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合成展开法求解圆薄板大挠度问题

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摘　要

本文用合成展开摄动法，把外场解和内层解结合起来，求解圆薄板大挠度问题．本文把Hencky的薄膜解当作外场解的一级近似解，并求出了外场解的二级近似解．利用边界内层坐标，求得了相应的各级内层解，即边界层解．本文采用最大位移和板厚之比的倒数作为小参数，所得结果大大改进了1948年作者所得的结果．

一、引　论

圆薄板在均佈载荷下的大挠度问题是由卡门方程处理的．它们用无量纲量可以写成

$$S_t = S_r + 2x\frac{dS_r}{dx} \tag{1.1a}$$

$$\frac{d^2}{dx^2}(xS_r) + \frac{1}{2}\left(\frac{dW}{dx}\right)^2 = 0 \tag{1.1b}$$

$$\frac{3}{16}P + \frac{3}{4}(1-\nu^2)S_r\frac{dW}{dx} = \frac{1}{4}\frac{d^2}{dx^2}\left(x\frac{dW}{dx}\right) \tag{1.1c}$$

式中$W(x)$，P，$S_r(x)$，$S_t(x)$都是无量纲量：

$$W(x) = \frac{w}{h},\quad P = \frac{a^4 q}{h^4 E}(1-\nu^2) \tag{1.2a,b}$$

$$S_r(x) = \frac{a^2}{Eh^3}N_r,\quad S_t(x) = \frac{a^2}{Eh^3}N_t \tag{1.2c,d}$$

而且x为无量纲坐标．

$$x = \frac{r^2}{a^2}\qquad 0 \leqslant x \leqslant 1 \tag{1.3}$$

其中h，a为板厚和板的半径．q和E为板受的均佈载荷和板材的杨氏模量．ν为泊桑比．r为径向坐标．N_r，N_t分别为径向和切向薄膜内力，w为板在r处的垂直位移（即挠度）．见图1和2．

对于固定边界条件而言，有

$$W = \frac{dW}{dx} = 0,\quad x = 1 \tag{1.4a,b}$$

$$2x\frac{dS_r}{dx}+(1-\nu)S_r=0,\quad x=1 \tag{1.4c}$$

$$\frac{dW}{dx},\ S_r\ 在x=0处保持有限 \tag{1.4d,e}$$

钱伟长在1948年首先用参数

$$\tau=\left[\frac{PC}{2(1-\nu^2)}\right]^{\frac{1}{3}} \tag{1.5}$$

把（1.1a,b,c）展开为外场解和内层解求解圆薄板大挠度问题．在这个方法中，把 Hencky 的薄膜解当作外场解的一级近似解，并求出了外场解的二级近似解，通过合成展开，利用内层坐标

$$\beta=\tau(1-x) \tag{1.6}$$

求得了相应的各级内层（边界层）解．(1.5)中的P为（1.2b）式中的无量纲载荷．C为薄膜解中由边界径向位移u为零所决定的常数，它和泊桑比ν有关．数值见钱伟长（1948）的文章．这个工作是合成展开法的最早文献．此后 Bromberg（1956）Срубщик(1960) 重复了这个工作，用相同的方法得到了类似的结果．最近周焕文（1979）重新用合成展开法研究了这个问题，用待定参数法决定了较为正确的参数选择．同时，固定边界条件是由合成解满足的．从而克服了钱伟长（1948）等单纯用薄膜解（已经满足固定边界条件$W=0$）所引起的一些疑虑．如果薄膜外场解满足$W=0$的固定条件，则内层解必然导致有一项常数，它在无穷远处不等于零．当然这个常数在合成解中并不能肯定它是属于外场解还是内层解的．最后计算并无区别．

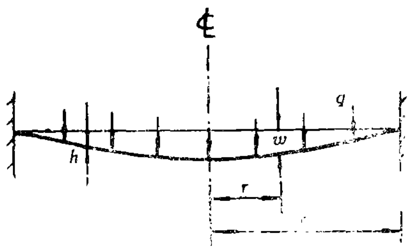

图1　圆板变形侧视图

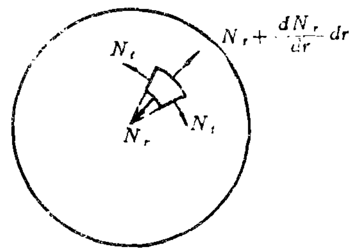

图2　圆板内的薄膜内力

本文将采用待定小参数ϵ通过钱伟长的合成展开法求本题的解．

二、圆薄板大挠度问题的外场解

我们采用任意小参数ϵ求解本题，引入新的待定量，设

$$P=\frac{1}{\epsilon^3}P^* \tag{2.1a}$$

$$W(x)=\frac{1}{\epsilon}W^*(x),\qquad S_r(x)=\frac{1}{\epsilon^2}S_r^*(x) \tag{2.1b,c}$$

（1.1b,c）可以写成：

$$\frac{3}{16}P^* + \frac{3}{4}(1-\nu^2)S_r^*\frac{dW^*}{dx} = \frac{1}{4}\epsilon^2\frac{d^2}{dx^2}\left(x\frac{dW^*}{dx}\right) \tag{2.2a}$$

$$\frac{d^2}{dx^2}(xS_r^*) + \frac{1}{2}\left(\frac{dW^*}{dx}\right)^2 = 0 \tag{2.2b}$$

边界条件化为

$$W^*=0,\quad \frac{dW^*}{dx}=0,\quad 2x\frac{dS_r^*}{dx}+(1-\nu)S_r^*=0 \qquad (在x=1) \tag{2.3a,b,c}$$

$$\frac{dW^*}{dx},\ S_r^*在x=0保持有限 \tag{2.3d,e}$$

现在求W^*，S_r^*的外场解，设

$$P^*=\alpha_0+\alpha_1\epsilon+\alpha_2\epsilon^2+\cdots \tag{2.4a}$$

$$W^*=W_0(x)+W_1(x)\epsilon+W_2(x)\epsilon^2+\cdots \tag{2.4b}$$

$$S_r^*=S_0(x)+S_1(x)\epsilon+S_2(x)\epsilon^2+\cdots \tag{2.4c}$$

代入(2.2a,b)，得各级递代方程：

(1) 零级方程

$$\frac{3}{16}\alpha_0+\frac{3}{4}(1-\nu^2)S_0\frac{dW_0}{dx}=0 \tag{2.5a}$$

$$\frac{d^2}{dx^2}(xS_0)+\frac{1}{2}\left(\frac{dW_0}{dx}\right)^2=0 \tag{2.5b}$$

(2) 一级方程

$$\frac{3}{16}\alpha_1+\frac{3}{4}(1-\nu^2)\left[S_1\frac{dW_0}{dx}+S_0\frac{dW_1}{dx}\right]=0 \tag{2.6a}$$

$$\frac{d^2}{dx^2}(xS_1)+\frac{dW_0}{dx}\frac{dW_1}{dx}=0 \tag{2.6b}$$

(3) 二级方程

$$\frac{3}{16}\alpha_2+\frac{3}{4}(1-\nu^2)\left[S_2\frac{dW_0}{dx}+S_1\frac{dW_1}{dx}+S_0\frac{dW_2}{dx}\right]=\frac{1}{4}\frac{d^2}{dx^2}\left(x\frac{dW_0}{dx}\right) \tag{2.7a}$$

$$\frac{d^2}{dx^2}(xS_2)+\frac{1}{2}\left(\frac{dW_1}{dx}\right)^2+\frac{dW_0}{dx}\frac{dW_2}{dx}=0 \tag{2.7b}$$

外场方程都是降阶的方程，我们不可能满足所有边界条件．(2.3d,e)中主要保持$W^{*\prime}(x=0)$，$S_r^*(x=0)$有限，除了在零级近似解中要求满足$W_0(1)=0$和$2S_0'(1)+(1-\nu)S_0(1)=0$外，其它各条件将和内层解合成在一起满足．

这里的ϵ是完全任意的，因此，我们可以把$W_0(0)$中的大小吸收进ϵ中去，设

$$W_0(0)=1 \tag{2.8}$$

这样假设，丝毫也影响不了最后结果．但在计算中，可以取得相当的方便．这里必须指出，这里和$W_0(0)=W_m$的规定是有区别的．这里只是把W的零级近似的最大值算作是$\frac{1}{\epsilon}$．

求 (2.5a,b) 在条件

$$\left.\begin{aligned}&W_0(1)=0,\ 2S_0'(1)+(1-\nu)S_0(1)=0\\&W_0'(0),S_0(0)\text{有限}\\&W_0(0)=1\end{aligned}\right\}\tag{2.9}$$

下的解.它和 Hencky 的薄膜解相同，即

$$W_0(x)=1-x\frac{g(cx)}{g(c)},\qquad \frac{dW_0}{dx}=-\frac{1}{g(c)}h(cx)\tag{2.10a,b}$$

$$S_0(x)=\frac{1}{2c}\frac{1}{g^2(c)}f(cx),\tag{2.10c}$$

$$\frac{3}{16}\alpha_0=\frac{3}{8}(1-\nu^2)\frac{1}{cg^3(c)}\tag{2.10d}$$

其中

$$f(x_1)=1-\frac{1}{2}x_1-\frac{1}{6}x_1^2-\frac{13}{144}x_1^3-\frac{17}{288}x_1^4-\frac{37}{864}x_1^5-\frac{1205}{36288}x_1^6$$

$$-\frac{218241}{8128512}x_1^7-\frac{6605069}{292626432}x_1^8+\cdots\tag{2.11a}$$

$$h(x_1)=[f(x_1)]^{-1}=1+\frac{1}{2}x_1+\frac{5}{12}x_1^2+\frac{55}{144}x_1^3+\frac{35}{96}x_1^4+\frac{205}{576}x_1^5$$

$$+\frac{17051}{48384}x_1^6+\frac{2863485}{8128512}x_1^7+\frac{103798265}{292626432}x_1^8+\cdots\tag{2.11b}$$

$$g(x_1)=\frac{1}{x_1}\int_0^{x_1}h(x_1)\,dx_1=1+\frac{1}{4}x_1+\frac{5}{36}x_1^2+\frac{55}{576}x_1^3+\frac{7}{96}x_1^4+\frac{205}{3456}x_1^5$$

$$+\frac{17051}{338688}x_1^6+\frac{2863485}{65028096}x_1^7+\frac{103798265}{2633637888}x_1^8+\cdots\tag{2.11c}$$

c 是由方程

$$2cf'(c)+(1-\nu)f(c)=0\tag{2.12}$$

决定的，所以是ν的函数，见钱伟长（1948）的结果，当$\nu=0.3$时，$c=0.390$.

现在求一级方程（2.6a,b）的解，我们暂时只要求它满足下列条件

$$W_1'(0),\ S_1(0)\quad\text{有限}\tag{2.13}$$

把（2.10）的零级解，代入（2.6a,b），得

$$\frac{dW_1}{dx}=-\frac{1}{2}\frac{\alpha_1c}{1-\nu^2}g^2(c)h(cx)+2cg(c)h^2(cx)S_1\tag{2.14a}$$

$$\frac{d^2}{dx^2}(xS_1)-\frac{1}{g(c)}h(cx)\frac{dW_1}{dx}=0\tag{2.14b}$$

把（2.14a）代入(2.14b)，消去$\frac{dW_1}{dx}$，得

$$\frac{d^2}{dx^2}(xS_1)-2ch^3(cx)S_1=-\frac{1}{2}\frac{\alpha_1c}{1-\nu^2}g(c)h^2(cx)\tag{2.15}$$

其解在$S_1(0)$有限的条件下，可以写成

$$S_1(x)=Bf_1(cx)+\frac{1}{2}\frac{\alpha_1}{1-\nu^2}g(c)f_2(cx) \tag{2.16}$$

其中B为待定常数，$f_1(cx)$为（2.15）的齐次解，（2.16）中的第二项为非齐次解，α_1也尚未决定。

$$f_1(x_1)=1+x_1+\frac{5}{6}x_1^2+\frac{13}{18}x_1^3+\frac{187}{288}x_1^4+\frac{259}{432}x_1^5+\frac{20485}{36288}x_1^6+\frac{1096205}{2032128}x_1^7+\frac{152580587}{292626432}x_1^8+\cdots \tag{2.17a}$$

$$f_2(x_1)=1+\frac{1}{2}x_1+\frac{1}{2}x_1^2+\frac{65}{144}x_1^3+\frac{119}{288}x_1^4+\frac{333}{864}x_1^5+\frac{13255}{36288}x_1^6+\frac{2850133}{8128512}x_1^7+\frac{99528955}{292626432}x_1^8+\cdots \tag{2.17b}$$

把（2.16）代入(2.14a)，得

$$\frac{dW_1}{dx}=\frac{\alpha_1 c}{2(1-\nu^2)}g^2(c)h_2(cx)+2Bcg(c)h_1(cx) \tag{2.18}$$

其中

$$h_2(cx)=2h^2(cx)f_2(cx)-h(cx) \tag{2.19a}$$

$$h_1(cx)=h^2(cx)f_1(cx) \tag{2.19b}$$

而

$$h_1(x_1)=1+2x_1+\frac{35}{12}x_1^2+\frac{275}{72}x_1^3+\frac{455}{36}x_1^4+\frac{205}{96}x_1^5+\frac{323969}{48384}x_1^6+\frac{31508335}{4064256}x_1^7+\frac{2596340625}{292626432}x_1^8+\cdots \tag{2.20a}$$

$$h_2(x_1)=1+\frac{5}{2}x_1+\frac{15}{4}x_1^2+\frac{715}{144}x_1^3+\frac{595}{96}x_1^4+\frac{4305}{576}x_1^5+\frac{426275}{48384}x_1^6+\frac{83067065}{8128512}x_1^7+\frac{3427144585}{292626432}x_1^8+\cdots \tag{2.20b}$$

把（2.18）积分，得

$$W_1(x)=-\frac{\alpha_1 c}{2(1-\nu^2)}g^2(c)[g_2(c)-xg_2(cx)]-2Bcg(c)[g_1(c)-xg_1(cx)]+R \tag{2.21}$$

其中R为一待定积分常数，$g_1(x)$，$g_2(x)$为

$$g_1(x_1)=1+x_1+\frac{35}{36}x_1^2+\frac{275}{288}x_1^3+\frac{91}{96}x_1^4+\frac{205}{216}x_1^5+\frac{323969}{338688}x_1^6+\frac{31508335}{32514048}x_1^7+\frac{2596340625}{2633637888}x_1^8+\cdots \tag{2.22a}$$

$$g_2(x_1)=1+\frac{5}{4}x_1+\frac{5}{4}x_1^2+\frac{715}{576}x_1^3+\frac{119}{96}x_1^4+\frac{1435}{1152}x_1^5+\frac{426275}{338688}x_1^6$$

$$+\frac{83067065}{65028096}x_1^7+\frac{3427144585}{2633637888}x_1^8+\cdots \tag{2.22b}$$

现在让我们积分二级方程(2.7a,b)，把(2.10),(2.16),(2.18)代入(2.7a,b)，从(2.7a)中解出$\frac{dW_2}{dx}$，得

$$\begin{aligned}\frac{dW_2}{dx}&=2cg(c)h^2(cx)S_2-\frac{\alpha_2 c}{2(1-\nu^2)}g^2(c)h(cx)\\&-4B^2c^2g^3(c)f_1(cx)h_1(cx)\,h(cx)-\frac{\alpha_1c^2B}{1-\nu^2}g^4(c)f_1(cx)h_2(cx)h(cx)\\&-2\frac{\alpha_1c^2B}{1-\nu^2}g^4(c)f_2(cx)h_1(cx)h(cx)-\frac{\alpha_1^2c^2}{2(1-\nu^2)^2}g^5(c)f_2(cx)h_2(cx)h(cx)\\&+\frac{2c}{3(1-\nu^2)}g^2(c)h(cx)\frac{d^2}{dx^2}\left(x\frac{dW_0}{dx}\right)\end{aligned} \tag{2.23}$$

把 (2.23) 代入(2.7b)，得决定S_2的方程：

$$\begin{aligned}&\frac{d^2}{dx^2}(xS_2)-2ch^3(cx)S_2\\&=-\frac{\alpha_1^2c^2}{8(1-\nu^2)^2}g^4(c)[h_2^2(cx)+4f_2(cx)h_2(cx)h^2(cx)]\\&-\frac{Bc^2\alpha_1}{1-\nu^2}g^3(c)[h_1(cx)h_2(cx)+f_1(cx)h_2(cx)h^2(cx)\\&+2h_1(cx)f_2(cx)h^2(cx)]\\&-2B^2c^2g^2(c)[h_1^2(cx)+2h_1(cx)f_1(cx)h^2(cx)]\\&-\frac{\alpha_2c}{2(1-\nu^2)}g(c)h^2(cx)+\frac{2c}{3(1-\nu^2)}g(c)h^2(cx)\frac{d^2}{dx^2}\left(x\frac{dW_0}{dx}\right)\end{aligned} \tag{2.24}$$

在利用了 (2.19a,b) 以后，上式可以简化为

$$\begin{aligned}&\frac{d^2}{dx^2}(xS_2)-2ch^3(cx)S_2\\&=-\frac{\alpha_1^2c^2}{8(1-\nu^2)^2}g^4(c)[3h_2(cx)+2h(cx)]h_2(cx)\\&-Bc^2\frac{\alpha_1}{1-\nu^2}g^3(c)[3h_2(cx)+h(cx)]h_1(cx)\\&-6B^2c^2g^2(c)h_1^2(cx)-\frac{\alpha_2c}{2(1-\nu^2)}g(c)h^2(cx)\\&-\frac{2c^2}{3(1-\nu^2)}h^2(cx)\left\{cx\frac{d^2h(cx)}{d(cx)^2}+2\frac{dh(cx)}{d(cx)}\right\}\end{aligned} \tag{2.25}$$

其级数解为

$$S_2=B^*f_1(cx)+\frac{\alpha_2}{2(1-\nu^2)}g(c)f_2(cx)+6B^2cg^2(c)f_{22}(cx)$$

$$+4B\frac{\alpha_1 c}{1-\nu^2}g^3(c)f_{23}(cx)+\frac{5}{8}\frac{\alpha_1^2 c}{(1-\nu^2)^2}g^4(c)f_{24}(cx)$$

$$+\frac{2c}{3(1-\nu^2)}f_{20}(cx) \tag{2.26}$$

其中$f_1(cx)$，$f_2(cx)$见(4.17a,b)，B^*为又一积分常数，$f_{20}(cx)$，$f_{22}(cx)$，$f_{23}(cx)$，$f_{24}(cx)$分别为

$$f_{20}(x_1)=1+\frac{1}{2}x_1+\frac{1}{12}x_1^2-\frac{5}{24}x_1^3-\frac{4}{9}x_1^4-\frac{5669}{8640}x_1^5-\frac{89003}{103680}x_1^6$$

$$-\frac{102527}{96768}x_1^7-\cdots \tag{2.27a}$$

$$f_{22}(x_1)=1+\frac{1}{2}x_1-\frac{13}{36}x_1^3-\frac{187}{288}x_1^4-\frac{777}{864}x_1^4-\frac{20485}{18144}x_1^6-\frac{5481025}{4064256}x_1^7$$

$$-\frac{457751761}{292626432}x_1^8-\cdots \tag{2.27b}$$

$$f_{23}(x_1)=1+\frac{1}{2}x_1-\frac{13}{36}x_1^3-\frac{187}{288}x_1^4-\frac{777}{864}x_1^5-\frac{20485}{18144}x_1^6-\frac{5481025}{4064256}x_1^7$$

$$-\frac{457751511}{292626432}x_1^8-\cdots \tag{2.27c}$$

$$f_{24}(x_1)=1+\frac{1}{2}x_1-\frac{1}{30}x_1^2-\frac{299}{720}x_1^3-\frac{1037}{1440}x_1^4-\frac{851}{864}x_1^5-\frac{44585}{36288}x_1^6$$

$$-\frac{59414311}{40642560}x_1^7-\frac{824836579}{485550720}x_1^8-\cdots \tag{2.27d}$$

把(2.26)代入(2.23)，得

$$\frac{dW_2}{dx}=B^*\cdot 2cg(c)h_1(cx)+\alpha_2\cdot\frac{c}{2(1-\nu^2)}g^2(c)h_2(cx)$$

$$+8B^2c^2g^3(c)h_{22}(cx)+5Bc^2\frac{\alpha_1}{1-\nu^2}g^4(c)h_{23}(cx)$$

$$+\frac{3}{4}\frac{\alpha_1^2 c^2}{(1-\nu^2)^2}g^5(c)h_{24}(cx)+\frac{2c^2}{3(1-\nu^2)}g(c)h_{20}(cx) \tag{2.28}$$

其中

$$h_{20}(x_1)=2h^2(x_1)f_{20}(x_1)-h(x_1)\left\{x_1\frac{d^2h(x_1)}{dx_1^2}+2\frac{dh(x_1)}{dx_1}\right\} \tag{2.29a}$$

$$h_{22}(x_1)=\frac{3}{2}h^2(x_1)f_{22}(x_1)-\frac{1}{2}f_1(x_1)h_1(x_1)h(x_1) \tag{2.29b}$$

$$h_{23}(x_1)=\frac{8}{5}h^2(x_1)f_{23}(x_1)-\frac{1}{5}f_1(x_1)h_2(x_1)h(x_1)-\frac{2}{5}f_2(x_1)h_1(x_1)h(x_1) \tag{2.29c}$$

$$h_{24}(x_1)=\frac{5}{3}h^2(x_1)f_{24}(x_1)-\frac{2}{3}f_2(x_1)h_2(x_1)h(x_1) \tag{2.29d}$$

把(2.27)，(2.20)，(2.17)，(2.11) 代入上式，得

$$h_{20}(x_1)=1-\frac{35}{12}x_1^2-\frac{1125}{144}x_1^3-\frac{4299}{288}x_1^4-\frac{424701}{17280}x_1^5-\frac{26947059}{725760}x_1^6-\frac{2154318295}{40642560}x_1^7-\cdots \tag{2.30a}$$

$$h_{22}(x_1)=1+\frac{1}{2}x_1-\frac{35}{24}x_1^2-\frac{1375}{288}x_1^3-\frac{455}{48}x_1^4-\frac{2255}{144}x_1^5-\frac{2267783}{96768}x_1^6-\frac{256108657}{3612672}x_1^7-\cdots \tag{2.30b}$$

$$h_{23}(x_1)=1+\frac{2}{5}x_1-\frac{7}{4}x_1^2-\frac{385}{72}x_1^3-\frac{1001}{96}x_1^4-\frac{205}{12}x_1^5-\frac{6155411}{241920}x_1^6-\frac{132340207}{4064256}x_1^7-\cdots \tag{2.30c}$$

$$h_{24}(x_1)=1+\frac{1}{6}x_1-\frac{85}{36}x_1^2-\frac{935}{144}x_1^3-\frac{3535}{288}x_1^4-\frac{34235}{1728}x_1^5-\frac{4245699}{145152}x_1^6-\frac{993978295}{24385536}x_1^7-\cdots \tag{2.30d}$$

(2.28) 可以积分，得

$$W_2=-B^*\cdot 2cg(c)[g_1(c)-xg_1(cx)]-\alpha_2\cdot\frac{c}{2(1-\nu^2)}g^2(c)[g_2(c)-xg_2(cx)]-8B^2c^2g^3(c)[g_{22}(c)-xg_{22}(cx)]-5Bc^2\frac{\alpha_1}{1-\nu^2}g^4(c)[g_{23}(c)-xg_{23}(cx)]-\frac{3}{4}\frac{\alpha_1^2c^2}{(1-\nu^2)^2}g^5(c)[g_{24}(c)-xg_{24}(cx)]-\frac{2c^2}{3(1-\nu^2)}g(c)[g_{20}(c)-xg_{20}(cx)]+R_2 \tag{2.31}$$

其中R_2为又一待定常数。$g_1(cx)$，$g_2(cx)$ 见 (2.22a,b)，$g_{20}(cx)$，$g_{22}(cx)$，$g_{23}(cx)$，$g_{24}(cx)$ 分别为

$$g_{20}(x_1)=1-\frac{35}{36}x_1^2-\frac{1125}{576}x_1^3-\frac{4299}{1440}x_1^4-\frac{141567}{34560}x_1^5-\frac{26947059}{5080320}x_1^6-\frac{2154318295}{325140480}x_1^7-\cdots \tag{2.32a}$$

$$g_{22}(x_1)=1+\frac{1}{4}x_1-\frac{35}{72}x_1^2-\frac{1375}{1152}x_1^3-\frac{91}{48}x_1^4-\frac{2255}{864}x_1^5-\frac{323969}{96768}x_1^6-\frac{256108657}{28901376}x_1^7-\cdots \tag{2.32b}$$

$$g_{23}(x_1)=1+\frac{1}{5}x_1-\frac{7}{12}x_1^2-\frac{385}{288}x_1^3-\frac{1001}{480}x_1^4-\frac{205}{72}x_1^5-\frac{6155411}{1693440}x_1^6$$

$$-\frac{132340207}{32514048}x_1^7-\cdots \tag{2.32c}$$

$$g_{24}(x_1)=1+\frac{1}{12}x_1-\frac{85}{108}x_1^2-\frac{935}{576}x_1^3-\frac{707}{288}x_1^4-\frac{34235}{10368}x_1^5-\frac{4245699}{1016064}x_1^6$$

$$-\frac{993978295}{195084288}x_1^7-\cdots \tag{2.32d}$$

现在我们有了S_0，W_0，S_1，W_1，S_2，W_2的三级外场解。除了满足$W_k'(0)$，$S_k'(0)$有限外，尚满足$W_0(1)=0$，和$2S_0'(1)+(1-\nu)S_0(1)=0$，$W_0(0)=1$。解中尚有待定常数α_1，α_2，B, B^*，R，R_2。所有这些常数都将与内层解在一起，通过满足一切尚待满足的边界条件来决定。

三、圆薄板大挠度问题的内层解和合成展开解

在内层解中，我们采用内层坐标ξ

$$\xi=\frac{1}{\epsilon}(1-x)\quad 或x=1-\epsilon\xi \tag{3.1}$$

其中

$$0\leqslant\xi\leqslant\frac{1}{\epsilon}\to\infty,\ 当\epsilon\to 0 \tag{3.2}$$

设P^*，W^*，S_r^*可以用合成解展开为ϵ的幂级数。设

$$P^*=\alpha_0+\alpha_1\epsilon+\alpha_2\epsilon^2+\cdots \tag{3.3a}$$

$$W^*=W_0(x)+[W_1(x)+\Omega_0(\xi)]\epsilon+[W_2(x)+\Omega_1(\xi)]\epsilon^2$$
$$+[W_3(x)+\Omega_2(\xi)]\epsilon^3+\cdots \tag{3.3b}$$

$$\frac{dW^*}{dx}=W_0'(x)-\Omega_0'(\xi)+[W_1'(x)-\Omega_1'(\xi)]\epsilon$$
$$+[W_2'(x)-\Omega_2'(\xi)]\epsilon^2+\cdots \tag{3.3c}$$

$$S_r^*=S_0(x)+[S_1(x)+\sigma_1(\xi)]\epsilon+[S_2(x)+\sigma_2(\xi)]\epsilon^2+\cdots \tag{3.3d}$$

其中$W_0(x)$，$W_1(x)$，$W_2(x)$，$S_0(x)$，$S_1(x)$，$S_2(x)$都是已知的外场解。$\Omega_k(\xi)$，$\sigma_k(\xi)$都是待定的内层解。$W_k(x)$，$S_k(x)$应该满足外场解的所有方程（2.5），（2.6），（2.7）等。

把（3.3）代入（2.2a,b），并用（2.5），（2.6），（2.7）消去一些项，得

$$-\frac{3}{4}(1-\nu^2)\{[S_0(x)+S_1(x)\epsilon+S_2(x)\epsilon^2+\cdots][\Omega_0'(\xi)+\Omega_1'(\xi)\epsilon+\Omega_2'(\xi)\epsilon^2+\cdots]\}$$

$$+\frac{3}{4}(1-\nu^2)\epsilon\{[\sigma_1(\xi)+\sigma_2(\xi)\epsilon+\cdots][W_0'(x)+W_1'(x)\epsilon+W_2'(x)\epsilon^2+\cdots]\}$$

$$-\frac{3}{4}(1-\nu^2)\epsilon\{[\sigma_1(\xi)+\sigma_2(\xi)\epsilon+\cdots][\Omega_0'(\xi)+\Omega_1'(\xi)\epsilon+\Omega_2'(\xi)\epsilon^2+\cdots]\}$$

$$=-\frac{1}{4}\{\Omega_0'''(\xi)+\Omega_1'''(\xi)\epsilon+\Omega_2'''(\xi)\epsilon^2+\cdots\}$$

$$+\frac{1}{4}\epsilon\left\{\frac{d^2}{d\xi^2}\xi\Omega_0'(\xi)+\epsilon\frac{d^2}{d\xi^2}\xi\Omega_1'(\xi)+\epsilon^2\frac{d^2}{d\xi^2}\xi\Omega_2'(\xi)+\cdots\right\} \tag{3.4a}$$

$$\left\{\frac{d^2\sigma_1}{d\xi^2}+\frac{d^2\sigma_2}{d\xi^2}\epsilon+\cdots\right\}-\epsilon\left\{\frac{d^2}{d\xi^2}(\xi\sigma_1)+\epsilon\frac{d^2}{d\xi^2}(\epsilon\sigma_2)+\cdots\right\}$$

$$-\epsilon[W_0'(x)+W_1'(x)\epsilon+W_2'(x)\epsilon^2+\cdots][\Omega_0'(\xi)+\Omega_1'(\xi)\epsilon+\Omega_2'(\xi)\epsilon^2+\cdots]$$

$$+\frac{1}{2}\epsilon\{\Omega_0'(\xi)+\Omega_1'(\xi)\epsilon+\Omega_2'(\xi)\epsilon^2+\cdots\}^2=0 \tag{3.4b}$$

我们还可以把$W_k'(x)$，$S_k(x)$展开为ϵ的幂级数

$$W_k'(x)=W_k'(1-\epsilon\xi)=W_k'(1)-W_k''(1)\xi\epsilon+\frac{1}{2}W_k'''(1)\xi^2\epsilon^2-\frac{1}{6}W_k''''(1)\xi^3\epsilon^3+\cdots \tag{3.5a}$$

$$S_k(x)=S_k(1-\epsilon\xi)=S_k(1)-S_k'(1)\xi\epsilon+\frac{1}{2}S_k''(1)\xi^2\epsilon^2-\frac{1}{6}S_k'''(1)\xi^3\epsilon^3+\cdots \tag{3.5b}$$

把（3.5a,b）代入（3.4a,b），按ϵ^k的次序重新排列，得各级摄动方程：

（1） 零级摄动方程：

$$\epsilon^0 \qquad -\frac{3}{4}(1-\nu^2)S_0(1)\Omega_0'(\xi)=-\frac{1}{4}\Omega_0'''(\xi) \tag{3.6a}$$

$$\epsilon^0 \qquad \sigma_1''(\xi)=0 \tag{3.6b}$$

（2） 一级摄动方程：

$$\epsilon \qquad -\frac{3}{4}(1-\nu^2)\{S_0(1)\Omega_1'(\xi)+[S_1(1)-S_0'(1)\xi+\sigma_1(\xi)]-W_0'(1)\sigma_1(\xi)\}$$

$$=-\frac{1}{4}\Omega_1'''(\xi)+\frac{1}{4}\frac{d^2}{d\xi^2}[\xi\Omega_0'(\xi)] \tag{3.7a}$$

$$\epsilon \qquad \sigma_2''(\xi)-\frac{d^2}{d\xi^2}[\xi\sigma_1(\xi)]+\frac{1}{2}[\Omega_0'(\xi)]^2-W_0'(1)\Omega_0'(\xi)=0 \tag{3.7b}$$

（3） 二级摄动方程

$$\epsilon^2 \qquad -\frac{3}{4}(1-\nu^2)\Big\{[S_2(1)-S_1'(1)\xi+\frac{1}{2}S_0''(1)\xi^2]\Omega_0'(\xi)$$

$$+[S_1(1)-S_0'(1)\xi]\Omega_1'(\xi)+S_0(1)\Omega_2'(\xi)-W_0'(1)\sigma_2(\xi)$$

$$+\sigma_1(\xi)\Omega_1'(\xi)+\sigma_2(\xi)\Omega_0'(\xi)-[W_1'(1)-W_0''(1)\xi]\sigma_1(\xi)\Big\}$$

$$=-\frac{1}{4}\Omega_2'''(\xi)+\frac{1}{4}\frac{d^2}{d\xi^2}[\xi\Omega_1'(\xi)] \tag{3.8a}$$

$$\epsilon^2 \qquad \frac{d^2}{d\xi^2}\sigma_2(\xi)-\frac{d^2}{d\xi^2}[\xi\sigma_2(\xi)]-W_0'(1)\Omega_1'(\xi)-[W_1'(1)-W_0''(1)\xi]\Omega_0'(\xi)$$

$$+\Omega_0'(\xi)\Omega_1'(\xi)=0 \tag{3.8b}$$

求解这些内层解的条件为

（1） $\xi\to\infty$的条件．设内层解在外场内逐步衰减．于是有

$$\Omega_k(\infty)=0,\quad \sigma_k(\infty)=0 \tag{3.9}$$

(2) 在边界上，应满足有关挠度的条件和有关径向位移固定的条件

$$-\Omega_0'(0)+W_0'(1)=0,\ W_1(1)+\Omega_0(0)=0,\ \sigma_1'(0)=0 \tag{3.10a,b,c}$$

$$\left.\begin{aligned}&-\Omega_1'(0)+W_1'(1)=0,\ W_2(1)+\Omega_1(0)=0,\\&2S_1'(1)+(1-\nu)S_1(1)-2\sigma_2'(0)+(1-\nu)\sigma_1(0)=0\end{aligned}\right\} \tag{3.11a,b,c}$$

$$\left.\begin{aligned}&-\Omega_2'(0)+W_2'(1)=0,\ W_3(1)+\Omega_2(0)=0,\\&2S_2'(1)+(1-\nu)S_2(1)-2\sigma_3'(0)+(1-\nu)\sigma_2(0)=0\end{aligned}\right\} \tag{3.12a,b,c}$$

现在让我们积分内层方程(3.6a,b)，根据(2.10c)，有

$$S_0(1)=\frac{1}{2c}\frac{1}{g^2(c)}f(c) \tag{3.13}$$

引进

$$\lambda^2(c)=3(1-\nu^2)S_0(1)=\frac{3}{2c}(1-\nu^2)f(c)g^{-2}(c) \tag{3.14}$$

(3.6a)可以写成

$$\Omega_0''(\xi)-\lambda^2(c)\Omega_0'(\xi)=0,\ \sigma_1''(\xi)=0 \tag{3.15}$$

根据条件(3.9)，(3.10a,b,c)，而且从(2.10)，(2.21)有

$$W_1(1)=R,\quad W_0'(1)=-\frac{h(c)}{g(c)} \tag{3.16}$$

(3.15)的解为

$$\Omega_0(\xi)=\frac{h(c)}{\lambda g(c)}e^{-\lambda\xi},\ \sigma_1(\xi)=0 \tag{3.17}$$

而且从 (3.10b) 定出了R.

$$R=-\frac{h(c)}{\lambda g(c)} \tag{3.18}$$

一级摄动方程(3.7)可以写为

$$\Omega_1''(\xi)-\lambda^2\Omega_1'(\xi)=[N(c)\xi+M(c)]e^{-\lambda\xi} \tag{3.19a}$$

$$\sigma_2''(\xi)=\frac{h^2(c)}{g^2(c)}\left[e^{-\lambda\xi}-\frac{1}{2}e^{-2\lambda\xi}\right] \tag{3.19b}$$

其中

$$N(c)=\frac{h(c)}{g^3(c)}\left[\frac{3}{2}(1-\nu^2)f'(c)-\lambda^2g^2(c)\right] \tag{3.20a}$$

$$M(c)=-\frac{h(c)}{g(c)}\left\{-2\lambda+3(1-\nu^2)\left[Bf_1(c)+\frac{\alpha_1}{2(1-\nu^2)}g(c)f_2(c)\right]\right\} \tag{3.20b}$$

有关边界条件为(3.11a,b,c)，(3.9).根据前面结果，我们有

$$\left.\begin{aligned}&W_1'(1)=2Bcg(c)h_1(c)+\frac{\alpha_1c}{2(1-\nu^2)}g^2(c)h_2(c),\ W_2(1)=R_2\\&S_1'(1)=cBf_1'(c)+\frac{\alpha_1c}{2(1-\nu^2)}g(c)f_2'(c),\ S_1(1)=Bf_1(c)+\frac{\alpha_1}{2(1-\nu^2)}g(c)f_2(c)\end{aligned}\right\} \tag{3.21}$$

(3.19a,b)的解可以写成

$$\sigma_2(\xi)=\frac{h^2(c)}{\lambda^2 g^2(c)}\left\{e^{-\lambda\xi}-\frac{1}{8}e^{-2\lambda\xi}\right\} \tag{3.22a}$$

$$\Omega_1(\xi)=\left\{\frac{N(c)}{4\lambda^2}\xi^2+\frac{1}{4\lambda^3}[2M(c)\lambda+3N(c)]\xi-R_2\right\}e^{-\lambda\xi} \tag{3.22b}$$

其中(3.9)，(3.11b)都已满足.(3.11a,c)给出下列B和α_1的关系式

$$B\{2cf_1'(c)+(1-\nu)f_1(c)\}+\frac{\alpha_1}{1-\nu^2}g(c)\left\{cf_2'(c)+\frac{1}{2}(1-\nu)f_2(c)\right\}$$

$$=-\frac{3}{2}\frac{h^2(c)}{g^2(c)}\frac{1}{\lambda} \tag{3.23a}$$

$$Bcg(c)[2h_1(c)+f_1(c)h^2(c)]+\frac{\alpha_1}{2(1-\nu^2)}cg^2(c)[h_2(c)+h^2(c)f_2(c)]$$

$$=\lambda R_2+\frac{3h(c)}{4\lambda g(c)}\left[\frac{1}{3}+ch(c)f'(c)\right] \tag{3.23b}$$

这两个方程共有三个待定常数B，α_1，R_2，为了决定这三个常数，尚缺一个条件，我们将从ϵ的定义的基础上来寻求这个条件.

从(2.1),(3.3)有

$$P=\epsilon^{-3}(\alpha_0+\alpha_1\epsilon+\alpha_2\epsilon^2+\cdots) \tag{3.24a}$$

$$W_m=\epsilon^{-1}[1+W_1(0)\epsilon+W_2(0)\epsilon^2+W_3(0)\epsilon^3+\cdots] \tag{3.24b}$$

其中我们业已利用了（2.8）式，即$W_0(0)=1$；W_m为中心挠度.如果从（3.24a,b）中消去ϵ，即得P用W_m^{-1}级数表达的表达式.从（3.24b）中解ϵ，用W_m^{-1}的级数表示，即得

$$\epsilon=W_m^{-1}+W_1(0)W_m^{-2}+[W_1^2(0)+W_2(0)]W_m^{-3}+[W_1^3(0)+2W_1(0)W_2(0)$$
$$+W_3(0)]W_m^{-4}+O(W_m^{-5}) \tag{3.25}$$

把（3.25）代入(3.24a)，展开为W_m^{-1}的级数，即得

$$\begin{aligned}P=W_m^3\{&\alpha_0+[\alpha_1-3W_1(0)\alpha_0]W_m^{-1}\\&+[\alpha_2-2W_1(0)\alpha_1+3(W_1^2(0)-W_2(0))\alpha_0]W_m^{-2}\\&+[\alpha_3-W_1(0)\alpha_2+(W_1^2(0)-2W_2(0))\alpha_1\\&-(W_1^3(0)-6W_1(0)W_2(0)+3W_3(0))\alpha_2]W_m^{-3}\}\\&+O(W_m^{-1})\end{aligned} \tag{3.26}$$

由于正的载荷和负的载荷产生的位移是反对称的，所以（3.26）式右侧的W_m^{-1}级数项一定是奇次项，其偶次项的系数一定恒等于零.于是，我们有下列各级的补充条件，即

$$\alpha_1-3W_1(0)\alpha_0=0 \tag{3.27a}$$

$$\alpha_3-W_1(0)\alpha_2+[W_1^2(0)-2W_2(0)]\alpha_1$$
$$-[W_1^3(0)-6W_1(0)W_2(0)+3W_3(0)]\alpha_0=0 \tag{3.27b}$$

根据(2.10d)，(2.21)，(3.18)

$$\alpha_1+6(1-\nu^2)\frac{1}{cg^3(c)}\left\{\frac{h(c)}{\lambda g(c)}+\frac{\alpha_1 c}{2(1-\nu^2)}g^2(c)g_2(c)+2Bcg(c)g_1(c)\right\}=0 \tag{3.28}$$

在利用了（3.14）、(2.11b）以后，上式简化为

$$\alpha_1\left[1+3\frac{g_2(c)}{g(c)}\right]+B\cdot 8c\lambda^2h(c)g_1(c)+4\lambda\frac{h^2(c)}{g^2(c)}=0 \tag{3.29}$$

这是α_1和B的一个补充关系式。于是，把（3.23a,b）和（3.29）放在一起，得求解α_1, B, R_2的三个联立方程组，其解为

$$\alpha_1=-4\frac{\lambda}{\Delta}\frac{h^2(c)}{g^2(c)}[2cf_1'(c)+(1-\nu)f_1(c)]+12\frac{\lambda c}{\Delta}\frac{h^3(c)}{g^2(c)}g_1(c) \tag{3.30}$$

$$B=-\left[1+3\frac{g_2(c)}{g(c)}\right]\frac{3h^2(c)}{2\lambda\Delta g^2(c)}+\frac{4\lambda h^2(c)}{\Delta g(c)(1-\nu^2)}\left[cf_2'(c)+\frac{1}{2}(1-\nu)f_2(c)\right] \tag{3.31}$$

$$R_2=-\frac{3h(c)}{4\lambda^2 g(c)}\left[\frac{1}{3}+ch(c)f'(c)\right]+\frac{\alpha_1 cg^2(c)}{2\lambda(1-\nu^2)}[h_2(c)+h^2(c)f_2(c)]$$

$$+\frac{Bcg(c)}{\lambda}[2h_1(c)+f_1(c)h^2(c)] \tag{3.32}$$

其中Δ为

$$\Delta=\left[1+3\frac{g_2(c)}{g(c)}\right][2cf_1'(c)+(1-\nu)f_1(c)]$$

$$-\frac{8c\lambda^2}{1-\nu^2}h(c)g(c)g_1(c)\left[cf_2'(c)+\frac{1}{2}(1-\nu)f_2(c)\right] \tag{3.33}$$

现在让我们研究下一级近似的内层解。(3.8a,b)可以写成

$$\Omega_2'''(\xi)-\lambda^2\Omega_2'(\xi)=(Q_0+Q_1\xi+Q_2\xi^2+Q_3\xi^3)e^{-\lambda\xi}$$

$$-\frac{3}{8}(1-\nu^2)\frac{h^3(c)}{\lambda^2 g^3(c)}[9e^{-2\lambda\xi}-e^{-3\lambda\xi}] \tag{3.34}$$

$$\sigma_3''(\xi)=(N_0+N_1\xi+N_2\xi^2)e^{-\lambda\xi}+(M_0+M_1\xi+M_2\xi^2)e^{-2\lambda\xi} \tag{3.35}$$

其中

$$Q_0=-\frac{2}{\lambda^2}[\lambda M(c)+N(c)+\lambda^4R_2]+3(1-\nu^2)\frac{h^3(c)}{\lambda^2 g^3(c)}$$

$$+3(1-\nu^2)\left\{\frac{1}{4\lambda^3}S_1(1)[2\lambda M(c)+3N(c)+4\lambda^4R_2]-S_2(1)\frac{h(c)}{g(c)}\right\} \tag{3.36a}$$

$$Q_1=\frac{1}{4\lambda}[10\lambda M(c)+N(c)+4\lambda^4R_2]$$

$$-3(1-\nu^2)\left\{\frac{S_0'(1)}{4\lambda^3}[2\lambda M(c)+3N(c)+4\lambda^4R_2]\right.$$

$$\left.+\frac{S_1(1)}{4\lambda^2}[2\lambda M(c)+N(c)]-S_1'(1)\frac{h(c)}{g(c)}\right\} \tag{3.36b}$$

$$Q_2=-\frac{1}{4}[2\lambda M(c)-5N(c)]$$

$$-3(1-\nu^2)\left\{-\frac{S_0'(1)}{4\lambda^2}[2\lambda M(c)+N(c)]+\frac{S_1(1)}{4\lambda}N(c)+\frac{1}{2}S_0''(1)\frac{h(c)}{g(c)}\right\} \tag{3.36c}$$

$$Q_3=\frac{3}{4\lambda}(1-\nu^2)S_0'(1)N(c)-\frac{1}{4}\lambda N(c) \tag{3.36d}$$

$$N_0=-\frac{2}{\lambda}\frac{h^2(c)}{g^2(c)}-W_1'(1)\frac{h(c)}{g(c)}-\frac{1}{4\lambda^3}\frac{h(c)}{g(c)}[2M(c)\lambda+3N(c)+4\lambda^4R_2] \tag{3.37a}$$

$$N_1=\frac{h^2(c)}{g^2(c)}+W''_0(1)\frac{h(c)}{g(c)}+\frac{1}{4\lambda^2}[2M(c)\lambda+N(c)] \tag{3.37b}$$

$$N_2=\frac{1}{4\lambda}\frac{h(c)}{g(c)}N(c) \tag{3.37c}$$

$$M_0=\frac{1}{2\lambda}\frac{h^2(c)}{g^2(c)}+\frac{1}{4\lambda^3}\frac{h(c)}{g(c)}[2M(c)\lambda+3N(c)+4\lambda^4R_2] \tag{3.38a}$$

$$M_1=-\frac{h^2(c)}{2g^2(c)}-\frac{1}{4\lambda^2}\frac{h(c)}{g(c)}[2M(c)\lambda+N(c)] \tag{3.38b}$$

$$M_2=-\frac{1}{4\lambda}\frac{h(c)}{g(c)}N(c) \tag{3.38c}$$

(3.34)，(3.35)的积分为

$$\begin{aligned}\Omega_2(\xi)=&\left\{\frac{1}{8\lambda^2}Q_3\xi^4+\left[\frac{1}{6\lambda^2}Q_2+\frac{3}{4\lambda^3}Q_3\right]\xi^3+\left[\frac{1}{4\lambda^2}Q_1+\frac{3}{4\lambda^3}Q_2+\frac{21}{8\lambda^4}Q_3\right]\xi^2\right.\\&\left.+\left[\frac{1}{2\lambda^2}Q_0+\frac{3}{4\lambda^3}Q_1+\frac{7}{4\lambda^4}Q_2+\frac{45}{8\lambda^5}Q_3\right]\xi+R_3\right\}e^{-\lambda\xi}\\&+\frac{1}{64}(1-\nu^2)\frac{h^3(c)}{\lambda^5g^3(c)}[36e^{-2\lambda\xi}-e^{-3\lambda\xi}]\end{aligned} \tag{3.39}$$

$$\begin{aligned}\sigma_3(\xi)=&\left\{\frac{N_2}{\lambda^2}\xi^2+\left(\frac{N_1}{\lambda^2}+\frac{4N_2}{\lambda^3}\right)\xi+\left(\frac{6N_2}{\lambda^4}+\frac{2N_1}{\lambda^3}+\frac{N_0}{\lambda^2}\right)\right\}e^{-\lambda\xi}\\&+\left\{\frac{M_2}{4\lambda^2}\xi^2+\left(\frac{M_1}{4\lambda^2}+\frac{M_2}{2\lambda^3}\right)\xi+\left(\frac{3M_2}{8\lambda^4}+\frac{M_1}{4\lambda^3}+\frac{M_0}{4\lambda^2}\right)\right\}e^{-2\lambda\xi}\end{aligned} \tag{3.40}$$

其中R_3为又一积分常数．

这里一共有三个待定常数，即R_3和$S_2(1)$中（见Q_0）出现的B^*和α_2．这里只有两个条件，即(3.12a,c)．在这两个条件中，$W'_2(1)$，$S'_2(1)$，$S_2(1)$内也有待定常数B^*和α_2．这两个方程可以写为

$$a_{21}\alpha_2+b_{21}B^*+\lambda R_3=F_1 \tag{3.41a}$$

$$a_{22}\alpha_2+b_{22}B^*=F_2 \tag{3.41b}$$

其中

$$\begin{aligned}F_1=&\frac{Q_0^*}{2\lambda^2}+\frac{3}{4\lambda^3}Q_1+\frac{7}{4\lambda^4}Q_2+\frac{45}{8\lambda^5}Q_3-\frac{69}{64}(1-\nu^2)\frac{h^3(c)}{\lambda^4g^3(c)}\\&-\frac{2c}{3(1-\nu^2)}g(c)h_{20}(c)-8B^2c^2g^3(c)h_{22}(c)\\&-5Bc^2\frac{\alpha_1}{1-\nu^2}g^4(c)h_{23}(c)-\frac{3}{4}\frac{\alpha_1^2c^2}{1-\nu^2}g^5(c)h_{24}(c)\\&+3(1-\nu^2)\frac{h(c)}{2\lambda^2}\left[6B^2cg(c)f_{22}(c)+4Bc\frac{\alpha_1}{1-\nu^2}g^2(c)f_{23}(c)\right.\\&\left.+\frac{5}{8}\frac{\alpha_1^2c}{(1-\nu^2)^2}g^3(c)f_{24}(c)+\frac{2c}{3(1-\nu^2)}\frac{1}{g(c)}f_{20}(c)\right]\end{aligned} \tag{3.42a}$$

$$F_2=-\left[\frac{2N_0}{\lambda}+\frac{2N_1}{\lambda^2}+\frac{4N_2}{\lambda^3}+\frac{M_0}{\lambda}+\frac{M_1}{2\lambda^2}+\frac{M_2}{2\lambda^3}\right]-\frac{7}{8}(1-\nu)\frac{h^2(c)}{\lambda^2g^2(c)}$$

$$-6B^2cg^2(c)[2cf'_{22}(c)+(1-\nu)f_{22}(c)]$$

$$-4Bc\frac{\alpha_1}{1-\nu^2}g^3(c)[2cf'_{23}(c)+(1-\nu)f_{23}(c)]$$

$$-\frac{5}{8}\frac{\alpha_1^2c}{(1-\nu^2)^2}g^4(c)[2cf'_{24}(c)+(1-\nu)f_{24}(c)]$$

$$-\frac{2c}{3(1-\nu^2)}[2cf'_{20}(c)+(1-\nu)f_{20}(c)] \tag{3.42b}$$

$$a_{21}=\frac{c}{2(1-\nu^2)}g^2(c)h_2(c)+\frac{3}{4\lambda}h(c)f_2(c) \tag{3.42c}$$

$$b_{21}=\frac{3}{2}(1-\nu^2)\frac{h(c)}{\lambda^2g(c)}f_1(c)+2cg(c)h_1(c) \tag{3.42d}$$

$$a_{22}=[2cf'_2(c)+(1-\nu)f_2(c)]\frac{g(c)}{1-\nu^2} \tag{3.42e}$$

$$b_{22}=2cf'_1(c)+(1-\nu)f_1(c) \tag{3.42f}$$

F_1式中的Q_0^*代表

$$Q_0^*=Q_0+3(1-\nu^2)\frac{h(c)}{g(c)}S_2(1)$$

$$=-\frac{2}{\lambda^2}[\lambda M(c)+N(c)+\lambda^4R_2]+3(1-\nu^2)\frac{h^3(c)}{\lambda^2g^3(c)}$$

$$+3(1-\nu^2)\frac{1}{4\lambda^3}S_1(1)[2\lambda M(c)+3N(c)+4\lambda^4R_2] \tag{3.43}$$

其中Q_k，N_k，M_k见(3.36)，(3.37)，(3.38)。这里还缺少一个方程。这个短缺的方程应该从ϵ的定义中寻找。ϵ和W_m的关系式(3.24b)，由于$W_2(0)$，$W_3(0)$等还没有确定，所以也只能说正确到$O(\epsilon)$，亦即是说，对于（3.24b）而言，如果略去了ϵ^3项，则还有$W_2(0)$未定。所以，在$O(\epsilon^2)$的精确度上，还有一定随意性。为了选定这种关系，让我们设

$$W_2(0)=0 \tag{3.44}$$

则W_m，ϵ的关系式（3.24b）就可以写成

$$W_m=\frac{1}{\epsilon}[1+W_1(0)\epsilon]+O(\epsilon^3) \tag{3.45}$$

而（3.25）式可以写成

$$\epsilon=W_m^{-1}+W_1(0)W_m^{-2}+W_1^2(0)W_m^{-3}+O(W_m^{-4}) \tag{3.46}$$

最后（3.26）式即P，W_m关系式可以写成

$$P=\alpha_0W_m^3+[\alpha_2-2W_1(0)\alpha_1+3W_1^2(0)\alpha_0]W_m+O(W_m^{-1}) \tag{3.47}$$

根据（2.31）式，我们可以把（3.44）式写成

$$a_{23}\alpha_2+b_{23}B^*=F_3 \tag{3.48}$$

其中

$$a_{23}=\frac{c}{2(1-\nu^2)}g^2(c)g_2(c) \tag{3.49a}$$

$$b_{23}=2cg(c)g_1(c) \tag{3.49b}$$

$$F_3=R_2-8B^2c^2g^3(c)g_{22}(c)-5Bc^2\frac{\alpha_1}{1-\nu^2}g^4(c)g_{23}(c)$$

$$-\frac{3}{4}\frac{\alpha_1^2c^2}{(1-\nu^2)^2}g^5(c)g_{24}(c)-\frac{2c}{3(1-\nu^2)}g(c)g_{20}(c) \tag{3.49c}$$

(3.41a,b)，(3.48)式在一起组成求解B^*，α_2，R_3的三个线性方程组，其解为

$$\alpha_2=\frac{1}{\Delta_1}(F_2b_{23}-F_3b_{22}) \tag{3.50a}$$

$$B^*=\frac{1}{\Delta_1}(F_3a_{22}-F_2a_{23}) \tag{3.50b}$$

$$R_3=\frac{F_1}{\lambda}-\frac{1}{\lambda\Delta_1}[a_{21}(F_2b_{23}-F_3b_{22})-b_{21}(F_3a_{22}-F_2a_{23})] \tag{3.50c}$$

其中

$$\Delta_1=a_{22}b_{23}-a_{23}b_{22} \tag{3.51}$$

以上，我们求得了内层解中的$\Omega_0(\xi)$，$\Omega_1(\xi)$，$\Omega_2(\xi)$，$\sigma_1(\xi)$，$\sigma_2(\xi)$，$\sigma_3(\xi)$，而且满足了全部有关边界条件，求得了P，W_m的刚度关系。

$$P=\alpha_0W_m^3+[\alpha_2-2W_1(0)+3W_1^2(0)]W_m+O(W_m^{-1}) \tag{3.52}$$

其中α_0，α_1，α_2，$W_1(0)$分别见(2.10d)，(3.30)，(3.50a)，(2.21)。这样，我们满意地求得了零级、一级、二级的合成解。

参 考 文 献

[1] Chien Wei-zang, Asymptotic behavior of a thin clamped circular plate under uniform normal pressure at very large deflection, *Science Reports (A) of National Tsing Hua University*, 5, 1 (1948), 1—24.

[2] Bromberg, E., Nonlinear bending of a circular plate under normal pressure, *Communication of Pure and Applied Mathematics*, 9 (1956), 633—659.

[3] Срубщик Л. С. и В. И. Юдович, Асимптоика уравнений большого прогиба, круглой симметрично загуженней пластины, *ДАН СССР*, 139, 2 (1961).

[4] O'Malley, R. E., Jr., Boundary layer methods for nonlinear initial value problems, *SIAM Review*, 13 (1971), 425—434.

[5] Chudov, L. A., Some shortcomings of classical boundary layer theory, NASA(transl) TTF-360, TT65-50138 (1966).

[6] Latta, G. E., *Advanced Ordinary Differential Equations*, Lecture Notes, Stanford University (1964).

[7] 周焕文，奇异摄动法在圆板大挠度问题中的应用，钱伟长主编《奇异摄动理论及其在力学中的应用》，科学出版社 (1982)，310—338.

The Solution of Large Deflection Problem of Thin Circular Plate by the Method of Composite Expansion

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Abstract

In this paper, the method of composite expansion in perturbation theory is used for the solution of large deflection problem of thin circular plate. In this method, the outer field solution and the inner boundary layer solution are combined togather to satisfy all the boundary conditions. In this paper, Hencky's membrane solution is used for the first approximation in outer field solution, and then the second approximate solution is obtained. The inner boundary layer solution is found on the bases of boundary layer coordinate. In this paper, the reciprocal ratio of maximum deflection and thickness of the plate is used as the small parameter. The results of the paper improves quite a bit in comparison with the results obtained in 1948 by Chien Wei-zang.