

关于带缺陷和裂纹的半无限平面加筋结构的应力强度因子的计算

欧阳鬯 周小康

(复旦大学, 1984年3月5日收到)

摘 要

本文研究两种不同材料、不同厚度、各带裂纹和椭圆孔的半无限平面加筋结构受均匀拉伸的问题。采用复变函数、摄动法以幂级数形式给出裂纹尖端应力强度因子的计算公式。本文的实际计算扩充了“应力强度因子手册”中的结果, 本文的特例, 计算结果与[1]、[3]一致。

一、引 言

在工程实际应用中, 广泛地采用着加筋板结构。然而, 在加工、装配与使用过程中, 这种结构内部难免会产生缺陷或裂纹, 这种复杂裂纹的应力强度因子计算不能不引起人们的关注。前些年, M. Isida^[1]利用复势罗朗展开方法解决了两块半无限板, 其中一板带裂纹, 另一板不带任何缺陷的加筋结构的单向拉伸问题。考虑到实际应用中, 两块板内均可能出现缺陷或裂纹, 我们研究两类板都带奇性的加筋结构问题。利用复势的积分表达式以及摄动方法, 最终以幂级数形式给出了应力强度因子的计算公式, 并对一些常用的加筋结构, 给出计算图表, 以供工程设计应用。

二、基 本 公 式

一般平面弹性问题的应力、位移可用两个复势表示。为简便起见, 在直角坐标系 XOY 中引进无量纲的复变量 z 与复势 $\varphi(z)$, $\psi(z)$, 有:

$$z = (X + iY)/d \quad (2.1)$$

$$\text{和} \quad \left. \begin{aligned} \sigma_{xx} + \sigma_{yy} &= 4\sigma \operatorname{Re} \varphi'(z) \\ \sigma_{xx} - \sigma_{yy} - 2i\sigma_{xy} &= -2\sigma [\bar{z}\varphi''(z) + \psi''(z)] \\ 2G(u - iv) &= \sigma d [\kappa \overline{\varphi(z)} - \bar{z}\varphi'(z) - \psi'(z)] \end{aligned} \right\} \quad (2.2)$$

其中 σ 为特征应力, d 为特征长度, G 为材料的剪切模量, 又

$$\kappa = \begin{cases} \frac{3-\nu}{1+\nu} & (\text{平面应力}) \\ 3-4\nu & (\text{平面应变}) \end{cases} \quad (2.3)$$

ν 为泊松比。

设裂纹长为 $2l$ ，记

$$\lambda = \frac{l}{d} \quad (2.4)$$

由[2]，裂纹尖端的应力强度因子 K 为

$$K = 2\sigma\sqrt{2d} \lim_{z \rightarrow \lambda} [\sqrt{z-\lambda} \varphi'(z)] \quad (2.5)$$

三、问题的描述

考虑两种不同类型的半无限板，区别为 I, II，其厚度，杨氏模量，泊松比分别记为 h, E, ν ，与 h', E', ν' 。板 I 的裂纹长为 $2l$ ，板 II 的椭圆孔长、短半轴为 a, b ，裂纹中心与椭圆中心到筋条的距离均为 d 。筋条的截面积为 S ，杨氏模量为 E_s ，平面弯曲惯性矩为 I_s 。设筋条的宽度远小于 d ，无穷远处应力分别为 σ, σ' 。建立两组坐标系 $XOY, X_1O_1Y_1$ ，如图 1 所示。

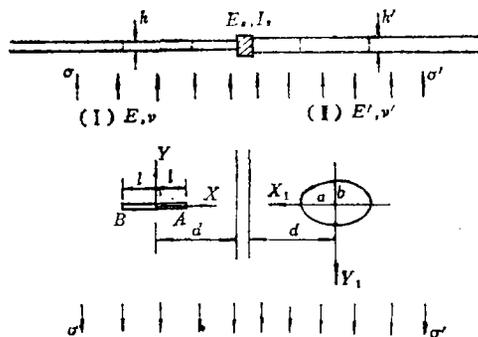


图 1

取 d 为特征长度， σ 与 σ' 分别为板 I, II 的特征应力，引进无量纲变量及参数为

$$z = (X + iY)/d, \quad z_1 = (X_1 + iY_1)/d \quad (3.1)$$

$$\lambda = -\frac{l}{d}, \quad c = \frac{a}{d}, \quad \alpha = \frac{E'h'}{Eh}, \quad \beta = \frac{E_s}{E} \frac{S}{dh}, \quad \gamma = \frac{E_s}{E} \frac{I_s}{d^3h} \quad (3.2)$$

边界条件为：在裂纹及椭圆孔边缘为应力自由；在无穷远处 ($Y = \pm\infty, Y_1 = \pm\infty$) 有

$$\left. \begin{aligned} \text{板 I: } & \sigma_{yy}^{\infty} = \sigma, \quad \sigma_{xx}^{\infty} = \sigma_{xy}^{\infty} = 0 \\ \text{板 II: } & \sigma_{y_1y_1}^{\infty} = \sigma', \quad \sigma_{x_1x_1}^{\infty} = \sigma_{x_1y_1}^{\infty} = 0 \end{aligned} \right\} \quad (3.3)$$

由于在无穷远处筋条两侧板变形协调， σ, σ' 应满足

$$\frac{\sigma}{E} = \frac{\sigma'}{E'} \quad (3.4)$$

板 I, II 可分别作为平面应力问题处理，设它们所对应的复势分别为 $\varphi(z), \psi(z)$ 与 $\Phi(z_1), \Psi(z_1)$ 。为方便起见，设 $\varphi(z) = \varphi_0(z) + \varphi_1(z) + \varphi_2(z)$ ，对 $\psi(z), \Phi(z_1), \Psi(z_1)$ 作同样分解，其中，指标为“0”的复势对应着无穷远处边界条件，易得

$$\varphi_0(z) = \frac{1}{4}z, \quad \psi_0(z) = \frac{1}{4}z^2, \quad \Phi_0(z_1) = \frac{1}{4}z_1, \quad \Psi_0(z_1) = \frac{1}{4}z_1^2 \quad (3.5)$$

指标为“1”的复势对应着板内部的奇性，可设

$$\left. \begin{aligned} \varphi_1(z) &= \sum_{n=0}^{\infty} F_n z^{-(n+1)}, & \psi_1(z) &= -D_0 \ln z + \sum_{n=1}^{\infty} D_n z^{-n} \\ \Phi_1(z_1) &= \sum_{n=0}^{\infty} \tilde{F}_n z_1^{-(n+1)}, & \Psi_1(z_1) &= -\tilde{D}_0 \ln z_1 + \sum_{n=1}^{\infty} \tilde{D}_n z_1^{-n} \end{aligned} \right\} \quad (3.6)$$

指标为“2”的复势为复势修正项，它的作用应使得总复势 $\varphi(z)$, $\psi(z)$ 与 $\Phi(z_1)$, $\Psi(z_1)$ 满足裂纹面与椭圆孔边缘的边界条件以及在筋条处板的连续性条件。考虑到问题的对称性，它可构造为如下的积分形式：

$$\left. \begin{aligned} \varphi_2(z) &= \frac{1}{2} \int_0^{\infty} B(m) \exp(mz) dm, & \psi_2(z) &= \int_0^{\infty} \left[A(m) + \frac{1}{2} z B(m) \right] \exp(mz) dm \\ \Phi_2(z_1) &= \frac{1}{2} \int_0^{\infty} H(m) \exp(mz_1) dm, & \Psi_2(z_1) &= \int_0^{\infty} \left[G(m) + \frac{1}{2} z_1 H(m) \right] \exp(mz_1) dm \end{aligned} \right\} \quad (3.7)$$

这里 $A(m)$, $B(m)$, $G(m)$, $H(m)$ 为待定函数。

由于问题关于 X 与 X_1 轴的对称性，可知表达式 (3.6) 的各系数 D_n , F_n , \tilde{D}_n , \tilde{F}_n 以及 (3.7) 中 $A(m)$, $B(m)$, $G(m)$, $H(m)$ 均为实的。

四、板在筋条处的连续性条件

为了分析板在筋条处的连续性条件，在 $z=1+iy$ 亦即 $z_1=1-iy$ 处选取一微元，如图 2 示。

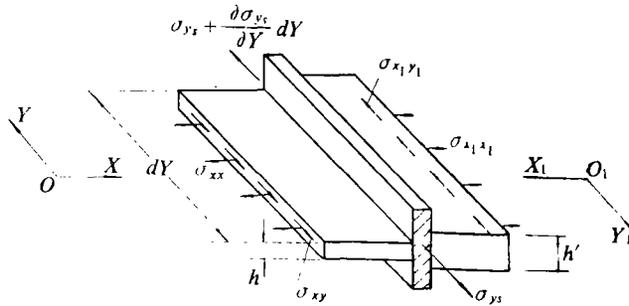


图 2

由板的位移连续性以及微元的平衡条件得

$$\left. \begin{aligned} (u-iv) \Big|_{z=1+iy} + (u'-iv') \Big|_{z_1=1-iy} &= 0 \\ h\sigma_{xy} \Big|_{z=1+iy} - h'\sigma_{x_1y_1} \Big|_{z_1=1-iy} &= S \frac{\partial \sigma_{y_1}}{\partial Y} = \frac{S}{d} \frac{\partial \sigma_{y_1}}{\partial y} \\ -h\sigma_{xx} \Big|_{z=1+iy} + h'\sigma_{x_1x_1} \Big|_{z_1=1-iy} &= E_s I_s \frac{\partial^4 u_s}{\partial Y^4} = \frac{E_s I_s}{d^4} \frac{\partial^4 u_s}{\partial y^4} \end{aligned} \right\} \quad (4.1)$$

其中 $u_s = u \Big|_{z=1+iy}$, $\sigma_{y_1} = E_s \varepsilon_s = E_s \varepsilon_{yy} \Big|_{z=1+iy} = \frac{E_s}{E} (\sigma_{yy} - \nu \sigma_{xx}) \Big|_{z=1+iy}$

$$\left. \begin{aligned} \sigma_1(y) + \sigma_2(y) &= 4\sigma \operatorname{Re} \varphi_1'(z) \Big|_{z=1+iy} \\ \sigma_1(y) - \sigma_2(y) - 2i\tau_1(y) &= -2\sigma [\bar{z}\varphi_1''(z) + \psi_1''(z)] \Big|_{z=1+iy} \\ u_1(y) - iv_1(y) &= \frac{\sigma d}{2G} \left[\frac{3-\nu}{1+\nu} \overline{\varphi_1(z)} - \bar{z}\varphi_1'(z) - \psi_1(z) \right] \Big|_{z=1+iy} \\ \sigma_1'(y) + \sigma_2'(y) &= 4\sigma' \operatorname{Re} \Phi_1'(z_1) \Big|_{z_1=1-iy} \\ \sigma_1'(y) - \sigma_2'(y) - 2i\tau_1'(y) &= -2\sigma' [\bar{z}_1\Phi_1''(z_1) + \Psi_1''(z_1)] \Big|_{z_1=1-iy} \\ u_1'(y) - iv_1'(y) &= \frac{\sigma'd}{2G'} \left[\frac{3-\nu'}{1+\nu'} \overline{\Phi_1(z_1)} - \bar{z}_1\Phi_1'(z_1) - \Psi_1(z_1) \right] \Big|_{z_1=1-iy} \end{aligned} \right\} \quad (4.2)$$

由应力、位移的复势表达式(2.2)以及(3.5)、(3.7), 从(4.1)可得到四个方程, 再作Fourier逆变换化为

$$\left. \begin{aligned}
 & (1+\nu)mA + [(1+\nu)m - (1-\nu)]B + (1+\nu')mG + [(1+\nu')m - (1-\nu')]H \\
 & = \frac{2Ee^{-m}}{\pi\sigma d} \int_0^{\infty} [u_1(y) + u_1'(y)] \cos my dy = P(m) \\
 & (1+\nu)mA + [2 + (1+\nu)m]B - (1+\nu')mG - [2 + (1+\nu')m]H \\
 & = -\frac{2Ee^{-m}}{\pi\sigma d} \int_0^{\infty} [v_1(y) + v_1'(y)] \sin my dy = -Q(m) \\
 & [1 + \beta m(1+\nu)]m^2A + [1 + m(1+2\beta) + \beta m^2(1+\nu)]mB + \alpha m^2G \\
 & + \alpha(m+1)mH = -\frac{2e^{-m}}{\pi\sigma} \left\{ \int_0^{\infty} \left[\tau_1(y) - \frac{h'}{h} \tau_1'(y) \right] \sin my dy \right. \\
 & \left. + \beta m \int_0^{\infty} [\sigma_2(y) - \nu\sigma_1(y)] \cos my dy \right\} = -R(m) \\
 & [1 + \gamma m^3(1+\nu)]m^2A + [1 - \gamma m^2(1-\nu) + \gamma m^3(1+\nu)]m^2B - \alpha m^2G - \alpha m^2H \\
 & = \frac{2e^{-m}}{\pi\sigma} \int_0^{\infty} \left[\sigma_1(y) - \frac{h'}{h} \sigma_1'(y) + \frac{\gamma E m^4}{d} u_1(y) \right] \cos my dy = T(m)
 \end{aligned} \right\} \quad (4.3)$$

这里, (4.3)中的第3、4式已施行了分部积分, 并且考虑到(3.6)、(4.2), 有

$$\left. \begin{aligned}
 & \sigma_i(y) \rightarrow 0, \quad \text{当 } y \rightarrow \infty \text{ 时} \quad (i=1, 2) \\
 & \left. \begin{aligned}
 & \frac{\partial u_1(y)}{\partial y} \Big|_{y=0} = \frac{\partial^3 u_1(y)}{\partial y^3} \Big|_{y=0} = 0
 \end{aligned} \right\} \quad (4.4)
 \end{aligned}$$

把(3.6)、(4.2)代入(4.3), 利用积分公式

$$\left. \begin{aligned}
 & \int_0^{\infty} \frac{\cos n\theta \cos my}{(1+y^2)^{n/2}} dy = \int_0^{\infty} \frac{\sin n\theta \sin my}{(1+y^2)^{n/2}} dy = \frac{\pi e^{-m} m^{n-1}}{2 \cdot (n-1)!} \\
 & \int_0^{\infty} \frac{\cos(n+2)\theta \cos my}{(1+y^2)^{n/2}} dy = \int_0^{\infty} \frac{\sin(n+2)\theta \sin my}{(1+y^2)^{n/2}} dy = \frac{\pi e^{-m} m^{n-1} (2m-n)}{2 \cdot n!} \\
 & \theta = \text{tg}^{-1} y
 \end{aligned} \right\} \quad (4.5)$$

得

$$\left. \begin{aligned}
 P(m) &= e^{-2m} \sum_{n=0}^{\infty} \left\{ \frac{(1+\nu)D_n + (1+\nu')\tilde{D}_n}{(n-1)!} \right. \\
 & \left. + \frac{[2m-n+2 + (2m-n-2)\nu]F_n + [(2m-n+2) + (2m-n-2)\nu']\tilde{F}_n}{n!} \right\} m^n \\
 Q(m) &= e^{-2m} \sum_{n=0}^{\infty} \left\{ \frac{(1+\nu)D_n - (1+\nu')\tilde{D}_n}{(n-1)!} \right. \\
 & \left. + \frac{[2m-n-4 + (2m-n)\nu]F_n - [(2m-n-4) + (2m-n)\nu']\tilde{F}_n}{n!} \right\} m^n \\
 R(m) &= e^{-2m} \sum_{n=0}^{\infty} \left\{ \frac{[-1 + \beta m(1+\nu)]D_n - \alpha\tilde{D}_n}{(n-1)!} \right. \\
 & \left. + \frac{[(2m-n)(-1 + \beta m(1+\nu)) + 2 - 4\beta m]F_n - \alpha(2m-n-2)\tilde{F}_n}{n!} \right\} m^{n+1}
 \end{aligned} \right\} \quad (4.6)$$

$$T(m) = e^{-2m} \sum_{n=0}^{\infty} \left\{ \frac{[-1 + \gamma m^3(1+\nu)] D_n + a \tilde{D}_n}{(n-1)!} + \frac{[(2m-n)(-1 + \gamma m^3(1+\nu)) + 2\gamma m^3(1-\nu)] F_n + \alpha(2m-n) \tilde{F}_n}{n!} \right\} m^{n+1}$$

式中规定

$$0! = (-1)! = 1$$

求解联立方程组(4.3), 把所得的 A, B, G, H 表达式代入(3.7), 利用 $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$,

且因我们仅对应力、应力强度因子感兴趣, 为此略去 $\varphi_2(z), \Phi_2(z_1)$ 的常数项以及 $\psi_2(z), \Psi_2(z_1)$ 的一次项, 可得

$$\left. \begin{aligned} \varphi_2(z) &= \sum_{n=0}^{\infty} M_n z^{n+1}, & \psi_2(z) &= \sum_{n=0}^{\infty} K_n z^{n+2} \\ \Phi_2(z_1) &= \sum_{n=0}^{\infty} \tilde{M}_n z_1^{n+1}, & \Psi_2(z_1) &= \sum_{n=0}^{\infty} \tilde{K}_n z_1^{n+2} \end{aligned} \right\} \quad (4.7)$$

其中

$$\left. \begin{aligned} K_n &= \sum_{p=0}^{\infty} (\alpha_p^n D_p + \beta_p^n F_p + \tilde{\alpha}_p^n \tilde{D}_p + \tilde{\beta}_p^n \tilde{F}_p) \\ M_n &= \sum_{p=0}^{\infty} (\gamma_p^n D_p + \delta_p^n F_p + \tilde{\gamma}_p^n \tilde{D}_p + \tilde{\delta}_p^n \tilde{F}_p) \\ \tilde{K}_n &= \sum_{p=0}^{\infty} (a_p^n D_p + b_p^n F_p + \tilde{a}_p^n \tilde{D}_p + \tilde{b}_p^n \tilde{F}_p) \\ \tilde{M}_n &= \sum_{p=0}^{\infty} (c_p^n D_p + d_p^n F_p + \tilde{c}_p^n \tilde{D}_p + \tilde{d}_p^n \tilde{F}_p) \end{aligned} \right\} \quad (4.8)$$

(4.8) 中诸系数 $a_p^n, b_p^n, \tilde{a}_p^n, \tilde{b}_p^n$ 等均为含参数 $n, p, \alpha, \beta, \gamma, \nu, \nu'$ 的关于 m 的积分式。因式子冗长, 从略。

五、应力强度因子计算

由(3.5)、(3.6)及(4.7)可得本问题的总复势表达式为

$$\left. \begin{aligned} \varphi(z) &= \frac{1}{4} z + \sum_{n=0}^{\infty} F_n z^{-(n+1)} + \sum_{n=0}^{\infty} M_n z^{n+1} \\ \psi(z) &= \frac{1}{4} z^2 - D_0 \ln z + \sum_{n=1}^{\infty} D_n z^{-n} + \sum_{n=0}^{\infty} K_n z^{n+2} \\ \Phi(z_1) &= \frac{1}{4} z_1 + \sum_{n=0}^{\infty} \tilde{F}_n z_1^{-(n+1)} + \sum_{n=0}^{\infty} \tilde{M}_n z_1^{n+1} \end{aligned} \right\} \quad (5.1)$$

$$\Psi(z_1) = \frac{1}{4} z_1^2 - \tilde{D}_0 \ln z_1 + \sum_{n=1}^{\infty} \tilde{D}_n z_1^{-n} + \sum_{n=0}^{\infty} \tilde{K}_n z_1^{n+2}$$

由上节板在筋条处的连续性条件得到上述级数的负幂次系数表示正幂次系数的一组关系式(4.8)。其次,根据裂纹与椭圆孔边缘上应力自由边界条件,可得到正幂次系数表示负幂次系数的一组关系式^[1]:

$$\left. \begin{aligned} D_n &= \frac{1}{2} P_0^n \lambda^{n+2} + \sum_{p=0}^{\infty} \lambda^{n+p+2} P_p^n (K_p + M_p) \\ F_n &= -\frac{1}{2} Q_0^n \lambda^{n+2} - \sum_{p=0}^{\infty} \lambda^{n+p+2} Q_p^n (K_p + M_p) \\ \tilde{D}_n &= \frac{1}{4} (\tilde{P}_0^n + \tilde{R}_0^n) \lambda^{n+2} + \sum_{p=0}^{\infty} \lambda^{n+p+2} (\tilde{P}_p^n \tilde{K}_p + \tilde{R}_p^n \tilde{M}_p) \\ \tilde{F}_n &= -\frac{1}{4} (\tilde{Q}_0^n + \tilde{S}_0^n) \lambda^{n+2} - \sum_{p=0}^{\infty} \lambda^{n+p+2} (\tilde{Q}_p^n \tilde{K}_p + \tilde{S}_p^n \tilde{M}_p) \end{aligned} \right\} \quad (5.2)$$

其中

$$\left. \begin{aligned} \tilde{P}_{2p}^0 &= [c^2(1-\varepsilon^2)]^{p+1} \frac{p+1}{2^{2p}} \binom{2p+1}{p}, & \tilde{R}_{2p}^0 &= \tilde{P}_{2p}^0 \frac{R^2 + R^{-2}}{2} \\ \tilde{P}_{2p}^{2n} &= -\frac{[c^2(1-\varepsilon^2)]^{n+p+1}}{2^{2p+2}} \left\{ 4(p+1) \binom{2p+1}{p} \left(1 + \frac{1}{n}\right) A_{n,1} \right. \\ &\quad \left. + \sum_{m=1}^{n,p+1} \binom{2p+2}{p-m+1} [(2p+1)R^{4m} - 1] A_{n-m,2m} \right\} \quad (n \geq 1) \\ \tilde{R}_{2p}^{2n} &= \frac{[c^2(1-\varepsilon^2)]^{n+p+1}}{2^{2p+1}} \left\{ (p+1) \binom{2p+1}{p} (R^2 + R^{-2}) \left(1 + \frac{1}{n}\right) A_{n,1} \right. \\ &\quad \left. + 2p \sum_{m=0}^{n,p} \binom{2p+1}{p-m} R^{4m+2} A_{n-m,2m+1} \right\} \quad (n \geq 1) \\ \tilde{Q}_{2p}^{2n} &= [c^2(1-\varepsilon^2)]^{n+p+1} \frac{p+1}{2^{2p}} \sum_{m=0}^{n,p} \binom{2p+1}{p-m} R^{4m+2} A_{n-m,2m+1} \\ \tilde{S}_{2p}^{2n} &= \frac{[c^2(1-\varepsilon^2)]^{n+p+1}}{2^{2p+1}} \left\{ (p+1) \binom{2p+1}{p} A_{n,1} + \sum_{m=0}^{n,p} \binom{2p+1}{p-m} A_{n-m,2m+1} \right. \\ &\quad \left. + 2(2p+1) \sum_{m=0}^{n+1,p} \binom{2p}{p-m} R^{4m} A_{n-m+1,2m} \right\} \quad (5.3) \\ \tilde{P}_{2p+1}^{2n+1} &= \frac{[c^2(1-\varepsilon^2)]^{n+p+2}}{2^{2p+3}} \left\{ (p+2) \binom{2p+3}{p+1} A_{n,1} + \sum_{m=0}^{n,p+1} \binom{2p+3}{p-m+1} \right. \end{aligned} \right.$$

$$\begin{aligned} & \cdot [(2p+2)R^{4m+2}-1]A_{n-m,2m+1}\} \\ \tilde{R}_{2p+1}^{2n+1} &= \frac{[c^2(1-\varepsilon^2)]^{n+p+2}}{2^{2p+2}} \left\{ (p+1) \binom{2p+1}{p} (R+R^{-1})^2 A_{n,1} \right. \\ & \quad \left. + (2p+1) \sum_{m=1}^{n+1,p+1} \binom{2p+2}{p-m+1} R^{4m} A_{n-m+1,2m} \right\} \\ \tilde{Q}_{2p+1}^{2n+1} &= [c^2(1-\varepsilon^2)]^{n+p+2} \frac{2p+3}{2^{2p+2}} \sum_{m=1}^{n+1,p+1} \binom{2p+2}{p-m+1} R^{4m} A_{n-m+1,2m} \\ \tilde{S}_{2p+1}^{2n+1} &= \frac{[c^2(1-\varepsilon^2)]^{n+p+2}}{2^{2p+2}} \left\{ \sum_{m=1}^{n+1,p+1} \binom{2p+2}{p-m+1} A_{n-m+1,2m} \right. \\ & \quad \left. + 2(2p+2) \sum_{m=0}^{n+1,p} \binom{2p+1}{p-m} R^{4m+2} A_{n-m+1,2m+1} \right\} \\ P_p^n &= \tilde{P}_p^n \Big|_{s=0, R=1, c=1}, \quad Q_p^n = \tilde{Q}_p^n \Big|_{s=0, R=1, c=1}. \end{aligned}$$

以上各式中之 c 由(3.2)表示, 又

$$\varepsilon = \frac{b}{a}, \quad R = \sqrt{\frac{1+\varepsilon}{1-\varepsilon}}, \quad A_{n,m} = \frac{m}{2^{m+2n}} \binom{m+2n}{n} \quad (5.4)$$

对于(5.4)中 Σ 的上指标必须取较小的一个, 且当 $n+p$ = 奇数时 $\tilde{P}_p^n = \tilde{R}_p^n = \tilde{Q}_p^n = \tilde{S}_p^n = 0$.

由复势展开式(5.1)的两组关系式, 即可计算板 I 的裂纹尖端的应力强度因子. 为了方便,

$$\text{令} \quad \left. \begin{aligned} K_n + M_n &= H_n \\ \alpha_p^n + \gamma_p^n &= \Gamma_p^n, & \beta_p^n + \delta_p^n &= \Delta_p^n \\ \tilde{\alpha}_p^n + \tilde{\gamma}_p^n &= \tilde{\Gamma}_p^n, & \tilde{\beta}_p^n + \tilde{\delta}_p^n &= \tilde{\Delta}_p^n \end{aligned} \right\} \quad (5.5)$$

则(4.8)、(5.2)可简化为

$$\left. \begin{aligned} H_n &= \sum_{p=0}^{\infty} (\Gamma_p^n D_p + \Delta_p^n F_p + \tilde{\Gamma}_p^n \tilde{D}_p + \tilde{\Delta}_p^n \tilde{F}_p) \\ \tilde{K}_n &= \sum_{p=0}^{\infty} (a_p^n D_p + b_p^n F_p + \tilde{a}_p^n \tilde{D}_p + \tilde{b}_p^n \tilde{F}_p) \\ \tilde{M}_n &= \sum_{p=0}^{\infty} (c_p^n D_p + d_p^n F_p + \tilde{c}_p^n \tilde{D}_p + \tilde{d}_p^n \tilde{F}_p) \end{aligned} \right\} \quad (5.6)$$

$$\left. \begin{aligned} D_n &= \frac{1}{2} P_0^n \lambda^{n+2} + \sum_{p=0}^{\infty} \lambda^{n+p+2} P_p^n H_p \\ F_n &= -\frac{1}{2} Q_0^n \lambda^{n+2} - \sum_{p=0}^{\infty} \lambda^{n+p+2} Q_p^n H_p \\ \tilde{D}_n &= \frac{1}{4} (\tilde{P}_0^n + \tilde{R}_0^n) \lambda^{n+2} + \sum_{p=0}^{\infty} \lambda^{n+p+2} (\tilde{P}_p^n \tilde{K}_p + \tilde{R}_p^n \tilde{M}_p) \end{aligned} \right\} \quad (5.7)$$

$$\tilde{F}_n = -\frac{1}{4}(\tilde{Q}_0^n + \tilde{S}_0^n)\lambda^{n+2} - \sum_{p=0}^{\infty} \lambda^{n+p+2}(\tilde{Q}_p^n \tilde{K}_p + \tilde{S}_p^n \tilde{M}_p)$$

选取 λ 为参数, 对(5.6)、(5.7)作摄动, 设

$$\left. \begin{aligned} (D_{2n}, F_{2n}, \tilde{D}_{2n}, \tilde{F}_{2n}) &= \sum_{q=n+1}^{\infty} (D_{2n}^{(2q)}, F_{2n}^{(2q)}, \tilde{D}_{2n}^{(2q)}, \tilde{F}_{2n}^{(2q)}) \lambda^{2q} \\ (D_{2n+1}, F_{2n+1}, \tilde{D}_{2n+1}, \tilde{F}_{2n+1}) &= \sum_{q=n+3}^{\infty} (D_{2n+1}^{(2q)}, F_{2n+1}^{(2q)}, \tilde{D}_{2n+1}^{(2q)}, \tilde{F}_{2n+1}^{(2q)}) \lambda^{2q} \\ (H_n, \tilde{K}_n, \tilde{M}_n) &= \sum_{q=1}^{\infty} (H_n^{(2q)}, \tilde{K}_n^{(2q)}, \tilde{M}_n^{(2q)}) \lambda^{2q} \end{aligned} \right\} \quad (5.8)$$

可推得它们之间的递推关系为

$$\left. \begin{aligned} D_{2n}^{(2n+2)} &= \frac{1}{2} P_0^{2n}, \quad F_{2n}^{(2n+2)} = -\frac{1}{2} Q_0^{2n} \\ \tilde{D}_{2n}^{(2n+2)} &= \frac{1}{4} (\tilde{P}_0^{2n} + \tilde{R}_0^{2n}), \quad \tilde{F}_{2n}^{(2n+2)} = -\frac{1}{4} (\tilde{Q}_0^{2n} + \tilde{S}_0^{2n}) \\ H_n^{(2q+2)} &= \sum_{p=0}^{2q} (F_p^n D_p^{(2q+2)} + A_p^n F_p^{(2q+2)} + \tilde{F}_p^n \tilde{D}_p^{(2q+2)} + \tilde{J}_p^n \tilde{F}_p^{(2q+2)}) \\ \tilde{K}_n^{(2q+2)} &= \sum_{p=0}^{2q} (a_p^n D_p^{(2q+2)} + b_p^n F_p^{(2q+2)} + \tilde{a}_p^n \tilde{D}_p^{(2q+2)} + \tilde{b}_p^n \tilde{F}_p^{(2q+2)}) \\ \tilde{M}_n^{(2q+2)} &= \sum_{p=0}^{2q} (c_p^n D_p^{(2q+2)} + d_p^n F_p^{(2q+2)} + \tilde{c}_p^n \tilde{D}_p^{(2q+2)} + \tilde{d}_p^n \tilde{F}_p^{(2q+2)}) \quad (q \geq 0) \\ D_{2n}^{(2n+2q+2)} &= \sum_{p=0}^{q-1} P_{2p}^{2n} H_{2p}^{(2q-2p)}, \quad F_{2n}^{(2n+2q+2)} = -\sum_{p=0}^{q-1} Q_{2p}^{2n} H_{2p}^{(2q-2p)} \\ \tilde{D}_{2n}^{(2n+2q+2)} &= \sum_{p=0}^{q-1} (\tilde{P}_{2p}^{2n} \tilde{K}_{2p}^{(2q-2p)} + \tilde{R}_{2p}^{2n} \tilde{M}_{2p}^{(2q-2p)}) \\ \tilde{F}_{2n}^{(2n+2q+2)} &= -\sum_{p=0}^{q-1} (\tilde{Q}_{2p}^{2n} \tilde{K}_{2p}^{(2q-2p)} + \tilde{S}_{2p}^{2n} \tilde{M}_{2p}^{(2q-2p)}) \quad (q \geq 1) \\ D_{2n+1}^{(2n+2q+2)} &= \sum_{p=0}^{q-2} P_{2p+1}^{2n+1} H_{2p+1}^{(2q-2p-2)}, \quad F_{2n+1}^{(2n+2q+2)} = -\sum_{p=0}^{q-2} Q_{2p+1}^{2n+1} H_{2p+1}^{(2q-2p-2)} \\ \tilde{D}_{2n+1}^{(2n+2q+2)} &= \sum_{p=0}^{q-2} (\tilde{P}_{2p+1}^{2n+1} \tilde{K}_{2p+1}^{(2q-2p-2)} + \tilde{R}_{2p+1}^{2n+1} \tilde{M}_{2p+1}^{(2q-2p-2)}) \end{aligned} \right\} \quad (5.9)$$

$$\tilde{F}_{2n+1}^{(2n+2q+2)} = - \sum_{p=0}^{q-2} (\tilde{Q}_{2p+1}^{2q+1} \tilde{K}_{2p+1}^{(2q-2p-2)} + \tilde{S}_{2p+1}^{2q+1} \tilde{M}_{2p+1}^{(2q-2p-2)}) \quad (q \geq 2)$$

其中 $D_{2n+1}^{(2n+4)} = F_{2n+1}^{(2n+4)} = \tilde{D}_{2n+1}^{(2n+4)} = \tilde{F}_{2n+1}^{(2n+4)} = 0$

最后由(2.5)以及[1]得到板 I 的裂纹尖端 A, B 的应力强度因子

$$K_{1A} = \sigma \sqrt{l} F_{1A}, \quad K_{1B} = \sigma \sqrt{l} F_{1B} \quad (5.10)$$

其中

$$F_{1A} = 1 + \sum_{q=1}^N \sum_{p=0}^{q-1} \frac{p+1}{2^{2p}} \binom{2p+1}{p} [(2p+2) H_{2p}^{(2q-2p)} \lambda^{2^q} + (2p+3) H_{2p+1}^{(2q-2p)} \lambda^{2^q+1}]$$

$$F_{1B} = 1 + \sum_{q=1}^N \sum_{p=0}^{q-1} \frac{p+1}{2^{2p}} \binom{2p+1}{p} [(2p+2) H_{2p}^{(2q-2p)} \lambda^{2^q} - (2p+3) H_{2p+1}^{(2q-2p)} \lambda^{2^q+1}]$$

这里 N 随计算精度要求而定。

六、典型加筋结构计算与分析

这里, 我们选算几个实例, 以说明方法的有效性, 这些算例的结果构成了对工程设计所用“应力强度因子手册”的补充。

(i) 裂纹—裂纹板加筋结构 (图3)

因裂纹是椭圆孔的特例, 为此只要在前面的结果中令 $b=0$, 即 $R=1, e=0$ 。考虑到在此情况下有 $\tilde{P}_p^* = \tilde{K}_p^*$ 与 $\tilde{Q}_p^* = \tilde{S}_p^*$, (5.6)、(5.7) 两组关系式可进一步简化, 又设

$$\left. \begin{aligned} \tilde{H}_n &= \tilde{K}_n + \tilde{M}_n, & e_p^* &= a_p^* + c_p^*, & f_p^* &= b_p^* + d_p^*, \\ \tilde{e}_p^* &= \tilde{a}_p^* + \tilde{c}_p^*, & \tilde{f}_p^* &= \tilde{b}_p^* + \tilde{d}_p^* \end{aligned} \right\} \quad (6.1)$$

可得

$$\left. \begin{aligned} H_n &= \sum_{p=0}^{\infty} (\Gamma_p^* D_p + \Delta_p^* F_p + \tilde{\Gamma}_p^* \tilde{D}_p + \tilde{\Delta}_p^* \tilde{F}_p) \\ \tilde{H}_n &= \sum_{p=0}^{\infty} (e_p^* D_p + f_p^* F_p + \tilde{e}_p^* \tilde{D}_p + \tilde{f}_p^* \tilde{F}_p) \\ D_n &= \frac{1}{2} P_0^* \lambda^{n+2} + \sum_{p=0}^{\infty} \lambda^{n+p+2} P_p^* H_p, & F_n &= -\frac{1}{2} Q_0^* \lambda^{n+2} - \sum_{p=0}^{\infty} \lambda^{n+p+2} Q_p^* H_p, \\ \tilde{D}_n &= \frac{1}{2} \tilde{P}_0^* \lambda^{n+2} + \sum_{p=0}^{\infty} \lambda^{n+p+2} \tilde{P}_p^* \tilde{H}_p, & \tilde{F}_n &= -\frac{1}{2} \tilde{Q}_0^* \lambda^{n+2} - \sum_{p=0}^{\infty} \lambda^{n+p+2} \tilde{Q}_p^* \tilde{H}_p \end{aligned} \right\} \quad (6.2)$$

其中的 $\Gamma, \Delta, \tilde{\Gamma}, \tilde{\Delta}$ 等系数通过(5.5)、(6.1)可以 α_p^*, β_p^* 诸系数之表达式计算。类似(5.8)对(6.2)作扰动, 相应的递推关系与(5.9)相仿, 最后板 I 的裂纹尖端的应力强度因子由(5.10)

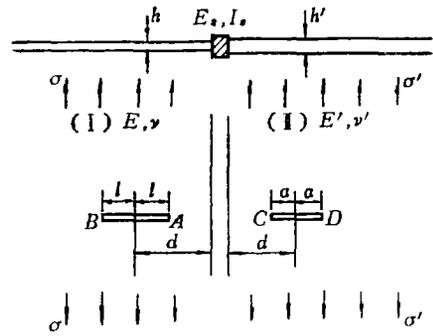


图 3

表出, 另外板 II 的裂纹尖端的应力强度因子可从

$$K_1^{(II)} = 2\sigma' \sqrt{\lambda} \lim_{z_1 \rightarrow \pm \lambda c} [\sqrt{z_1 \mp \lambda c} \Phi'(z_1)] \quad (6.3)$$

得

$$K_{1C} = \sigma' \sqrt{a} F_{1C}, \quad K_{1D} = \sigma' \sqrt{a} F_{1D} \quad (6.4)$$

其中

$$F_{1C} = 1 + \sum_{q=1}^N \sum_{p=0}^{q-1} \frac{p+1}{2^{2p}} \binom{2p+1}{p} [(2p+2)c^{2p} \tilde{H}_{2p}^{(2q-2p)} \lambda^{2q} + (2p+3)c^{2p+1} \tilde{H}_{2p+1}^{(2q-2p)} \lambda^{2q+1}]$$

$$F_{1D} = 1 + \sum_{q=1}^N \sum_{p=0}^{q-1} \frac{p+1}{2^{2p}} \binom{2p+1}{p} [(2p+2)c^{2p} \tilde{H}_{2p}^{(2q-2p)} \lambda^{2q} - (2p+3)c^{2p+1} \tilde{H}_{2p+1}^{(2q-2p)} \lambda^{2q+1}]$$

取 $\nu = \nu' = 0.3$, 对此种加筋板结构作了数值计算. 计算中利用 Laguerre-Gauss 公式处理 α_n^* 等的积分. 计算误差来自于两方面: 一是数值积分计算, 二是截断级数 N 的选取. 数值计算表明, F_1 的 λ 级数比几何级数收敛速度更快. 因此, 适当地选取数值积分的节点数以及 N 值, 就能期望获得满意的计算精度.

图4所示的曲线为 $\beta = \gamma = 0$, $a = l$ 情况, 即无筋条、等长裂纹、不同厚度板受单向拉伸的问题. 考虑到这里板 I、II 的互换性, 亦即板 II 在 C, D 点相应于 a 的 $F_1^{(II)}$ 值分别与在 A, B 点相应于 $1/a$ 的 $F_1^{(I)}$ 值相等, 因此我们不再绘制 $F_1^{(I)}$ 的曲线, 它的值可以从图中相应的曲线上找到. 其次要说明的是, 图中 $F_{1B}^{(I)}$ 的曲线

(即图中的虚线), 这里仅给出了 $0 \leq a \leq 1$ 的部分, 当 $a > 1$ 时, 应力强度因子值随 a 增加, 但增量相当小, 而且基本上落在 $a = 0.1$ 与 $a = 1$ 的虚线所围的三角形区域内, 为了保持图象的清晰性, 略去了这部分曲线.

图5所绘制的曲线为 $a = 1$, $\gamma = 0.01$, $a = l$ 情况, 即有筋条、等长裂纹、等厚度板受单向拉伸的问题. 注意其中 $\beta = 0$ 的曲线, 此时筋条已丧失了抗拉能力, 仅能承受弯曲 (由 $\gamma \neq 0$ 表明). 然而由于 $a = 1$, 即结构与载荷的布局关于筋条对称, 筋条并不产生弯曲, 因此, 在这种情况下筋条对 F 值没有贡献. 数值计算结果也指出, 图5中 $\beta = 0$ ($a = 1, \gamma = 0.01$) 曲线与图4中 $a = 1$ ($\beta = \gamma = 0$) 的曲线相同. 这样, 物理上所期望的结果得到了计算的证实. 另外, 考虑到结构关于筋条的对称性, $F_{A(B)}^{(I)} = F_{D(C)}^{(II)}$, 为此图中只画了 $F^{(I)}$ 的曲线.

本文对 $\beta = \gamma = 0$, $a = 1$, 即无筋条、等厚度板、不等长裂纹问题也同样作了数值计算,

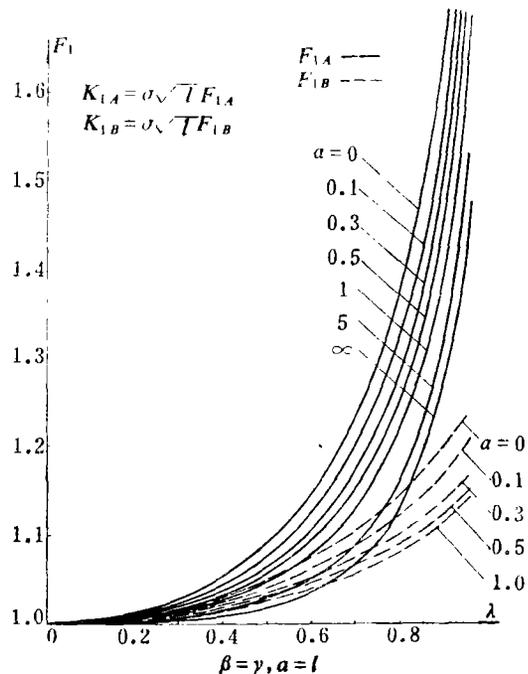


图 4

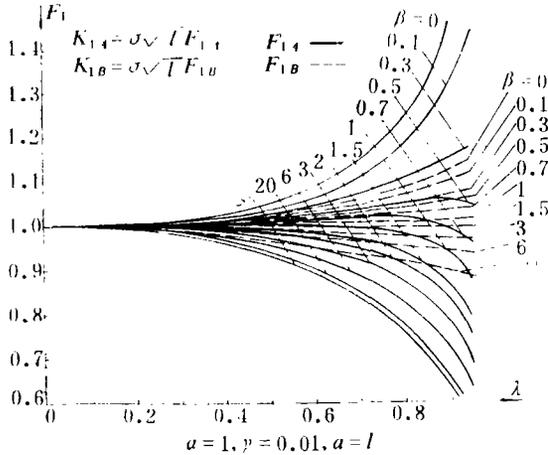


图 5

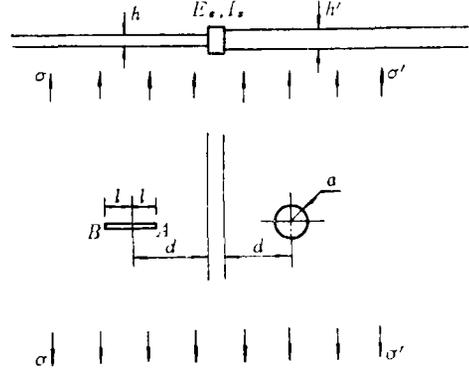


图 6

其结果与[3]一致, 在此不再重复.

(ii) 裂纹—圆孔板加筋结构(图6)

考虑到板 II 在圆孔边缘应力自由的边界条件, 由[1], 把(5.7)之第3、4式改为

$$\left. \begin{aligned} \tilde{D}_0 &= \frac{1}{2}(c\lambda)^2 + 2(c\lambda)^2 \tilde{M}_0, & \tilde{D}_1 &= (c\lambda)^4 \tilde{M}_1 \\ \tilde{D}_n &= (n-1)(c\lambda)^{2n} \left(\frac{1}{4} \delta_{n,2} + \tilde{K}_{n-2} \right) + n(c\lambda)^{2n+2} \tilde{M}_n & (n \geq 2) \\ \tilde{F}_n &= -(n+2)(c\lambda)^{2n+2} \left(\frac{1}{4} \delta_{n,0} + \tilde{K}_n \right) - (n+3)(c\lambda)^{2n+4} \tilde{M}_{n+2} \end{aligned} \right\} \quad (6.5)$$

这里 $\delta_{m,n}$ 为 Kronecker 记号.

然后, 利用(5.8)表达式与前类似地对(5.6)、(5.9)_{1,2}、(6.5)的关系式作摄动, 最后板 I 裂纹的 K_I 值仍由(5.10)表示.

取 $\nu = \nu' = 0.3$, 采用(i)中类似的数值计算方法处理了以下三种特例.

a) $\beta = \gamma = 0, a = l$, 即无筋条、不同厚度的情况. 计算结果示如图7. 图中出现了十分有趣的现象: 当 α 相当大时, 在 λ 的某一范围内, 出现了 F_{I_B} 超过 F_{I_A} 的情况. 这是由于板 II 具有很大的刚度, 从而约束了 A 处的应力强度所致. 造成 F_{I_A} 低于 F_{I_B} . 当 λ 越出该范围后, 由于板 II 的刚度削弱, 导致 F_{I_A} 超出了 F_{I_B} . 还有一点, 就是对于充分大的 α , 随 λ 增大 F_{I_A} 出现先减少后增加的现象. 这是因为 λ 开始增加时, 意味着裂纹尖端 A 向刚度大的板 II 逼近, 从而降低了 F_{I_A} 值. 但当 λ 充分大时, 由圆孔的同时增大, 削弱了板 II 的刚度, 这样, F_{I_A} 值又迅速增加起来. 当 α 愈大, 以上现象愈明显.

b) $\alpha = 1, \nu = 0.01, a = l$, 即带筋条且等厚度板情况, 结果见图8. 曲线的分布形状与图5相同. 从图可知, 对同一 λ 值, 当筋条与板的拉伸刚度之比 β 增加时, F 值下降了, 当 β 值相当大时, F 下降十分缓慢. 根据本文对其它不同 γ 值所作的数值计算表明, 弯曲刚度之比 γ 增大, F 值下降并不显著. 也就是说, 此种结构中影响应力强度因子的主要因素为 β . 因此, 在实际结构设计中, 为了防止裂纹的失稳扩展, 增大 β 值是有效的措施.

c) $\alpha = 1, \beta = \gamma = 0$, 即无筋条、等厚度、孔径与裂纹长不等的情况. 其数值结果与

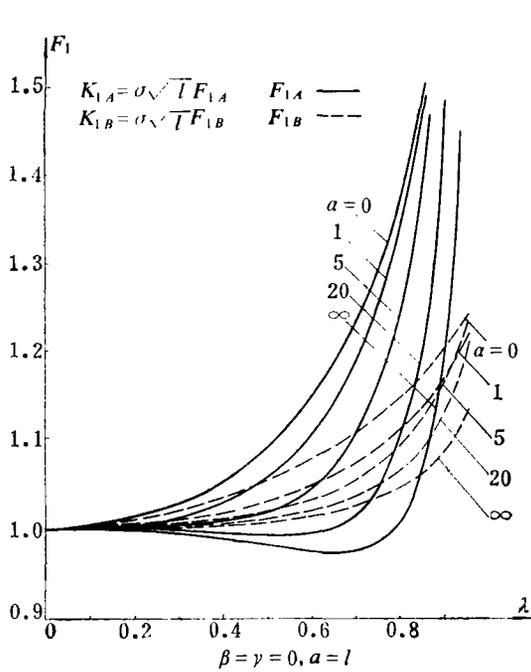


图 7

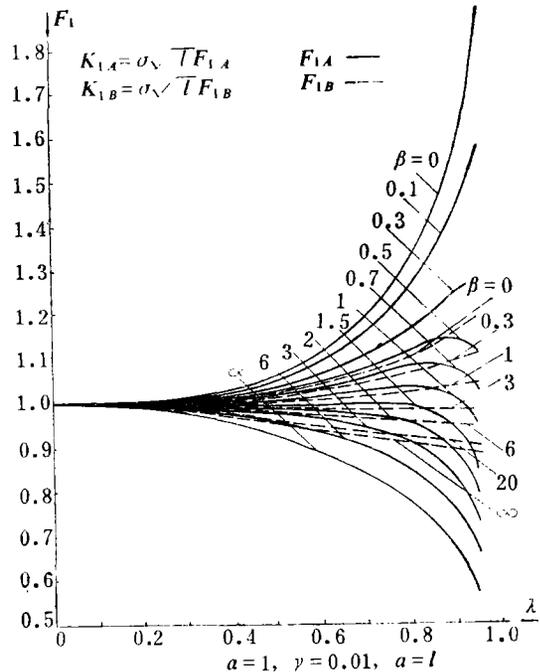


图 8

[1]一致, 图表从略.

应该指出, 本文所讨论、分析的有关裂纹—椭圆孔的加劲板问题还包含如下一些简单、特殊的结构:

例如, 一条裂纹的加劲板问题, 即板 I 有一条裂纹, 板 II 不带奇性的加劲板, 此情况对应于文中 $\Phi_1(z_1) = \Psi_1(z_1) = 0$ 或 $\tilde{D}_n = \tilde{F}_n = 0$. 又如, 带一条裂纹的半无限板问题 (边缘上包括自由、固支及筋条加劲等情况)^[1], 相应于本文中 $\Phi(z_1) = \Psi(z_1) = 0$.

最后, 可利用实际计算结果对精度估计作一些讨论. 由理论推导知, 裂纹尖端的应力强度因子为

$$K_{1A} = \sigma\sqrt{l}F_{1A}, \quad F_{1A} = 1 + \sum_{q=1}^{2N+1} C_q \lambda^q \quad (6.6)$$

实际计算表明, 级数(6.6)中 C_q 的绝对值是快速下降的, (6.6)中级数以较几何级数更快的速度收敛. 在适当地选取数值积分的节点数, 以保证积分精度的前提下, 级数(6.6)的计算误差小于 $C_{2N+1}\lambda^{2N+2}/(1-\lambda^2)$. 即使 $\lambda=0.9$, 它仍具有很高的精度. 对(i)、(ii)的一些实例, 本文数值计算中均采用 $N=20$. 在图4对应的实例中, 当取 $\lambda=0.9$ 时, 误差小于 0.19%, 而且当 α 增大, 其误差越来越小, 最小达 0.13%. 图5所对应的实例, 取 $\lambda=0.9$ 时, 其误差小于 0.15%; 图7的误差小于 0.13%; 图8为 0.16%. 然而, 当 $\lambda \geq 0.95$ 时, 其精度下降了, 可是这样的结构在实际工程中是罕见的. 从(5.10)还可知, F_{1A} 的计算精度与 F_{1B} 完全相同.

本文是在中国科学院科学基金委员会的资助下完成的.

参 考 文 献

- [1] Isida, M., On the determination of stress intensity factors for some common problems, *Engng. Fract. Mech.*, 2 (1973), 61. *Methods of Analysis and Solutions of Crack Problems*, edited by G. C. Sih, Noordhoff Int. Pub. (1973), 80.
- [2] Sih, G. C., P. C. Paris and F. Erdogan, Crack tip stress intensity factors for plane extension and plate bending problems, *J. Appl. Mech., Trans. ASME*, ser. E, 29 (1962), 306.
- [3] Sih, G. C., *Handbook of Stress Intensity Factors*, Lehigh Univ., Bethlehem, Pa., U.S.A. (1973).

On Computations of Stress Intensity Factors for Stiffened Half-Planes with Imperfections and Cracks

Ouyang Chang Zhou Shiao-kang

(Department of Mathematics, Fudan University, Shanghai)

Abstract

In this paper we consider uniform extension problems for joined two half-planes with different thickness and material behavior and one of which contains an elliptical hole, the other contains a crack. Along the boundary of these half planes there is a stiffening stringer. Computational formulas are given in power series form by complex variable-perturbation method. Results obtained here give extension to those of "Handbook of Stress Intensity Factors". Numerical results of special cases in this paper coincide with those of references [1], [3].