frac{w}{h} + \frac{1}{2} k\rho^2 \\ \varphi &= -\frac{dy}{d\rho}, \quad S = \frac{a^2}{D\rho} N_r, \quad k = \sqrt{12\lambda_1\lambda_2} \frac{a^2}{Rh} \\ Q &= \sqrt{3\lambda_1\lambda_2} \frac{a^4 q}{Dh} \end{aligned} \quad (2.12)$$

其中

$$\lambda_1 = 1 - \nu, \quad \lambda_2 = 1 + \nu \quad (2.13)$$

利用这些无量纲量, 方程(2.9), (2.1b)及边界条件(2.10), (2.11)简化为

$$\left. \begin{aligned} L(\rho\varphi) &= S\varphi + Q(\alpha^2 - \rho^2) \\ L(\rho S) &= \frac{1}{2}(k^2\rho^2 - \varphi^2) \end{aligned} \right\} \quad (2.14a, b)$$

当 $\rho=1$ 时,

$$y = \frac{1}{2} k, \quad \varphi = -k, \quad \frac{dS}{d\rho} - \nu S = 0 \quad (2.15)$$

当 $\rho=\alpha$ 时,

$$\varphi = -k\alpha, \quad \rho \frac{dS}{d\rho} - \nu S = 0 \quad (2.16)$$

其中 L 是微分算子

$$L = \rho \frac{d}{d\rho} \frac{1}{\rho} \frac{d}{d\rho} \quad (2.17)$$

这样一来, 问题就化为在边界条件(2.15), (2.16)下求解非线性微分方程组(2.14)~(2.16).

三、边值问题的求解

我们使用修正迭代法解此问题. 在第一次近似中, 我们略去方程(2.14a)中的非线性项

$S\varphi$, 便得到如下的线性边值问题

$$\left. \begin{aligned} L(\rho\varphi_1) &= Q(\alpha^2 - \rho^2) \\ L(\rho S_1) &= \frac{1}{2}(k^2\rho^2 - \varphi_1^2) \end{aligned} \right\} \quad (3.1a, b)$$

当 $\rho=1$ 时,

$$y_1 = \frac{1}{2}k, \quad \varphi_1 = -k, \quad \frac{dS_1}{d\rho} - \nu S_1 = 0 \quad (3.2a, b, c)$$

当 $\rho=\alpha$ 时,

$$\varphi_1 = -k\alpha, \quad \rho \frac{dS_1}{d\rho} - \nu S_1 = 0 \quad (3.3a, b)$$

应用式(3.2b)和(3.3a)积分方程(3.1a), 就得到小挠度理论的解

$$\begin{aligned} \varphi_1 = & -\frac{1}{8(1-\alpha^2)} \{ \alpha^2(1-\alpha^2+4\alpha^2\ln\alpha)Q\rho^{-1} - [(Q-8k)+8k\alpha^2 \\ & - Q\alpha^4+4Q\alpha^4\ln\alpha]\rho - 4\alpha^2(1-\alpha^2)Q\rho\ln\rho + (1-\alpha^2)Q\rho^3 \} \end{aligned} \quad (3.4)$$

以扁球壳的无量纲内边缘挠度 W_m 作迭代参数

$$W_m = \sqrt{12(1-\nu^2)} \frac{w}{h} \Big|_{r=b} \quad (3.5)$$

利用式(2.12)及(3.2a)可得到

$$W_m = \int_a^1 (\varphi + k\rho) d\rho \quad (3.6)$$

再将(3.4)代入上式, 即得到一次近似下壳体的特征关系

$$Q = 8\beta_1^{-1}\beta_2 W_m \quad (3.7)$$

其中

$$\begin{aligned} \beta_1 &= (1-\alpha^2)^{-1} \\ \beta_2 &= 4(1-5\alpha^2+4\alpha^2\ln\alpha+7\alpha^4-4\alpha^4\ln\alpha+16\alpha^4\ln^2\alpha-3\alpha^6)^{-1} \end{aligned} \quad (3.8)$$

将式(3.7)代入式(3.4)中, 得到

$$\begin{aligned} \varphi_1 = & -k\rho - \beta_2 W_m [(\alpha^2 - \alpha^4 + 4\alpha^4\ln\alpha)\rho^{-1} - (1-\alpha^4 \\ & + 4\alpha^4\ln\alpha)\rho - 4\alpha^2(1-\alpha^2)\rho\ln\rho + (1-\alpha^2)\rho^3] \end{aligned} \quad (3.9)$$

应用式(3.9)及边界条件(3.2c)、(3.3b), 积分方程(3.1b), 便得到无量纲薄膜内力的近似解为

$$\begin{aligned} S_1 = & k\beta_1\beta_2 W_m \left[B\rho^{-1} + A\rho - \frac{1}{2}\alpha^2(1-2\alpha^2+4\alpha^2\ln\alpha+\alpha^4-4\alpha^4\ln\alpha)\rho\ln\rho + \frac{1}{8}(1-4\alpha^2 \right. \\ & \left. + 5\alpha^4+4\alpha^4\ln\alpha-2\alpha^6-4\alpha^6\ln\alpha)\rho^3 + \frac{1}{2}\alpha^2(1-\alpha^2)^2\rho^3\ln\rho - \frac{1}{24}(1-\alpha^2)^2\rho^5 \right] \\ & + \beta_1\beta_2 W_m \left[D\rho^{-1} + \frac{1}{4}\alpha^4(1-3\alpha^2+8\alpha^2\ln\alpha+3\alpha^4-16\alpha^4\ln\alpha+16\alpha^4\ln^2\alpha-\alpha^6 \right. \\ & \left. + 8\alpha^6\ln\alpha-16\alpha^6\ln^2\alpha)\rho^{-1}\ln\rho + C\rho + \frac{1}{2}\alpha^2(1-4\alpha^2+4\alpha^2\ln\alpha+6\alpha^4-8\alpha^4\ln\alpha-4\alpha^6 \right. \\ & \left. + 4\alpha^6\ln\alpha+16\alpha^6\ln^2\alpha+\alpha^8-16\alpha^8\ln^2\alpha)\rho\ln\rho + \alpha^4(1-3\alpha^2+4\alpha^2\ln\alpha+3\alpha^4-8\alpha^4\ln\alpha \right. \end{aligned}$$

$$\begin{aligned}
& -\alpha^6 + 4\alpha^6 \ln \alpha) \rho \ln^2 \rho - \frac{1}{16} (1 - 5\alpha^2 + 18\alpha^4 + 16\alpha^4 \ln \alpha - 34\alpha^6 \\
& - 48\alpha^8 \ln \alpha + 29\alpha^8 + 48\alpha^8 \ln \alpha + 16\alpha^8 \ln^2 \alpha - 9\alpha^{10} - 16\alpha^{10} \ln \alpha - 16\alpha^{10} \ln^2 \alpha) \rho^3 \\
& - \frac{1}{2} \alpha^2 (1 - 5\alpha^2 + 9\alpha^4 + 4\alpha^4 \ln \alpha - 7\alpha^6 - 8\alpha^8 \ln \alpha + 2\alpha^8 + 4\alpha^8 \ln \alpha) \rho^3 \ln \rho \\
& - \alpha^4 (1 - \alpha^2)^3 \rho^3 \ln^2 \rho + \frac{1}{72} (3 - 11\alpha^2 + 15\alpha^4 + 12\alpha^4 \ln \alpha - 9\alpha^6 - 24\alpha^6 \ln \alpha \\
& + 2\alpha^8 + 12\alpha^8 \ln \alpha) \rho^5 + \frac{1}{6} \alpha^2 (1 - \alpha^2)^3 \rho^5 \ln \rho - \frac{1}{96} (1 - \alpha^2)^3 \rho^7) \quad (3.10)
\end{aligned}$$

其中

$$\begin{aligned}
A &= -\frac{1}{\lambda_1} \left(\frac{2-\nu}{12} - \frac{11-4\nu}{12} \alpha^2 + \frac{7-\nu}{8} \alpha^4 - \nu \alpha^4 \ln \alpha + \frac{4-5\nu}{12} \alpha^6 \right. \\
&\quad \left. + \nu \alpha^8 \ln \alpha + 2\lambda_1 \alpha^8 \ln^2 \alpha - \frac{11-7\nu}{24} \alpha^8 \right) \\
B &= -\frac{1}{\lambda_2} \left(\frac{2-\nu}{12} \alpha^2 - \frac{19-11\nu}{24} \alpha^4 + \frac{\lambda_1}{2} \alpha^4 \ln \alpha + \frac{13-8\nu}{12} \alpha^6 \right. \\
&\quad \left. - \frac{\lambda_1}{2} \alpha^8 \ln \alpha + 2\lambda_1 \alpha^8 \ln^2 \alpha - \frac{11-7\nu}{24} \alpha^8 \right) \\
C &= \frac{1}{\lambda_1} \left[\frac{5-3\nu}{96} - \frac{73-23\nu}{144} \alpha^2 + \frac{679-209\nu}{288} \alpha^4 + \frac{5-19\nu}{12} \alpha^4 \ln \alpha - \frac{275-67\nu}{72} \alpha^6 \right. \\
&\quad \left. + \frac{1+13\nu}{3} \alpha^8 \ln \alpha + (1-5\nu) \alpha^8 \ln^2 \alpha + \frac{461+125\nu}{288} \alpha^8 - \frac{23+47\nu}{12} \alpha^8 \ln \alpha \right. \\
&\quad \left. - (2-6\nu) \alpha^8 \ln^2 \alpha + \frac{151-197\nu}{144} \alpha^{10} + \frac{7}{6} \lambda_2 \alpha^{10} \ln \alpha - 8\nu \alpha^8 \ln^3 \alpha + \lambda_1 \alpha^{10} \ln^2 \alpha \right. \\
&\quad \left. - \frac{211-173\nu}{288} \alpha^{12} + 4\lambda_1 \alpha^{10} \ln^3 \alpha \right] \\
D &= \frac{1}{\lambda_2} \left[\frac{5-3\nu}{96} \alpha^2 + \frac{1+4\nu}{18} \alpha^4 + \frac{1-3\nu}{4} \alpha^4 \ln \alpha + \frac{73-311\nu}{288} \alpha^6 + \frac{11+5\nu}{3} \alpha^6 \ln \alpha \right. \\
&\quad \left. + (1-5\nu) \alpha^8 \ln^2 \alpha - \frac{155-213\nu}{96} \alpha^8 - \frac{97+13\nu}{12} \alpha^8 \ln \alpha + 7\lambda_2 \alpha^8 \ln^2 \alpha \right. \\
&\quad \left. + \frac{143-139\nu}{72} \alpha^{10} + \frac{25+\nu}{6} \alpha^{10} \ln \alpha - 8\nu \alpha^8 \ln^3 \alpha - \frac{211-173\nu}{288} \alpha^{12} \right. \\
&\quad \left. - (8+2\nu) \alpha^{10} \ln^2 \alpha + 4\lambda_1 \alpha^{10} \ln^3 \alpha \right]
\end{aligned}$$

在二次近似中, 我们有下述边值问题

$$L(\rho \varphi_2) = S_1 \varphi_1 + Q(\alpha^2 - \rho^2) \quad (3.11)$$

当 $\rho=1$ 时,

$$y_2 = \frac{1}{2} k, \quad \varphi_2 = -k \quad (3.12a, b)$$

当 $\rho = \alpha$ 时,

$$\varphi_2 = -k\alpha \quad (3.13)$$

将方程(3.11)积分, 并利用边界条件(3.12b)及(3.13), 就得到 φ 的二次近似解

$$\begin{aligned} \varphi_2 = & \varphi_1 + k^2 \beta_1 \beta_2 W_m [\phi_1(\rho) + A_1 \rho + B_1 \rho^{-1}] + k \beta_1 \beta_2^2 W_m^2 [\phi_1(\rho) \\ & + A_2 \rho + B_2 \rho^{-1}] + \beta_1 \beta_2^3 W_m^3 [\phi_3(\rho) + A_3 \rho + B_3 \rho^{-1}] \end{aligned} \quad (3.14)$$

其中

$$\begin{aligned} A_n &= \frac{\alpha \phi_n(\alpha) - \phi_n(1)}{1 - \alpha^2} \\ B_n &= \frac{\alpha^2 \phi_n(1) - \alpha \phi_n(\alpha)}{1 - \alpha^2} \quad (n=1, 2, 3) \end{aligned}$$

$$\begin{aligned} \phi_1(\rho) = & -\frac{B}{2} \rho \ln \rho - \frac{A}{8} \rho^3 + \frac{1}{64} \alpha^2 (1 - 2\alpha^2 + 4\alpha^2 \ln \alpha + \alpha^4 - 4\alpha^4 \ln \alpha) \cdot (4\rho^3 \ln \rho - 3\rho^3) \\ & - \frac{1}{192} (1 - 4\alpha^2 + 5\alpha^4 + 4\alpha^4 \ln \alpha - 2\alpha^6 - 4\alpha^6 \ln \alpha) \rho^5 \\ & - \frac{1}{576} \alpha^2 (1 - \alpha^2)^2 (12\rho^5 \ln \rho - 5\rho^5) + \frac{1}{1152} (1 - \alpha^2)^2 \rho^7 \\ \phi_2(\rho) = & -A \left[\frac{1}{2} \alpha^2 (1 - \alpha^2 + 4\alpha^2 \ln \alpha) \rho \ln \rho - \frac{1}{8} (1 - 3\alpha^2 + 2\alpha^4 + 4\alpha^4 \ln \alpha) \rho^3 \right. \\ & \left. - \frac{1}{2} \alpha^2 (1 - \alpha^2) \rho^3 \ln \rho + \frac{1}{24} (1 - \alpha^2) \rho^5 \right] + B \left[\frac{1}{2} \alpha^2 (1 - \alpha^2 + 4\alpha^2 \ln \alpha) \rho^{-1} \ln \rho \right. \\ & \left. + \frac{1}{2} (1 - 2\alpha^2 + \alpha^4 + 4\alpha^4 \ln \alpha) \rho \ln \rho + \alpha^2 (1 - \alpha^2) \rho \ln^2 \rho - \frac{1}{8} (1 - \alpha^2) \rho^3 \right] \\ & - \frac{1}{2} D \rho \ln \rho - \frac{1}{8} C \rho^3 - \frac{1}{64} \alpha^2 (1 - 5\alpha^2 + 4\alpha^2 \ln \alpha + 9\alpha^4 - 12\alpha^4 \ln \alpha - 7\alpha^6 + 12\alpha^6 \ln \alpha \\ & + 16\alpha^6 \ln^2 \alpha + 2\alpha^8 - 4\alpha^8 \ln \alpha - 16\alpha^8 \ln^2 \alpha) \rho^3 + \frac{1}{1152} (9 - 37\alpha^2 + 72\alpha^4 + 104\alpha^4 \ln \alpha \\ & - 84\alpha^6 - 280\alpha^6 \ln \alpha + 55\alpha^8 + 248\alpha^8 \ln \alpha + 144\alpha^8 \ln^2 \alpha - 15\alpha^{10} - 72\alpha^{10} \ln \alpha \\ & - 144\alpha^{10} \ln^2 \alpha) \rho^5 - \frac{1}{3456} (15 - 62\alpha^2 + 96\alpha^4 + 60\alpha^4 \ln \alpha - 66\alpha^6 - 120\alpha^6 \ln \alpha + 17\alpha^8 \\ & + 60\alpha^8 \ln \alpha) \rho^7 + \frac{1}{1536} (1 - \alpha^2)^3 \rho^9 + \frac{1}{16} \alpha^4 (1 - 3\alpha^2 + 8\alpha^2 \ln \alpha + 3\alpha^4 - 16\alpha^4 \ln \alpha \\ & + 16\alpha^4 \ln^2 \alpha - \alpha^6 + 8\alpha^6 \ln \alpha - 16\alpha^6 \ln^2 \alpha) (\rho \ln^2 \rho - \rho \ln \rho) - \frac{1}{64} \alpha^2 (2 - 5\alpha^2 + 8\alpha^2 \ln \alpha \\ & + 3\alpha^4 - 4\alpha^4 \ln \alpha + \alpha^6 - 16\alpha^6 \ln \alpha + 32\alpha^6 \ln^2 \alpha - \alpha^8 + 12\alpha^8 \ln \alpha - 32\alpha^8 \ln^2 \alpha) (4\rho^3 \ln \rho \\ & - 3\rho^3) + \frac{1}{576} \alpha^2 (4 - 15\alpha^2 + 4\alpha^2 \ln \alpha + 21\alpha^4 + 4\alpha^4 \ln \alpha - 13\alpha^6 - 20\alpha^6 \ln \alpha + 3\alpha^8 \\ & + 12\alpha^8 \ln \alpha) (12\rho^5 \ln \rho - 5\rho^5) - \frac{5}{6912} \alpha^2 (1 - \alpha^2)^3 (24\rho^7 \ln \rho - 7\rho^7) - \frac{3}{64} \alpha^4 (1 - 3\alpha^2 \\ & + 4\alpha^2 \ln \alpha + 3\alpha^4 - 8\alpha^4 \ln \alpha - \alpha^6 + 4\alpha^6 \ln \alpha) (8\rho^3 \ln^2 \rho - 12\rho^3 \ln \rho + 7\rho^3) \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{576} \alpha^4 (1-\alpha^2)^3 (72\rho^5 \ln^2 \rho - 60\rho^5 \ln \rho + 19\rho^5) \\
\phi_3(\rho) = & -C \left[\frac{1}{2} \alpha^2 (1-\alpha^2 + 4\alpha^2 \ln \alpha) \rho \ln \rho - \frac{1}{8} (1-3\alpha^2 + 2\alpha^4 + 4\alpha^4 \ln \alpha) \rho^3 \right. \\
& - \frac{1}{2} \alpha^2 (1-\alpha^2) \rho^3 \ln \rho + \left. \frac{1}{24} (1-\alpha^2) \rho^5 \right] + D \left[\frac{1}{2} \alpha^2 (1-\alpha^2 + 4\alpha^2 \ln \alpha) \rho^{-1} \ln \rho \right. \\
& + \left. \frac{1}{2} (1-2\alpha^2 + \alpha^4 + 4\alpha^4 \ln \alpha) \rho \ln \rho + \alpha^2 (1-\alpha^2) \rho \ln^2 \rho - \frac{1}{8} (1-\alpha^2) \rho^3 \right] \\
& + \frac{1}{128} \alpha^2 (1-6\alpha^2 + 4\alpha^2 \ln \alpha + 23\alpha^4 - 4\alpha^4 \ln \alpha - 52\alpha^6 + 8\alpha^6 \ln \alpha + 64\alpha^6 \ln^2 \alpha + 63\alpha^8 \\
& - 40\alpha^8 \ln \alpha - 176\alpha^8 \ln^2 \alpha - 38\alpha^{10} + 52\alpha^{10} \ln \alpha + 160\alpha^{10} \ln^2 \alpha + 9\alpha^{12} - 20\alpha^{12} \ln \alpha \\
& + 64\alpha^{10} \ln^3 \alpha - 48\alpha^{12} \ln^2 \alpha - 64\alpha^{12} \ln^3 \alpha) \rho^3 - \frac{1}{3456} (9 - 39\alpha^2 + 125\alpha^4 + 204\alpha^4 \ln \alpha \\
& - 209\alpha^6 - 676\alpha^6 \ln \alpha + 51\alpha^8 + 984\alpha^8 \ln \alpha + 816\alpha^8 \ln^2 \alpha + 247\alpha^{10} - 936\alpha^{10} \ln \alpha \\
& - 2064\alpha^{10} \ln^2 \alpha - 265\alpha^{12} + 604\alpha^{12} \ln \alpha + 1680\alpha^{12} \ln^2 \alpha + 81\alpha^{14} - 180\alpha^{14} \ln \alpha \\
& + 576\alpha^{12} \ln^3 \alpha - 432\alpha^{14} \ln^2 \alpha - 576\alpha^{14} \ln^3 \alpha) \rho^5 + \frac{1}{13824} (30 - 149\alpha^2 + 450\alpha^4 \\
& + 396\alpha^4 \ln \alpha - 910\alpha^6 - 1460\alpha^6 \ln \alpha + 1070\alpha^8 + 2004\alpha^8 \ln \alpha + 480\alpha^8 \ln^2 \alpha - 645\alpha^{10} \\
& - 1212\alpha^{10} \ln \alpha - 960\alpha^{10} \ln^2 \alpha + 154\alpha^{12} + 272\alpha^{12} \ln \alpha + 480\alpha^{12} \ln^2 \alpha) \rho^7 - \frac{1}{4608} (3 - 13\alpha^2 \\
& + 22\alpha^4 + 12\alpha^4 \ln \alpha - 18\alpha^6 - 36\alpha^6 \ln \alpha + 7\alpha^8 + 36\alpha^8 \ln \alpha - \alpha^{10} - 12\alpha^{10} \ln \alpha) \rho^9 \\
& + \frac{1}{11520} (1-\alpha^2)^4 \rho^{11} + \frac{1}{15360} \alpha^2 (1-\alpha^2)^4 (9\rho^9 - 40\rho^9 \ln \rho) + \frac{1}{16} \alpha^6 (1-4\alpha^2 \\
& + 12\alpha^2 \ln \alpha + 6\alpha^4 - 36\alpha^4 \ln \alpha + 48\alpha^4 \ln^2 \alpha - 4\alpha^6 + 36\alpha^6 \ln \alpha - 96\alpha^6 \ln^2 \alpha + \alpha^8 \\
& - 12\alpha^8 \ln \alpha + 64\alpha^8 \ln^3 \alpha + 48\alpha^8 \ln^2 \alpha - 64\alpha^8 \ln^3 \alpha) (\rho^{-1} \ln \rho + \rho^{-1} \ln^2 \rho) - \frac{1}{16} \alpha^4 (1 \\
& - 7\alpha^2 + 8\alpha^2 \ln \alpha + 18\alpha^4 - 44\alpha^4 \ln \alpha + 16\alpha^4 \ln^2 \alpha - 22\alpha^6 + 84\alpha^6 \ln \alpha - 48\alpha^6 \ln^2 \alpha \\
& + 13\alpha^8 - 68\alpha^8 \ln \alpha + 48\alpha^8 \ln^2 \alpha - 3\alpha^{10} + 20\alpha^{10} \ln \alpha + 64\alpha^8 \ln^3 \alpha - 16\alpha^{10} \ln^2 \alpha \\
& - 64\alpha^{10} \ln^3 \alpha) \cdot (\rho \ln^2 \rho - \rho \ln \rho) - \frac{1}{3456} \alpha^2 (15 - 76\alpha^2 + 24\alpha^2 \ln \alpha + 169\alpha^4 \\
& + 32\alpha^4 \ln \alpha - 216\alpha^6 - 312\alpha^6 \ln \alpha + 96\alpha^6 \ln^2 \alpha + 169\alpha^8 + 504\alpha^8 \ln \alpha - 48\alpha^8 \ln^2 \alpha \\
& - 76\alpha^{10} - 320\alpha^{10} \ln \alpha - 192\alpha^{10} \ln^2 \alpha + 15\alpha^{12} + 72\alpha^{12} \ln \alpha + 144\alpha^{12} \ln^2 \alpha) (12\rho^5 \ln \rho \\
& - 5\rho^5) + \frac{1}{20736} \alpha^2 (15 - 77\alpha^2 + 158\alpha^4 + 60\alpha^4 \ln \alpha - 162\alpha^6 - 180\alpha^6 \ln \alpha + 83\alpha^8 \\
& + 180\alpha^8 \ln \alpha - 17\alpha^{10} - 60\alpha^{10} \ln \alpha) (24\rho^7 \ln \rho - 7\rho^7) + \frac{3}{64} \alpha^4 (1 - 4\alpha^2 + 4\alpha^2 \ln \alpha \\
& + 6\alpha^4 - 8\alpha^4 \ln \alpha - 4\alpha^6 + 16\alpha^6 \ln^2 \alpha + \alpha^8 + 8\alpha^8 \ln \alpha - 32\alpha^8 \ln^2 \alpha - 4\alpha^{10} \ln \alpha + 16\alpha^{10} \ln^2 \alpha) \\
& (8\rho^3 \ln^2 \rho - 12\rho^3 \ln \rho + 7\rho^3) - \frac{1}{1728} \alpha^4 (4 - 19\alpha^2 + 4\alpha^2 \ln \alpha + 36\alpha^4 - 34\alpha^6 \\
& - 24\alpha^6 \ln \alpha + 16\alpha^8 + 32\alpha^8 \ln \alpha - 3\alpha^{10} - 12\alpha^{10} \ln \alpha) (72\rho^5 \ln^2 \rho - 60\rho^5 \ln \rho + 19\rho^5)
\end{aligned}$$

$$\begin{aligned}
& + \frac{5}{41472} \alpha^4 (1-\alpha^2)^4 \cdot (288\rho^7 \ln^2 \rho - 168\rho^7 \ln \rho + 37\rho^7) + \frac{1}{64} \alpha^6 (1-4\alpha^2 + 4\alpha^2 \ln \alpha \\
& + 6\alpha^4 - 12\alpha^4 \ln \alpha - 4\alpha^6 + 12\alpha^6 \ln \alpha + \alpha^8 - 4\alpha^8 \ln \alpha) \cdot (32\rho^8 \ln^3 \rho - 72\rho^8 \ln^2 \rho \\
& + 84\rho^8 \ln \rho - 45\rho^8) - \frac{1}{1728} \alpha^6 (1-\alpha^2)^4 (288\rho^5 \ln^3 \rho - 360\rho^5 \ln^2 \rho + 228\rho^5 \ln \rho - 65\rho^5) \\
& + \frac{1}{128} \alpha^2 (2-7\alpha^2 + 8\alpha^2 \ln \alpha + 2\alpha^4 - 8\alpha^4 \ln \alpha + 22\alpha^6 - 40\alpha^6 \ln \alpha + 48\alpha^6 \ln^2 \alpha - 38\alpha^8 \\
& + 88\alpha^8 \ln \alpha - 32\alpha^8 \ln^2 \alpha + 25\alpha^{10} - 64\alpha^{10} \ln \alpha - 80\alpha^{10} \ln^2 \alpha - 6\alpha^{12} + 16\alpha^{12} \ln \alpha \\
& + 128\alpha^{10} \ln^3 \alpha + 64\alpha^{12} \ln^2 \alpha - 128\alpha^{12} \ln^3 \alpha) (4\rho^9 \ln \rho - 3\rho^9)
\end{aligned}$$

将式(3.14)代入式(3.5), 便得到壳体的二次近似特征关系式

$$Q = (a_1 + a_2 k^2) W_m + a_3 k W_m^2 + a_4 W_m^3 \quad (3.15)$$

其中

$$a_1 = 8\beta_1^{-1} \beta_2$$

$$\begin{aligned}
a_2 = & -\lambda_1^{-1} \lambda_2^{-1} \beta_1 \beta_2^2 \left[\frac{227 + 144\nu - 83\nu^2}{3456} \alpha^{14} - \frac{35 + 20\nu - 15\nu^2}{48} \alpha^{12} \ln \alpha \right. \\
& + \frac{557 - 888\nu + 283\nu^2}{1152} \alpha^{12} - \frac{13 - 48\nu + 23\nu^2}{6} \alpha^{10} \ln^2 \alpha + \frac{37 - 10\nu + \nu^2}{24} \alpha^{10} \ln \alpha \\
& - \frac{3025 - 3456\nu + 1007\nu^2}{1152} \alpha^{10} + 8\lambda_1^2 \alpha^8 \ln^4 \alpha - 2\lambda_1^2 \alpha^8 \ln^3 \alpha + (5 - 17\nu + 8\nu^2) \alpha^8 \ln^2 \alpha \\
& - \frac{17 - 552\nu + 295\nu^2}{144} \alpha^8 \ln \alpha + \frac{14945 - 17784\nu + 5287\nu^2}{3456} \cdot \alpha^8 + 2\lambda_1^2 \alpha^6 \ln^3 \alpha \\
& - \frac{7 - 20\nu + 9\nu^2}{2} \alpha^6 \ln^2 \alpha - \frac{77 + 232\nu - 133\nu^2}{48} \alpha^6 \ln \alpha - \frac{1155 - 1744\nu + 557\nu^2}{384} \cdot \alpha^6 \\
& + \frac{2 - 3\nu + \nu^2}{3} \alpha^4 \ln^2 \alpha + \frac{25 + 46\nu - 27\nu^2}{24} \alpha^4 \ln \alpha + \frac{811 - 2376\nu + 845\nu^2}{1152} \alpha^4 \\
& \left. - \frac{19 + 12\nu - 7\nu^2}{144} \alpha^2 \ln \alpha + \frac{299 + 1440\nu - 587\nu^2}{3456} \alpha^2 - \frac{35 + 24\nu - 11\nu^2}{1152} \right]
\end{aligned}$$

$$\begin{aligned}
a_3 = & \lambda_1^{-1} \lambda_2^{-1} \beta_1 \beta_2^2 \left[\frac{2989 + 1610\nu - 1379\nu^2}{17280} \alpha^{18} - \frac{2407 + 1132\nu - 1275\nu^2}{576} \alpha^{16} \ln \alpha \right. \\
& + \frac{7599 - 10625\nu + 3496\nu^2}{2880} \alpha^{16} + \frac{74\nu - 46\nu^2}{3} \alpha^{14} \ln^3 \alpha + \frac{289 + 332\nu - 333\nu^2}{24} \alpha^{14} \ln^2 \alpha \\
& + \frac{2637 + 3820\nu - 2273\nu^2}{576} \alpha^{14} \ln \alpha - \frac{44671 - 54310\nu + 17059\nu^2}{2880} \alpha^{14} + (24 - 96\nu \\
& + 72\nu^2) \alpha^{12} \ln^5 \alpha - (68 - 40\nu + 20\nu^2) \alpha^{12} \ln^4 \alpha + \frac{237 - 304\nu + 155\nu^2}{6} \alpha^{12} \ln^3 \alpha \\
& - \frac{479 + 656\nu - 595\nu^2}{12} \alpha^{12} \ln^2 \alpha + \frac{4945 - 852\nu - 1573\nu^2}{192} \alpha^{12} \ln \alpha \\
& \left. + \frac{45682 - 61485\nu + 20813\nu^2}{1440} \alpha^{12} + (24 - 96\nu + 72\nu^2) \alpha^{10} \ln^5 \alpha + (48 + 24\nu - 24\nu^2) \alpha^{10} \ln^4 \alpha \right]
\end{aligned}$$

$$\begin{aligned}
 & -\frac{151-24\nu-7\nu^2}{2} \alpha^{10} \ln^3 \alpha + \frac{1127+1996\nu-1595\nu^2}{24} \alpha^{10} \ln^2 \alpha \\
 & -\frac{37645+5380\nu-16137\nu^2}{576} \alpha^{10} \ln \alpha - \frac{21879-38335\nu+14636\nu^2}{720} \alpha^{10} + (20-64\nu \\
 & + 44\nu^2) \alpha^8 \ln^4 \alpha + \frac{195+176\nu-139\nu^2}{6} \alpha^8 \ln^3 \alpha - \frac{523+1436\nu-975\nu^2}{24} \alpha^8 \ln^2 \alpha \\
 & + \frac{33735+10300\nu-16523\nu^2}{576} \alpha^8 \ln \alpha + \frac{17953-56530\nu+24817\nu^2}{1440} \alpha^8 \\
 & + \frac{21-92\nu+55\nu^2}{6} \alpha^6 \ln^3 \alpha + \frac{13+118\nu-65\nu^2}{6} \alpha^6 \ln^2 \alpha - \frac{12693+6588\nu-7257\nu^2}{576} \alpha^6 \ln \alpha \\
 & + \frac{351+49850\nu-25021\nu^2}{2880} \alpha^6 + \frac{13-52\nu+23\nu^2}{24} \alpha^4 \ln^2 \alpha \\
 & + \frac{1613+1580\nu-1185\nu^2}{576} \alpha^4 \ln \alpha - \frac{2202+6355\nu-3467\nu^2}{1440} \alpha^4 - \frac{75+44\nu-31\nu^2}{576} \alpha^2 \ln \alpha \\
 & + \left. \frac{1897+3290\nu-1727\nu^2}{5760} \alpha^2 - \frac{227+145\nu-82\nu^2}{8640} \right] \\
 a_4 = & -\lambda_1^{-1} \lambda_2^{-1} \beta_1 \beta_2 \left[\frac{238954+98800\nu-140154\nu^2}{2073600} \alpha^{22} - \frac{81726+26720\nu-55006\nu^2}{17280} \alpha^{20} \ln \alpha \right. \\
 & + \frac{249384-349552\nu+119912\nu^2}{82944} \alpha^{20} + \frac{157\nu-127\nu^2}{3} \alpha^{18} \ln^3 \alpha \\
 & + \frac{5949-1839\nu-1848\nu^2}{108} \alpha^{18} \ln^2 \alpha - \frac{62768-162720\nu+68272\nu^2}{8640} \alpha^{18} \ln \alpha \\
 & - \frac{3879361-5326680\nu+1879079\nu^2}{207360} \alpha^{18} + \frac{64-384\nu+320\nu^2}{3} \alpha^{16} \ln^6 \alpha \\
 & - (184-144\nu-40\nu^2) \alpha^{16} \ln^5 \alpha + \frac{224-612\nu-92\nu^2}{3} \alpha^{16} \ln^4 \alpha - \\
 & - \frac{187+192\nu-1027\nu^2}{9} \alpha^{16} \ln^3 \alpha - \frac{32069-2524\nu-13137\nu^2}{144} \alpha^{16} \ln^2 \alpha \\
 & + \frac{1000932-577640\nu-118412\nu^2}{8640} \alpha^{16} \ln \alpha + \frac{8666966-14289720\nu+5900914\nu^2}{207360} \alpha^{16} \\
 & + \frac{64-384\nu+1088\nu^2}{3} \alpha^{14} \ln^6 \alpha - (24-48\nu+216\nu^2) \alpha^{14} \ln^5 \alpha + \frac{74+816\nu+310\nu^2}{3} \alpha^{14} \ln^4 \alpha \\
 & + \frac{61-1310\nu-563\nu^2}{6} \alpha^{14} \ln^3 \alpha + \frac{37290+11139\nu-21291\nu^2}{108} \alpha^{14} \ln^2 \alpha \\
 & - \frac{5279806-1813360\nu-1321646\nu^2}{17280} \alpha^{14} \ln \alpha - \frac{4221089-10798200\nu+5492071\nu^2}{103680} \alpha^{14} \\
 & + \frac{64-384\nu+320\nu^2}{3} \alpha^{12} \ln^6 \alpha + (184-48\nu+56\nu^2) \alpha^{12} \ln^5 \alpha - \frac{790-48\nu+218\nu^2}{3} \alpha^{12} \ln^4 \alpha \\
 & + \frac{1721+4812\nu+211\nu^2}{18} \alpha^{12} \ln^3 \alpha - \frac{110445+107076\nu-94521\nu^2}{432} \alpha^{12} \ln^2 \alpha
 \end{aligned}$$

$$\begin{aligned}
& + \frac{6486112 - 1257360\nu - 2127472\nu^2}{17280} \alpha^{12} \ln \alpha + \frac{3280898 - 5042460\nu + 32440552\nu^2}{518400} \alpha^{12} \\
& + (24 - 144\nu + 120\nu^2) \alpha^{10} \ln^5 \alpha + (154 - 32\nu - 42\nu^2) \alpha^{10} \ln^4 \alpha \\
& - \frac{817 + 382\nu - 107\nu^2}{6} \alpha^{10} \ln^3 \alpha + \frac{3455 + 7397\nu - 4662\nu^2}{36} \alpha^{10} \ln^2 \alpha \\
& - \frac{2106857 - 53560\nu - 864417\nu^2}{8640} \alpha^{10} \ln \alpha + \frac{4709378 + 12210480\nu - 9934418\nu^2}{207360} \alpha^{10} \\
& + (10 - 52\nu + 42\nu^2) \alpha^8 \ln^4 \alpha + \frac{299 - 52\nu - 79\nu^2}{6} \alpha^8 \ln^3 \alpha - \frac{3307 + 9500\nu - 5471\nu^2}{144} \alpha^8 \ln^2 \alpha \\
& + \frac{88726 + 20785\nu - 47421\nu^2}{1080} \alpha^8 \ln \alpha - \frac{4456264 + 5005680\nu - 4876504\nu^2}{207360} \alpha^8 \\
& + \frac{4 - 23\nu + 17\nu^2}{3} \alpha^5 \ln^3 \alpha + \frac{528 + 573\nu - 495\nu^2}{108} \alpha^5 \ln^2 \alpha \\
& - \frac{37099 + 26120\nu - 28259\nu^2}{2880} \alpha^5 \ln \alpha + \frac{3437218 + 2918640\nu - 2937778\nu^2}{414720} \alpha^5 \\
& + \frac{5 - 28\nu + 15\nu^2}{48} \alpha^4 \ln^2 \alpha + \frac{8123 + 8920\nu - 7843\nu^2}{8640} \alpha^4 \ln \alpha \\
& - \frac{599344 + 585200\nu - 497984\nu^2}{414720} \alpha^4 - \frac{77 + 40\nu - 37\nu^2}{2880} \alpha^2 \ln \alpha \\
& + \left. \frac{119227 + 147800\nu - 101027\nu^2}{1036800} \alpha^2 - \frac{173 + 100\nu - 73\nu^2}{34560} \right]
\end{aligned}$$

我们以 $\alpha=0.3$, $\nu=0.3$ 的情况为例, 按公式(3.15)在图3上绘出了不同几何参数 k 值下的特征曲线。由图3看出, 只有当 k 大于某一值 k_0 后, 曲线才呈现出迴形线形态, 这时壳体出现了失稳现象。

对式(3.15)应用极值条件

$$\frac{dQ}{dW_m} = 0 \quad (3.16)$$

就得到产生失稳时的临界无量纲内缘挠度公式

$$W_m^* = \frac{-a_3 k \pm \sqrt{(a_3^2 - 3a_2 a_4) k^2 - 3a_1 a_4}}{3a_4} \quad (3.17)$$

将此 W_m^* 代入式(3.15), 就得到二次近似的临界载荷公式

$$Q^* = (a_1 + a_2 k^2) W_m^* + a_3 k W_m^{*2} + a_4 W_m^{*3} \quad (3.18)$$

这里, 对应于(3.17)式中的正、负号, 所得到的 Q^* 分别是下临界、上临界载荷, 它们分别对应于特征曲线上的极小值和极大值点。

按照式(3.18)进行数值计算, 我们得到了几个不同 α 值的上、下临界载荷数据, 并将结果绘在图4中。从图上看, 上临界载荷是几何参数 k 的单调上升函数, 而下临界载荷只在一段范围内是 k 的增值函数, 随后随着 k 的增大而迅速下降。当 k 值较大后, 下临界载荷变为负值, 表明若无反向作用, 则壳体不能恢复原状。

我们现在研究壳体的屈曲临界点, 也即上、下临界载荷相重合的点。记此时壳体的几何参数为 k_0 , 相应的临界载荷为 Q_0^* 。由根的判别式为零, 得到决定临界几何参数 k_0 的方程

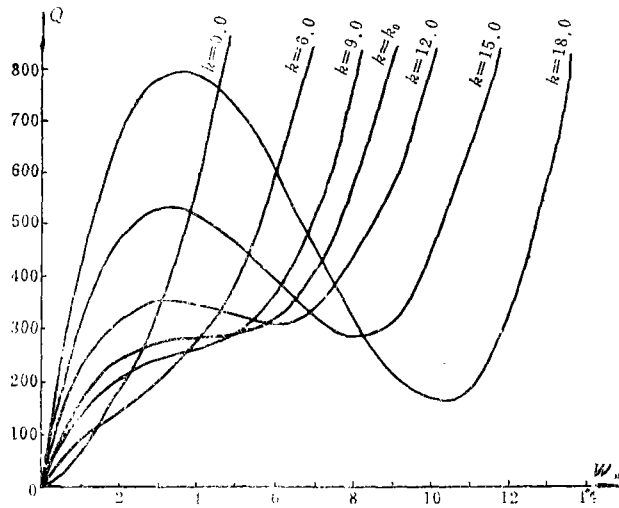


图3 各种k值下的特征曲线 ($\nu=0.3, \alpha=0.3$)

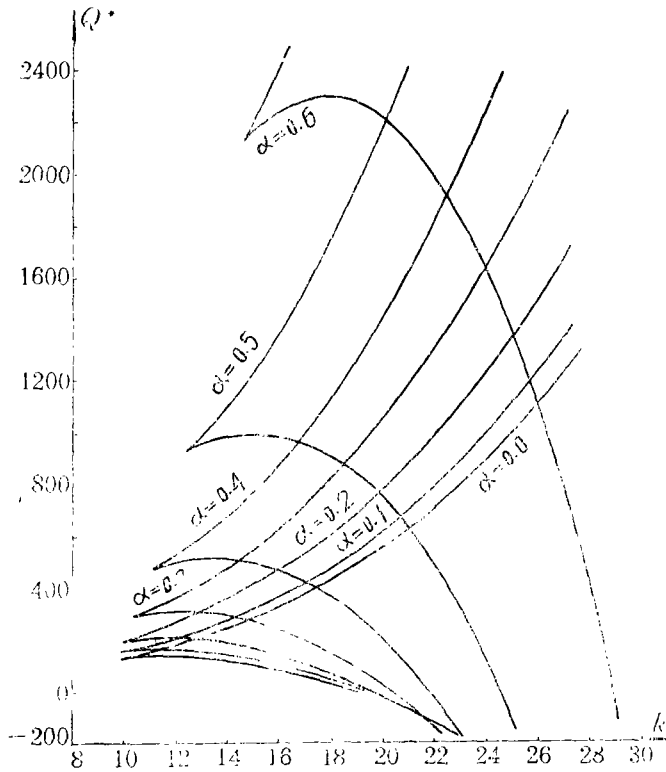


图4 各种alpha值下的稳定曲线 ($\nu=0.3$)

$$(a_3^2 - 3a_2a_4)k_0^2 - 3a_1a_4 = 0 \tag{3.19}$$

由此求出

$$k_0 = \sqrt{\frac{3a_1a_4}{a_3^2 - 3a_2a_4}} \tag{3.20}$$

显而易见，当 $k < k_0$ 时，壳体不会产生失稳。当 $k \geq k_0$ ，壳体就出现失稳。所以， k_0 是确定壳体失稳与否的分界点。图5及图6给出了 $k_0 - \alpha$ 以及 $Q_0^* - \alpha$ 的关系曲线。

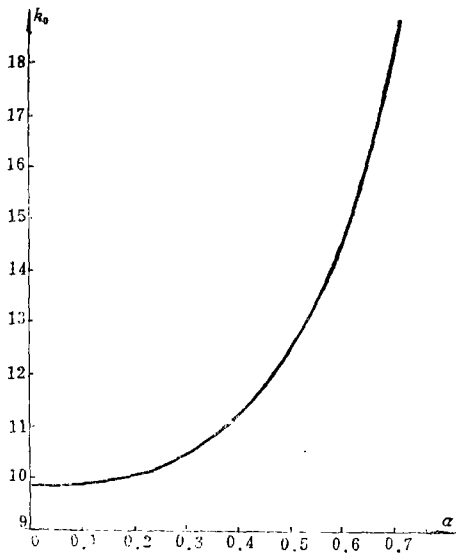


图5 临界几何参数 k_0 与 α 的关系曲线 ($\nu=0.3$)

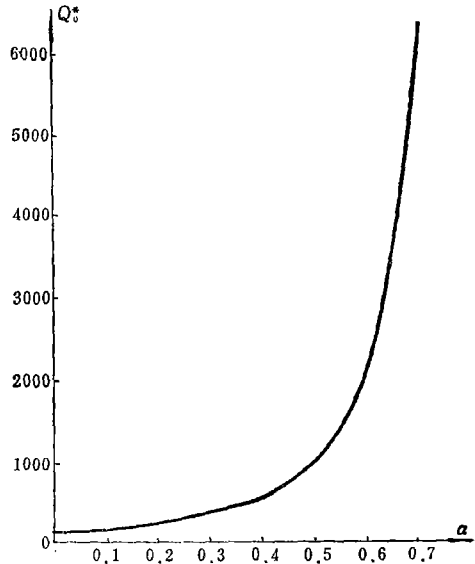


图6 临界点载荷 Q_0^* 与 α 的关系曲线 ($\nu=0.3$)

最后，在式 (3.18) 中令 $\alpha \rightarrow 0$ ，便得到封顶扁球壳在均布载荷作用下的临界载荷计算公式。在 $\nu=0.3$ 情况下，其计算结果绘在图 7 中，并与实验结果及以前的理论分析结果作一比较。从图上看，我们的理论结果与实验结果吻合较好。

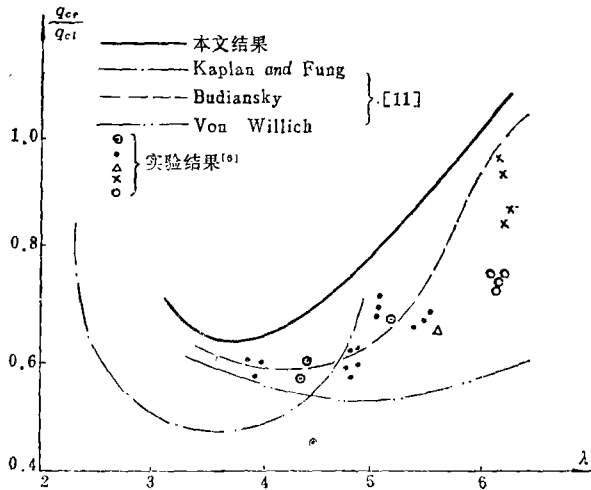


图7 理论结果与实验结果的比较 ($\nu=0.3$)

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On the Nonlinear Stability of a Truncated Shallow Spherical Shell under a Uniformly Distributed Load

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Abstract

A problem of practical interest for nonlinear axisymmetrical stability of a truncated shallow spherical shell with a nondeformable rigid body under uniformly distributed load is studied in this paper. By using modified iteration method, some important analytic results are obtained and the corresponding numerical results are given in figures.