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大小涡旋分开考虑的模式理论

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摘　要

近年来，k-ε 模式普遍流行，但它是一种梯度形式的模式理论．由于湍流涡旋的衰减时间极长，在一般的流动问题中，上游产生的大涡旋或附近产生的大涡旋在到达当地时还远远没有衰减掉，因此会对当地的流动产生很大的影响．根据不可逆热力学理论的基本原理，必须准平衡、准定常和小偏离的情形下，才能使用梯度形式的流和力的关系式．一般说来这对大涡旋是并不满足的，所以k-ε 模式的应用有着很大的局限性．本文根据湍流的实际情况，把湍流脉动分成大涡旋和小涡旋两个组成部份，并且把大涡旋部份再分成局部产生的和上游流来或扩散过来的两个组成部份，这样就得到了由三个部份组成的湍流模式理论．

一、引　　言

近年来发展起来的湍流模式理论很多，其中最著名的为 k-ε 二方程模式理论．但它仍然是一种把大涡旋和小涡旋合併在一起考虑的梯度形式的模式理论．大家都知道湍流中的大涡旋是各向异性的，并且和边界形状及平均流动密切相关，而小涡旋则基本上是各向同性的．因为动量的传输，传热和传质等主要是大涡旋在起作用，所以只有在大涡旋十分微弱时候，小涡旋在这些现象上的贡献才会表现出来．与此相反，湍流动能的耗散则是小涡旋在起主导作用．湍流脉动的值通常是由平均流动传给湍流的能量和湍流动能的耗散相互抵消来确定的．为了更好地描述湍流运动的性质，我们有必要把大小两种涡旋分开考虑．同时由于湍流涡旋的衰减时间很长，大涡旋中除了当地产生的涡旋以外，还有上游来流带来的涡旋和邻近区域扩散过来的涡旋．因为有最后一种外来的大涡旋的存在，这就使得雷诺应力不可能用当地平均速度梯度的形式表示出来．不仅如此，雷诺应力也不可能用当地的所有物理量的组合表示出来，而必须考虑这些湍流的历史过程．为了如实地反映湍流这方面的性质，我们必须把湍流大涡旋再分解成当地产生和邻近传来两个组成部份，这样我们就得到了一种和以前常见的模式相差很大的模式理论．

二、大小涡旋脉动速度所满足的方程式

对于某一个物理量A，我们可以把它写成由三个部份组成，即

$$A=\bar{A}+A^{l}+A^{s}$$

式中$\bar{A}$为A的雷诺平均值，A^l为A相应于大涡旋的部份，A^s为A相应于小涡旋的部份。现在同时存在有两种平均，一种是对大小涡旋一起的总的平均，也就是雷诺平均，我们用符号"—"表示，另一种则是仅仅对小涡旋取平均，我们用符号"～"来表示。这样我们有

$$\left.\begin{array}{l}\tilde{\bar{A}}=\bar{A}\\ \tilde{\tilde{A}}=\bar{A}+A^l,\ \tilde{\tilde{A^s}}=0,\ \bar{A^s}=\tilde{A^s}=0\\ \tilde{A}=\bar{A}+\bar{A^l}+\bar{A^s},\ \bar{A^l}+\bar{A^s}=0,\ \bar{A^l}=0\\ \overline{A^lB^s}=\widetilde{\widetilde{A^lB^s}}=A^l\widetilde{\widetilde{B^s}}=0\end{array}\right\} \tag{2.1}$$

这样，不可压缩流体的速度V_i和压力p等于

$$V_i=\bar{V}_i+V_i^l+V_i^s,\quad p=\bar{p}+p^l+p^s$$

于是我们有N-S方程和连续方程为

$$\frac{\partial V_i}{\partial t}+V_j\frac{\partial V_i}{\partial x_j}=-\frac{1}{\rho}\frac{\partial p}{\partial x_i}+\nu\nabla^2 V_i,\qquad \frac{\partial V_j}{\partial x_j}=0 \tag{2.2}$$

同时有平均流动的雷诺方程和连续方程为

$$(\mathrm{I})\quad\left\{\begin{array}{l}\dfrac{\partial\bar{V}_i}{\partial t}+\bar{V}_j\dfrac{\partial\bar{V}_i}{\partial x_j}=-\dfrac{1}{\rho}\dfrac{\partial\bar{p}}{\partial x_i}+\nu\nabla^2\bar{V}_i-\dfrac{\partial}{\partial x_j}\overline{(V_i^lV_j^l)}\\ \qquad\qquad -\dfrac{\partial}{\partial x_j}\overline{(V_i^sV_j^s)}\\ \dfrac{\partial\bar{V}_j}{\partial x_j}=0\end{array}\right. \tag{2.3}$$

把两式相减，得到

$$\left.\begin{array}{l}\dfrac{\partial}{\partial t}(V_i^l+V_i^s)+\bar{V}_j\dfrac{\partial}{\partial x_j}(V_i^l+V_i^s)+(V_j^l+V_j^s)\dfrac{\partial\bar{V}_i}{\partial x_j}\\ \quad+(V_j^l+V_j^s)\dfrac{\partial}{\partial x_j}(V_i^l+V_i^s)\\ =-\dfrac{1}{\rho}\dfrac{\partial}{\partial x_i}(p^l+p^s)+\nu\nabla^2(V_i^l+V_i^s)+\dfrac{\partial}{\partial x_j}\overline{(V_i^lV_j^l)}\\ \quad+\dfrac{\partial}{\partial x_j}\overline{(V_i^sV_j^s)}\\ \dfrac{\partial}{\partial x_j}(V_j^l+V_j^s)=0\end{array}\right\} \tag{2.4}$$

同样，可以得到经过小涡旋平均以后的运动方程式和连续方程式

$$\left.\begin{array}{l}\dfrac{\partial}{\partial t}(\bar{V}_i+V_i^l)+(\bar{V}_j+V_j^l)\dfrac{\partial}{\partial x_j}(\bar{V}_i+V_i^l)\\ =-\dfrac{1}{\rho}\dfrac{\partial}{\partial x_i}(\bar{p}+p^l)+\nu\nabla^2(\bar{V}_i+V_i^l)-\dfrac{\partial}{\partial x_j}\widetilde{(V_i^sV_j^s)}\\ \dfrac{\partial}{\partial x_j}(\bar{V}_j+V_j^l)=0\end{array}\right\} \tag{2.5}$$

从N-S方程和连续方程(2.2)中减去这个方程组(2.5)，就得到小涡旋所满足的方程组

$$
(\text{II})\quad
\begin{cases}
\dfrac{\partial}{\partial t}V_i^s+(\bar{V}_j+V_j^l)\dfrac{\partial V_i^s}{\partial x_j}+V_j^s\dfrac{\partial}{\partial x_j}(\bar{V}_i+V_i^l)+V_j^s\dfrac{\partial}{\partial x_j}V_i^s \\
\qquad =-\dfrac{1}{\rho}\dfrac{\partial p^s}{\partial x_i}+\nu\nabla^2 V_i^s+\dfrac{\partial}{\partial x_j}\widetilde{(V_i^sV_j^s)} \\
\dfrac{\partial V_j^s}{\partial x_j}=0
\end{cases}
\tag{2.6}
$$

从(2.4)式减去(2.6)式，我们就得到大涡旋所满足的方程组

$$
(\text{III})\quad
\begin{cases}
\dfrac{\partial}{\partial t}V_i^l+\bar{V}_j\dfrac{\partial V_i^l}{\partial x_j}+V_j^l\dfrac{\partial}{\partial x_j}\bar{V}_i+V_j^l\dfrac{\partial}{\partial x_j}V_i^l \\
\qquad =-\dfrac{1}{\rho}\dfrac{\partial p^l}{\partial x_i}+\nu\nabla^2V_i^l+\dfrac{\partial}{\partial x_j}\overline{V_i^lV_j^l}+\dfrac{\partial}{\partial x_j}\overline{V_i^sV_j^s} \\
\qquad\quad -\dfrac{\partial}{\partial x_j}\widetilde{V_i^sV_j^s} \\
\dfrac{\partial V_j^l}{\partial x_j}=0
\end{cases}
\tag{2.7}
$$

三、雷诺应力各部份所满足的方程式

按照周培源教授的办法，我们把(2.6)式的第一式乘上V_k^s，再加上相应的V_k^s的方程式乘以V_i^s，我们取整个式子的雷诺平均，就得到

$$
\begin{aligned}
&\frac{\partial}{\partial t}\overline{V_i^sV_k^s}+\bar{V}_j\frac{\partial}{\partial x_j}\overline{V_i^sV_k^s}+\frac{\partial}{\partial x_j}(\overline{V_j^lV_i^sV_k^s}) \\
&+\frac{\partial}{\partial x_j}(\overline{V_j^sV_i^sV_k^s})+\overline{V_j^sV_k^s}\frac{\partial\bar{V}_i}{\partial x_j}+\overline{V_j^sV_i^s}\frac{\partial\bar{V}_k}{\partial x_j} \\
&+\overline{V_j^sV_k^s\frac{\partial}{\partial x_j}V_i^l}+\overline{V_j^sV_i^s\frac{\partial}{\partial x_j}V_k^l} \\
=&-\frac{1}{\rho}\left[\overline{\frac{\partial p^s}{\partial x_i}V_k^s}+\overline{\frac{\partial p^s}{\partial x_k}V_i^s}\right]+\nu\nabla^2\overline{V_i^sV_k^s} \\
&-2\nu\overline{\frac{\partial V_i^s}{\partial x_\alpha}\frac{\partial V_k^s}{\partial x_\alpha}}+\overline{V_k^s\frac{\partial}{\partial x_j}\widetilde{(V_i^sV_j^s)}}+\overline{V_i^s\frac{\partial}{\partial x_j}\widetilde{(V_k^sV_j^s)}}
\end{aligned}
\tag{3.1$'$}
$$

把(2.7)式的第一式乘上V_k^l，再加上相应的V_k^l的方程式乘以V_i^l，再取整个式子的雷诺平均，我们就得到

$$
\begin{aligned}
&\frac{\partial}{\partial t}\overline{V_i^lV_k^l}+\bar{V}_j\frac{\partial}{\partial x_j}\overline{V_i^lV_k^l}+\overline{V_j^lV_k^l}\frac{\partial\bar{V}_i}{\partial x_j}+\overline{V_j^lV_i^l}\frac{\partial\bar{V}_k}{\partial x_j}+\frac{\partial}{\partial x_j}(\overline{V_j^lV_i^lV_k^l}) \\
&=-\frac{1}{\rho}\left(\overline{\frac{\partial p^l}{\partial x_i}V_k^l}+\overline{\frac{\partial p^l}{\partial x_k}V_i^l}\right)+\nu\nabla^2\overline{V_i^lV_k^l} \\
&\quad-2\nu\overline{\frac{\partial V_i^l}{\partial x_\alpha}\frac{\partial V_k^l}{\partial x_\alpha}}-\left[\overline{V_k^l\frac{\partial}{\partial x_j}\widetilde{V_i^sV_j^s}}+\overline{V_i^l\frac{\partial}{\partial x_j}\widetilde{V_k^sV_j^s}}\right]
\end{aligned}
\tag{3.2}
$$

由于

$$
\overline{V_k^s\frac{\partial}{\partial x_j}\widetilde{(V_i^sV_j^s)}}=\overline{\widetilde{V_k^s\frac{\partial}{\partial x_j}\widetilde{(V_i^sV_j^s)}}}=\overline{\widetilde{V_k^s}\frac{\partial}{\partial x_j}\widetilde{(V_i^sV_j^s)}}=0
$$

所以(3.1)′式可以简化为

$$\frac{\partial}{\partial t}\overline{V_i'V_k'}+\bar{V}_j\frac{\partial}{\partial x_j}\overline{V_i'V_k'}+\frac{\partial}{\partial x_j}(V_j'\widetilde{V_i'V_k'})+\frac{\partial}{\partial x_j}(\overline{V_j'V_i'V_k'})+\overline{\widetilde{V_j'V_k'}}\frac{\partial\bar{V}_i}{\partial x_j}$$

$$+\overline{\widetilde{V_j'V_i'}}\frac{\partial\bar{V}_k}{\partial x_j}+\widetilde{V_j'V_k'}\frac{\partial}{\partial x_j}V_i'+\widetilde{V_j'V_i'}\frac{\partial V_k'}{\partial x_j}$$

$$=-\frac{1}{\rho}\left[\overline{\frac{\partial p^s}{\partial x_i}V_k'}+\overline{\frac{\partial p^s}{\partial x_k}V_i'}\right]+\nu\nabla^2\overline{V_i'V_k'}-2\nu\overline{\frac{\partial V_i'}{\partial x_\alpha}\frac{\partial V_k'}{\partial x_\alpha}} \tag{3.1}$$

四、小涡旋的准均匀各向同性假设与方程式的简化

我们假设小涡旋是接近于均匀各向同性的，这时关联函数的表达式可用周培源教授采用过的表达式

$$\left.\begin{aligned}
\widetilde{V_i'V_k'}'=&\widetilde{V_i'V_k'}+A_1(\widetilde{V_i'V_k'})_{,j}\xi_j+A_2(\widetilde{V_j'V_k'})_{,i}\xi_j+A_3(\widetilde{V_j'V_i'})_{,k}\xi_j\\
&+A_4(\widetilde{V_j'V_k'})_{,j}\xi_i+A_5(\widetilde{V_i'V_j'})_{,j}\xi_k+\frac{q^2}{3\lambda^2}\Big\{\frac{1}{2}(1+4G)\xi_i\xi_k\\
&-\frac{1}{2}\Big[(2+5G)r^2-\frac{1}{3}(k+11G)R_{mn}\xi_m\xi_n\Big]\delta_{ik}\\
&-\frac{1}{6}(k-4G)r^2R_{ik}-G(R_{il}\xi_l\xi_k+R_{kl}\xi_l\xi_i)\Big\}+\cdots\\
&+\frac{q^2}{4!\lambda^4}E_{ikjlmn}\xi_j\xi_l\xi_m\xi_n+\cdots\\
\widetilde{V_i'V_j'V_k'}'=&\widetilde{V_i'V_j'V_k'}+B_1(\widetilde{V_i'V_j'V_k'})_{,l}\xi_l+\cdots\\
&+\frac{F(q^2)^{\frac{3}{2}}}{3!3\sqrt{3}\lambda^3}\Big[2\xi_i\xi_j\xi_k-\frac{5}{2}(\delta_{ik}\xi_j+\delta_{jk}\xi_i)r^2+\delta_{ij}\xi_kr^2\Big]+\cdots\\
\widetilde{V_i'V_m'V_n'}'=&\widetilde{V_i'V_m'V_n'}+B_1'(\widetilde{V_i'V_m'V_n'})_{,l}\xi_l+\cdots\\
&-\frac{F(q^2)^{\frac{3}{2}}}{3!3\sqrt{3}\lambda^3}\Big[2\xi_i\xi_m\xi_n-\frac{5}{2}(\delta_{mi}\xi_n+\delta_{ni}\xi_m)r^2+\delta_{mn}\xi_ir^2\Big]\\
&+\cdots
\end{aligned}\right\} \tag{4.1}$$

式中

$$\xi_i=x_i'-x_i,\quad r^2=\xi_i\xi_i,\quad q^2=\widetilde{V_i'V_i'}=\overline{V_i'V_i'},\quad R_{ik}=\frac{3}{q^2}\overline{V_i'V_k'}$$

λ为Taylor湍流微尺度。这样做了以后，我们再引入

$$\widetilde{V_i'V_k'}=\frac{1}{3}q^2\delta_{ik}-\nu_T\left(\frac{\partial\bar{V}_i}{\partial x_k}+\frac{\partial V_i'}{\partial x_k}+\frac{\partial\bar{V}_k}{\partial x_i}+\frac{\partial V_k'}{\partial x_i}\right) \tag{4.2}$$

$$\frac{1}{\rho}\left(\widetilde{\frac{\partial p^s}{\partial x_i}V_k'}+\widetilde{\frac{\partial p^s}{\partial x_k}V_i'}\right)+\frac{\partial}{\partial x_j}(\widetilde{V_i'V_j'V_k'})$$

$$\approx-\frac{\partial}{\partial x_j}\left(c_s\nu_T\frac{\partial}{\partial x_j}\widetilde{V_i'V_k'}\right)-\phi_{ik}' \tag{4.3}$$

$$\phi_{ik}'\equiv\widetilde{\frac{p^s}{\rho}\left(\frac{\partial V_i'}{\partial x_k}+\frac{\partial V_k'}{\partial x_i}\right)} \tag{4.4}$$

$$\overline{\frac{\partial V^s_i}{\partial x_a}\frac{\partial V^s_k}{\partial x_a}}\approx\frac{1}{3}\frac{q^2}{\lambda^2}[(5-k)\delta_{ik}+kR_{ik}]\tag{4.5}$$

$$\nu_T=q\lambda f(R_\lambda)\tag{4.6}$$

式中c_s为无量纲系数。而

$$R_\lambda=\frac{q\lambda}{\nu}$$

$f(R_\lambda)$为R_λ的已知函数，$f(R_\lambda)$为常数相应于k-ω模式；$f(R_\lambda)$正比于R_λ，相应于k-ε模式。把这些关系式代入(3.1)，并且把指标i和k收缩，就得到

$$\begin{aligned}&\frac{\partial}{\partial t}q^2+\bar V_j\frac{\partial}{\partial x_j}q^2-2\nu_T\left(\frac{\partial\bar V_j}{\partial x_k}+\frac{\partial\bar V_k}{\partial x_j}\right)\frac{\partial\bar V_k}{\partial x_j}\\&\quad-2\nu_T\overline{\left(\frac{\partial V^l_j}{\partial x_k}+\frac{\partial V^l_k}{\partial x_j}\right)\frac{\partial V^l_k}{\partial x_j}}\\&\quad\approx-10\nu\frac{q^2}{\lambda^2}+\nu\nabla^2q^2+\frac{\partial}{\partial x_j}\left(c_s\nu_T\frac{\partial q^2}{\partial x_j}\right)\end{aligned}\tag{4.7}$$

或改写成

$$\begin{aligned}&\frac{\partial}{\partial t}q^2+\bar V_j\frac{\partial q^2}{\partial x_j}-2\nu_T\frac{\partial\bar V_k}{\partial x_j}\frac{\partial\bar V_k}{\partial x_j}-2\nu_T\frac{\partial^2}{\partial x_j\partial x_k}(\bar V_j\bar V_k)\\&\quad-2\nu_T\overline{\frac{\partial V^l_k}{\partial x_j}\frac{\partial V^l_k}{\partial x_j}}-2\nu_T\frac{\partial^2}{\partial x_j\partial x_k}\overline{V^l_jV^l_k}\\&\quad\approx-10\nu\frac{q^2}{\lambda^2}+\nu\nabla^2q^2+\frac{\partial}{\partial x_j}\left(c_s\nu_T\frac{\partial}{\partial x_j}q^2\right)\end{aligned}\tag{4.7$'$}$$

由于存在有q^2和λ^2两个变数，现在只有一个方程式(4.7)，所以还必须补充一个方程式。我们就加进$\overline{\omega^2}$或ε的方程式。我们令

$$\bar\Omega_{ik}=\frac{\partial\bar V_i}{\partial x_k}-\frac{\partial\bar V_k}{\partial x_i},\quad\omega^l_{ik}=\frac{\partial V^l_i}{\partial x_k}-\frac{\partial V^l_k}{\partial x_i}$$

$$\omega_{ik}=\frac{\partial V^s_i}{\partial x_k}-\frac{\partial V^s_k}{\partial x_i},\quad\overline{\omega^2}=\frac{1}{2}\overline{\omega_{ik}\omega_{ik}}$$

我们先写出$\overline{\omega^2}$的方程式，得到

$$\begin{aligned}&\frac{\partial}{\partial t}\overline{\omega_{ik}\omega_{ik}}+\bar V_j\frac{\partial}{\partial x_j}\overline{\omega_{ik}\omega_{ik}}+\frac{\partial}{\partial x_j}\overline{(V^l_j\widetilde{\omega_{ik}\omega_{ik}})}+2\frac{\partial\bar V_j}{\partial x_k}\overline{\frac{\partial V^s_i}{\partial x_j}\omega_{ik}}\\&\quad+2\frac{\partial\bar V_i}{\partial x_j}\overline{\frac{\partial V^s_j}{\partial x_k}\omega_{ik}}+2\overline{\frac{\partial V^l_j}{\partial x_k}\widetilde{\frac{\partial V^s_i}{\partial x_j}\omega_{ik}}}+2\overline{\frac{\partial V^l_i}{\partial x_j}\widetilde{\frac{\partial V^s_j}{\partial x_k}\omega_{ik}}}\\&\quad-2\frac{\partial\bar V_j}{\partial x_i}\overline{\frac{\partial V^s_k}{\partial x_j}\omega_{ik}}-2\frac{\partial\bar V_k}{\partial x_j}\overline{\frac{\partial V^s_j}{\partial x_i}\omega_{ik}}-2\overline{\frac{\partial V^l_j}{\partial x_i}\widetilde{\frac{\partial V^s_k}{\partial x_j}\omega_{ik}}}\\&\quad-2\overline{\frac{\partial V^l_k}{\partial x_j}\widetilde{\frac{\partial V^s_j}{\partial x_i}\omega_{ik}}}+2\overline{V^s_j\omega_{ik}}\frac{\partial}{\partial x_j}\bar\Omega_{ik}+2\overline{V^s_j\omega_{ik}\frac{\partial}{\partial x_j}\omega^l_{ik}}\\&\quad+\frac{\partial}{\partial x_j}\overline{(V^s_j\omega_{ik}\omega_{ik})}+2\overline{\left(\frac{\partial V^s_j}{\partial x_k}\frac{\partial V^s_i}{\partial x_j}-\frac{\partial V^s_j}{\partial x_i}\frac{\partial V^s_k}{\partial x_j}\right)\omega_{ik}}\\&=\nu\nabla^2\overline{\omega_{ik}\omega_{ik}}-2\nu\overline{\frac{\partial}{\partial x_j}\omega_{ik}\frac{\partial}{\partial x_j}\omega_{ik}}\end{aligned}\tag{4.8}$$

我们再写出 ε 的方程式。我们令

$$\varepsilon^*=\nu\frac{\partial V_i'}{\partial x_k}\frac{\partial V_i'}{\partial x_k},\ \varepsilon=\widetilde{\varepsilon^*}=\nu\widetilde{\frac{\partial V_i'}{\partial x_k}\frac{\partial V_i'}{\partial x_k}}=\nu\overline{\frac{\partial V_i'}{\partial x_k}\frac{\partial V_i'}{\partial x_k}}=\bar{\varepsilon}^*$$

我们可以得到

$$\begin{aligned}&\frac{\partial\varepsilon}{\partial t}+\bar{V}_j\frac{\partial\varepsilon}{\partial x_j}+\frac{\partial}{\partial x_j}\overline{V_j'\varepsilon^*}+2\nu\frac{\partial\bar{V}_i}{\partial x_j}\overline{\frac{\partial V_i'}{\partial x_k}\frac{\partial V_j'}{\partial x_k}}+2\nu\frac{\partial\bar{V}_j}{\partial x_k}\overline{\frac{\partial V_i'}{\partial x_j}\frac{\partial V_i'}{\partial x_k}}\\&+2\nu\frac{\partial V_i'}{\partial x_j}\overline{\frac{\partial V_i'}{\partial x_k}\frac{\partial V_j'}{\partial x_k}}+2\nu\frac{\partial V_j'}{\partial x_k}\overline{\frac{\partial V_i'}{\partial x_j}\frac{\partial V_i'}{\partial x_k}}+2\nu\overline{V_j'\frac{\partial V_i'}{\partial x_k}}\frac{\partial^2\bar{V}_i}{\partial x_j\partial x_k}\\&+2\nu\overline{V_j'\frac{\partial V_i'}{\partial x_k}\frac{\partial^2 V_i'}{\partial x_j\partial x_k}}+\frac{\partial}{\partial x_j}V_j'\varepsilon^*+2\nu\overline{\frac{\partial V_j'}{\partial x_k}\frac{\partial V_i'}{\partial x_j}\frac{\partial V_i'}{\partial x_k}}\\&=-\frac{2\nu}{\rho}\overline{\frac{\partial^2 p^s}{\partial x_i\partial x_k}\frac{\partial V_i'}{\partial x_k}}+\nu\nabla^2\varepsilon-2\nu^2\overline{\frac{\partial^2 V_i'}{\partial x_j\partial x_k}\frac{\partial^2 V_i'}{\partial x_j\partial x_k}}\end{aligned}\tag{4.9}$$

因为

$$\begin{aligned}\overline{\omega_{ik}\omega_{ik}}&=2\left(\overline{\frac{\partial V_i'}{\partial x_k}\frac{\partial V_i'}{\partial x_k}}-\frac{\partial^2}{\partial x_i\partial x_k}\overline{V_i'V_k'}\right)\\&\approx\left\{\frac{\varepsilon}{\nu}-\frac{\partial^2}{\partial x_i\partial x_k}\left[\frac{1}{3}q^2\delta_{ik}-\nu_T\left(\frac{\partial\bar{V}_i}{\partial x_k}+\frac{\partial\bar{V}_k}{\partial x_i}\right)\right]\right\}\\&\approx2\left(\frac{\varepsilon}{\nu}-\frac{1}{3}\nabla^2q^2+\cdots\right)\\&\approx\frac{2\varepsilon}{\nu}\end{aligned}$$

所以(4.8)和(4.9)在准均匀各向同性条件下，基本上只差一个常数因子$2/\nu$。我们对(4.8)和(4.9)取近似值，略去很多小项，最后可以化简为

$$\begin{aligned}&\frac{\partial}{\partial t}\left(\frac{q^2}{\lambda^2}\right)+\bar{V}_j\frac{\partial}{\partial x_j}\left(\frac{q^2}{\lambda^2}\right)+\frac{14}{5}G\frac{\nu_T}{\lambda^2}\frac{\partial\bar{V}_j}{\partial x_k}\frac{\partial\bar{V}_j}{\partial x_k}\\&+\frac{14}{5}G\frac{\nu_T}{\lambda^2}\overline{\frac{\partial V_j'}{\partial x_k}\frac{\partial V_j'}{\partial x_k}}-\frac{14F}{3\sqrt{3}}\frac{q^3}{\lambda^3}\\&=-\frac{2}{15}\nu E\frac{q^2}{\lambda^4}+\nu\nabla^2\left(\frac{q^2}{\lambda^2}\right)+\frac{\partial}{\partial x_j}\left[c_\varepsilon\nu_T\frac{\partial}{\partial x_j}\left(\frac{q^2}{\lambda^2}\right)\right]\end{aligned}\tag{4.10}$$

式中F就是展开式(4.1)中的系数，而

$$E=E_{kkllmm}$$

c_ε为无量纲的系数。

五、关于大涡旋的假定和大涡旋方程式的简化

在利用了第四节的假定以后，大涡旋的方程式(3.2)可以简化为

$$\begin{aligned}&\frac{\partial}{\partial t}\overline{V_i'V_k'}+\bar{V}_j\frac{\partial}{\partial x_j}\overline{V_i'V_k'}+\overline{V_j'V_k'}\frac{\partial\bar{V}_i}{\partial x_j}+\overline{V_j'V_i'}\frac{\partial\bar{V}_k}{\partial x_j}\\&+\frac{\partial}{\partial x_j}(\overline{V_j'V_i'V_k'})=-\frac{1}{\rho}\left(\overline{\frac{\partial p^l}{\partial x_i}V_k'}+\overline{\frac{\partial p^l}{\partial x_k}V_i'}\right)\\&+(\nu_T+\nu)\nabla^2\overline{V_i'V_k'}-2(\nu_T+\nu)\overline{\frac{\partial V_i'}{\partial x_a}\frac{\partial V_k'}{\partial x_a}}\end{aligned}\tag{5.1}$$

接着我们把大涡旋分成彼此不相干扰的两个组成部份。一部份是上游流过来的或邻近扩散过来的涡旋，另一部份是当地产生的涡旋。不同涡旋的物理量是彼此独立的，但涡旋经过一段时间以后，可能会从这一类变到另一类。因为所谓外来的涡旋也就是早些时候在邻近地区或上游产生的涡旋。这样我们把脉动速度和脉动压力也相应地拆成当地产生的V_i^P和p^P及外来的V_i^N和p^N。于是

$$V_i^l=V_i^P+V_i^N,\quad p^l=p^P+p^N$$

我们假定外来涡旋和当地产生的涡旋之间没有能量的交换。于是我们有外来涡旋的运动方程式

$$\left.\begin{aligned}&\frac{\partial V_i^N}{\partial t}+\bar{V}_j\frac{\partial V_i^N}{\partial x_j}+V_j^l\frac{\partial V_i^N}{\partial x_j}+V_j^N\frac{\partial \bar{V}_i}{\partial x_j}+V_j^N\frac{\partial V_i^P}{\partial x_j}\\&\quad=-\frac{1}{\rho}\frac{\partial p^N}{\partial x_i}+(\nu_T+\nu)\nabla^2V_i^N\\&\frac{\partial V_j^N}{\partial x_j}=0\end{aligned}\right\}\tag{5.2}$$

从(5.2)式，还是用同样的办法，我们可以组成相应的方程式

$$\begin{aligned}&\frac{\partial}{\partial t}\overline{V_i^NV_k^N}+\bar{V}_j\frac{\partial}{\partial x_j}\overline{V_i^NV_k^N}+\overline{V_j^NV_k^N}\frac{\partial\bar{V}_i}{\partial x_j}+\overline{V_j^NV_i^N}\frac{\partial\bar{V}_k}{\partial x_j}\\&\quad+\frac{\partial}{\partial x_j}\overline{V_j^lV_i^NV_k^N}=-\frac{1}{\rho}\left(\overline{\frac{\partial p^N}{\partial x_i}V_k^N}+\overline{\frac{\partial p^N}{\partial x_k}V_i^N}\right)\\&\quad+(\nu_T+\nu)\nabla^2\overline{V_i^NV_k^N}-2(\nu_T+\nu)\overline{\frac{\partial V_i^N}{\partial x_a}\frac{\partial V_k^N}{\partial x_a}}\end{aligned}\tag{5.3}$$

由于有上面两种涡旋之间没有能量交换这一假定，我们得到

$$\overline{V_i^lV_k^l}=\overline{V_i^NV_k^N}+\overline{V_i^PV_k^P}\tag{5.4}$$

从(5.1)式中减去(5.3)式，我们得到

$$\begin{aligned}&\frac{\partial}{\partial t}\overline{V_i^PV_k^P}+\bar{V}_j\frac{\partial}{\partial x_j}\overline{V_i^PV_k^P}+\overline{V_j^PV_i^P}\frac{\partial\bar{V}_k}{\partial x_j}+\overline{V_j^PV_k^P}\frac{\partial\bar{V}_i}{\partial x_j}\\&\quad+\frac{\partial}{\partial x_j}(\overline{V_j^lV_i^lV_k^l}-\overline{V_j^lV_i^NV_k^N})\\&=-\frac{1}{\rho}\left(\overline{\frac{\partial p^l}{\partial x_i}V_k^l}+\overline{\frac{\partial p^l}{\partial x_k}V_i^l}\right)+\frac{1}{\rho}\left(\overline{\frac{\partial p^N}{\partial x_i}V_k^N}+\overline{\frac{\partial p^N}{\partial x_k}V_i^N}\right)\\&\quad+(\nu_T+\nu)\nabla^2\overline{V_i^PV_k^P}-2(\nu_T+\nu)\left(\overline{\frac{\partial V_i^l}{\partial x_a}\frac{\partial V_k^l}{\partial x_a}}-\overline{\frac{\partial V_i^N}{\partial x_a}\frac{\partial V_k^N}{\partial x_a}}\right)\end{aligned}\tag{5.5$'$}$$

利用两种涡旋物理量相互独立的假定，上列方程化简为

$$\begin{aligned}&\frac{\partial}{\partial t}\overline{V_i^PV_k^P}+\bar{V}_j\frac{\partial}{\partial x_j}\overline{V_i^PV_k^P}+\overline{V_j^PV_i^P}\frac{\partial\bar{V}_k}{\partial x_j}+\overline{V_j^PV_k^P}\frac{\partial\bar{V}_i}{\partial x_j}+\frac{\partial}{\partial x_j}(\overline{V_j^lV_i^PV_k^P})\\&=-\frac{1}{\rho}\left(\overline{\frac{\partial p^P}{\partial x_i}V_k^P}+\overline{\frac{\partial p^P}{\partial x_k}V_i^P}\right)+(\nu_T+\nu)\nabla^2\overline{V_i^PV_k^P}-(\nu_T+\nu)\overline{\frac{\partial V_i^P}{\partial x_a}\frac{\partial V_k^P}{\partial x_a}}\end{aligned}\tag{5.5}$$

我们现在来引进假定，使方程(5.3)和(5.5)封闭。首先我们引入大涡旋的尺度l，这样我们可以使

$$\left.\begin{aligned}
&\overline{\frac{\partial V_i^P}{\partial x_a}\frac{\partial V_k^P}{\partial x_a}}=c_1\frac{\overline{V_i^PV_k^P}}{l^2}\\
&\overline{\frac{\partial V_i^N}{\partial x_a}\frac{\partial V_k^N}{\partial x_a}}=c_2\frac{\overline{V_i^NV_k^N}}{l^2}\\
&\overline{\frac{\partial V_i'}{\partial x_a}\frac{\partial V_k'}{\partial x_a}}=c_1\frac{\overline{V_i^PV_k^P}}{l^2}+c_2\frac{\overline{V_i^NV_k^N}}{l^2}
\end{aligned}\right\}\qquad(5.6)$$

其次，我们把(5.3)和(5.5)式中的扩散都归併在一起。由于在湍流运动中，大涡旋衰减时间很长，通常的状态远远没有达到准平衡、准定常和小偏离的情况，所以不能应用不可逆热力学中的梯度形式的广义流和广义力之间的关系式。我们就假设

$$\left.\begin{aligned}
&\frac{\partial}{\partial x_j}\overline{V_j'V_i^PV_k^P}+\frac{1}{\rho}\left(\overline{\frac{\partial p^P}{\partial x_i}V_k^P}+\overline{\frac{\partial p^P}{\partial x_k}V_i^P}\right)\\
&\quad=-\sqrt{\overline{V_a^PV_a^P}}\,\alpha_j\frac{\partial}{\partial x_j}\overline{V_i^PV_k^P}-\phi^P_{ik}\\
&\phi^P_{ik}=\frac{1}{\rho}\overline{p^P\left(\frac{\partial V_i^P}{\partial x_k}+\frac{\partial V_k^P}{\partial x_i}\right)}\\
&\frac{\partial}{\partial x_j}\overline{V_j'V_i^NV_k^N}+\frac{1}{\rho}\left(\overline{\frac{\partial p^N}{\partial x_i}V_k^N}+\overline{\frac{\partial p^N}{\partial x_k}V_i^N}\right)\\
&\quad=-\sqrt{\overline{V_a^NV_a^N}}\,\beta_j\frac{\partial}{\partial x_j}\overline{V_i^NV_k^N}-\sqrt{\overline{V_a^PV_a^P}}\,c_j\frac{\partial}{\partial x_j}\overline{V_i^PV_k^P}-\phi^N_{ik}\\
&\phi^N_{ik}=-A\frac{\sqrt{\overline{V_a^PV_a^P}}}{l}\left(\overline{V_i^PV_k^P}-\frac{1}{3}\delta_{ik}\overline{V_\beta^PV_\beta^P}\right)\\
&\qquad+B\left[\overline{V_j^PV_k^P}\frac{\partial\bar V_i}{\partial x_j}+\overline{V_j^PV_i^P}\frac{\partial\bar V_k}{\partial x_j}-\frac{2}{3}\delta_{ik}\overline{V_j^PV_a^P}\frac{\partial\bar V_a}{\partial x_j}\right]\\
&\qquad-A'\frac{\sqrt{\overline{V_a^NV_a^N}}}{l}\left(\overline{V_i^NV_k^N}-\frac{1}{3}\delta_{ik}\overline{V_\beta^NV_\beta^N}\right)\\
&\qquad+B'\left[\overline{V_j^NV_k^N}\frac{\partial\bar V_i}{\partial x_j}+\overline{V_j^NV_i^N}\frac{\partial\bar V_k}{\partial x_j}-\frac{2}{3}\delta_{ik}\overline{V_j^NV_a^N}\frac{\partial\bar V_a}{\partial x_j}\right]
\end{aligned}\right\}\qquad(5.7)$$

式中有关外来涡旋的表达式中出现局部产生涡旋的原因是：所谓外来涡旋有一部份就是邻近扩散过来的涡旋，它们的平均效果近似地可以用当地涡旋的物理量表示出来。式中α_i，c_i为$\overline{V_a^PV_a^P}$梯度方向的无量纲矢量，这意味着涡旋沿着与梯度方向相反的方向扩散，而扩散速度和$\sqrt{\overline{V_a^PV_a^P}}$成正比。在湍流场均匀的时候，扩散量为零。同样，$\beta_i$为$\overline{V_a^NV_a^N}$梯度方向的无量纲矢量。包含$c_j$和$\phi^N_{ik}$中包含脉动速度$V_i^P$的项是从周围扩散过来涡旋的影响。通过这些假定，我们就得到除l^2以外的完整的方程组

$$\begin{aligned}
&\frac{\partial}{\partial t}\overline{q^2}+\bar V_j\frac{\partial\overline{q^2}}{\partial x_j}-2\nu_T\frac{\partial\bar V_k}{\partial x_j}\frac{\partial\bar V_k}{\partial x_j}-2\nu_T\frac{\partial^2}{\partial x_j\partial x_k}(\bar V_j\bar V_k)\\
&\quad-\frac{2\nu_T}{l^2}\left(c_1\overline{V_k^PV_k^P}+c_2\overline{V_k^NV_k^N}\right)-2\nu_T\frac{\partial^2}{\partial x_j\partial x_k}\left(\overline{V_j^PV_k^P}+\overline{V_j^NV_k^N}\right)\\
&=-10\nu\frac{\overline{q^2}}{\lambda^2}+\nu\nabla^2\overline{q^2}+\frac{\partial}{\partial x_j}\left(c_s\nu_T\frac{\partial\overline{q^2}}{\partial x_j}\right)
\end{aligned}\qquad(5.8)$$

$$\frac{\partial}{\partial t}\left(\frac{q^2}{\lambda^2}\right)+\bar{V}_j\frac{\partial}{\partial x_j}\left(\frac{q^2}{\lambda^2}\right)+\frac{14}{5}G\frac{\nu_T}{\lambda^2}\frac{\partial \bar{V}_j}{\partial x_k}\frac{\partial \bar{V}_j}{\partial x_k}$$

$$+\frac{14}{5}G\frac{\nu_T}{\lambda^2}\frac{1}{l^2}\left(c_1\overline{V^P_kV^P_k}+c_2\overline{V^N_kV^N_k}\right)-\frac{14}{3\sqrt{3}}\frac{q^3}{\lambda^3}$$

$$=-\frac{2}{15}E\nu\frac{q^2}{\lambda^4}+\nu\nabla^2\left(\frac{q^2}{\lambda^2}\right)+\frac{\partial}{\partial x_j}\left[c_s\nu_T\frac{\partial}{\partial x_j}\left(\frac{q^2}{\lambda^2}\right)\right] \tag{5.9}$$

$$\frac{\partial}{\partial t}\overline{V^N_iV^N_k}+\bar{V}_j\frac{\partial}{\partial x_j}\overline{V^N_iV^N_k}+\overline{V^N_jV^N_k}\frac{\partial \bar{V}_i}{\partial x_j}+\overline{V^N_jV^N_i}\frac{\partial \bar{V}_k}{\partial x_j}$$

$$-\sqrt{\overline{V^N_\alpha V^N_\alpha}}\,\beta_j\frac{\partial}{\partial x_j}\overline{V^N_iV^N_k}-\sqrt{\overline{V^P_\alpha V^P_\alpha}}\,c_j\frac{\partial \overline{V^P_iV^P_k}}{\partial x_j}-\phi^N_{ik}$$

$$=(\nu_T+\nu)\nabla^2\overline{V^N_iV^N_k}-2(\nu_T+\nu)c_2\frac{\overline{V^N_iV^N_k}}{l^2} \tag{5.10}$$

$$\frac{\partial}{\partial t}\overline{V^P_iV^P_k}+\bar{V}_j\frac{\partial}{\partial x_j}\overline{V^P_iV^P_k}+\overline{V^P_jV^P_k}\frac{\partial \bar{V}_i}{\partial x_j}+\overline{V^P_jV^P_i}\frac{\partial \bar{V}_k}{\partial x_j}$$

$$-\sqrt{\overline{V^P_\alpha V^P_\alpha}}\,\alpha_j\frac{\partial}{\partial x_j}\overline{V^P_iV^P_k}-\phi^P_{ik}$$

$$=(\nu_T+\nu)\nabla^2\overline{V^P_iV^P_k}-2(\nu_T+\nu)c_1\frac{\overline{V^P_iV^P_k}}{l^2} \tag{5.11$'$}$$

现在我们来解决 l^2 的方程式问题。直到现在为止，我们还没有用到 V^P_i 是当地产生的大涡旋的脉动速度这个性质。我们现在来利用这一个性质。所谓当地产生的大涡旋，实际上就是由于流动不稳定性引起的非定常运动减去平均运动以后的剩余部份，所以从涡旋线本身来说，实际上是和平均运动分不开的。因此这种大涡旋的涡量除了方向有所不同以外，是和平均运动的涡量成正比例的，有时甚至是一样的。设平均流动的涡量为

$$\bar{\Omega}_{ik}=\frac{\partial \bar{V}_i}{\partial x_k}-\frac{\partial \bar{V}_k}{\partial x_i}$$

则当地产生的大涡旋的涡量为

$$\omega^P_{ik}=bA_{i\alpha}A_{k\beta}\bar{\Omega}_{\alpha\beta}$$

式中 b 为比例系数。$A_{i\alpha}$为某一个正交变换的变换矩阵。设这个大涡旋的脉动速度为 V^P_i，则

$$V^P_i=l\gamma_k\omega^P_{ik}=bl\gamma_kA_{i\alpha}A_{k\beta}\bar{\Omega}_{\alpha\beta}$$

式中γ_k为某一个无量纲的随机向量。这样

$$\overline{V^P_iV^P_k}=l^2b^2\overline{\gamma_jA_{i\alpha}A_{j\beta}\gamma_mA_{k\delta}A_{m\epsilon}}\,\bar{\Omega}_{\alpha\beta}\bar{\Omega}_{\delta\epsilon}=l^2B_{ik\alpha\beta\delta\epsilon}\bar{\Omega}_{\alpha\beta}\bar{\Omega}_{\delta\epsilon} \tag{5.12}$$

式中$B_{ik\alpha\beta\delta\epsilon}$为无量纲量，随着空间各点的位置的差异，边界形状的不同而有所不同。在相似理论成立的区域，则为无量纲常数。把 i 和 k 收缩，得到

$$\overline{V^P_kV^P_k}=l^2b^2\overline{\gamma_jA_{j\beta}\gamma_mA_{m\epsilon}}\,\bar{\Omega}_{\alpha\beta}\bar{\Omega}_{\alpha\epsilon}=l^2B_{\beta\epsilon}\bar{\Omega}_{\alpha\beta}\bar{\Omega}_{\alpha\epsilon} \tag{5.13}$$

其中$B_{\beta\epsilon}$为无量纲量。我们把(5.11)′缩併，就得到

$$\frac{\partial}{\partial t}\overline{V^P_kV^P_k}+\bar{V}_j\frac{\partial}{\partial x_j}\overline{V^P_kV^P_k}+2\overline{V^P_jV^P_k}\frac{\partial \bar{V}_k}{\partial x_j}-\sqrt{\overline{V^P_\alpha V^P_\alpha}}\,\alpha_j\frac{\partial}{\partial x_j}\overline{V^P_kV^P_k}$$

$$=(\nu_T+\nu)\nabla^2\overline{V^P_kV^P_k}-2(\nu_T+\nu)c_1\frac{\overline{V^P_kV^P_k}}{l^2} \tag{5.11}$$

把(5.12)和(5.13)代入(5.11)，我们就得到 l^2 所满足的方程式。这样(5.8)，(5.9)，(5.10)和(5.11)就组成了一个封闭的方程组。

六、结论和讨论

通过上面的处理，我们可以把一些定性的看法建立起一个封闭的模式理论体系。这个理论体系可以克服大涡旋的衰减时间长，广义流不能用广义力线性表示出来的困难，以及一系列由表达不确当所产生的矛盾现象。如平均速度梯度为零时，雷诺剪应力不为零；湍流粘性系数为负值等现象。

在上游或周围外来的涡旋可以略去的情况下，在湍流强产生区里，可以得到普通的混合长度理论。

$$\overline{V_1V_2}\approx V_1^pV_2^p=(B_{121212}+B_{122121}-B_{121221}-B_{122112})l^2\left(\frac{dU}{dy}\right)^2$$

而在接近均匀各向同性的区域里，我们有

$$\nu_T\approx q\lambda f(R_\lambda)$$

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A New Turbulence Model with the Separate Consideration of Large and Small Vortexes

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Abstract

Recently the k-ε model has been widely used, but it is a kind of gradient model. Because the life-time of turbulence vortexes is very long, in common flow problems the influence of up-stream vortexes must be important, and the vortexes are not in quasi-equilibrium. So the usefulness of the k-ε model and other gradient models is limited. In this paper, according to actual cases of the turbulence, the velocity fluctuations are separated into large and small vortexes, and the large vortexes consist of two parts, one comes from up-stream and around, the other is locally generated. Thus we get a turbulence model, which consists of three parts.