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解平行四边形薄板弯曲问题的样条有限条法

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摘　要

Y．K．Cheung等（1982）用样条有限条方法成功地求解了矩形板和壳的问题．本文将此法作少许的修正用于分析平行四边形板，这一修正只增加少许的计算量，但仍保持有限条法的良好性质（如带状性质等）．本文还给出方法的误差估计和数值算例．

一、引　言

O．C．Zienkiewicz[1]等首先建立了有限元方法，随着Y．K．Cheung发展了解矩形薄板弯曲问题的有限条方法，收到节省计算机内存和计算时间的良好效果；其后该法在结构分析中得到充分的发挥[2]．1982年，Y．K．Cheung等人进一步将这一方法和样条函数结合起来，形成"样条有限条方法"[3]．迄今为止，在他们的大量的算例中，只涉及规则区域的问题，严格地讲只限于一类"直角形区域"。在薄板弯曲问题中，他们未曾用于分析平行四边形的斜板．本文试图将样条有限条法用到任意平行四边形板的弯曲问题，其结果表明这一方法对非规则区域的问题仍能发挥一定的作用．实际上，对曲边平行四边形板，甚至任意的四边形板的问题，样条有限条法也能发挥作用，此将另文讨论．

二、问题的陈述

考虑平行四边形的薄板Ω：

$$ky<x<ky+a \qquad (0<y<b)$$

此处a，$b>0$，$k=\mathrm{tg}\theta$（图1）．Ω的边界记为$\partial\Omega$，并以$(\partial\Omega)_1$，$(\partial\Omega)_2$，$(\partial\Omega)_3$分别表示固支、简支和自由的边界

$$\partial\Omega=\bigcup_{j=1}^{3}(\partial\Omega)_j$$

对于Kirchhoff板，总的能量为：

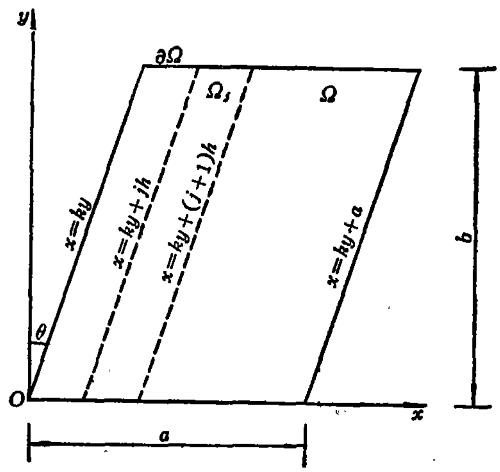

图1 板$\Omega=\{(x,\ y)\,|\,ky<x<ky+a,\ 0<y<b\}$，$a>0$，$b>0$，$k=\mathrm{tg}\theta$

$$J(w)=\frac{D}{2}\iint_{\Omega}\left\{\left(\frac{\partial^2 w}{\partial x^2}\right)^2+\left(\frac{\partial^2 w}{\partial y^2}\right)^2+2\nu\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}+2(1-\nu)\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2\right\}dxdy-\iint_{\Omega}qw\,dxdy$$

其中w——挠度函数，q——外荷载，D——弯曲刚度，ν——泊桑比．薄板弯曲问题是要寻找在Ω上适当光滑的函数w，满足边界条件：

$$w=\frac{\partial w}{\partial n}=0 \qquad 于(\partial\Omega)_1 \tag{2.1}$$

$$w=0 \qquad 于(\partial\Omega)_2 \tag{2.2}$$

并且使

$$J(w)\leqslant J(w^*) \qquad \forall w^* \tag{2.3}$$

或等价地使

$$D\iint_{\Omega}\left\{\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w^*}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\frac{\partial^2 w^*}{\partial y^2}+\nu\left(\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w^*}{\partial y^2}+\frac{\partial^2 w^*}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right)+2(1-\nu)\frac{\partial^2 w}{\partial x\partial y}\frac{\partial^2 w^*}{\partial x\partial y}\right\}dx\,dy=\iint_{\Omega}qw^*dx\,dy \qquad \forall w^* \tag{2.4}$$

此处w^*具有与w相同的光滑度，满足同样的边界条件：

$$w^*=\frac{\partial w^*}{\partial n}=0 \qquad 于(\partial\Omega)_1 \tag{2.5}$$

$$w^*=0 \qquad 于(\partial\Omega)_2 \tag{2.6}$$

三、坐标变换与离散化

选取变换

$$x=a\xi+kb\eta,\quad y=b\eta \tag{3.1}$$

在此变换下，域Ω变为$\xi—\eta$平面的单位正方形$\hat{\Omega}=[0,\ 1]\times[0,\ 1]$．原变数的二阶导数与新变数的二阶导数之间有下述关系：

$$\begin{bmatrix}\dfrac{\partial^2}{\partial x^2}\\[2mm] \dfrac{\partial^2}{\partial y^2}\\[2mm] 2\dfrac{\partial^2}{\partial x\partial y}\end{bmatrix}=\begin{bmatrix}\dfrac{1}{a^2} & 0 & 0\\[2mm] \dfrac{k^2}{a^2} & \dfrac{1}{b^2} & -\dfrac{k}{ab}\\[2mm] -\dfrac{2k}{a^2} & 0 & \dfrac{1}{ab}\end{bmatrix}\begin{bmatrix}\dfrac{\partial^2}{\partial \xi^2}\\[2mm] \dfrac{\partial^2}{\partial \eta^2}\\[2mm] 2\dfrac{\partial^2}{\partial \xi\partial \eta}\end{bmatrix} \tag{3.2}$$

板的总能量可改写成：

$$J(\hat{w})=\frac{D}{2}\iint_{\hat{\Omega}}\left\{\frac{b}{a^3}(1+k^2)^2\left(\frac{\partial^2\hat{w}}{\partial\xi^2}\right)^2+\frac{a}{b^3}\left(\frac{\partial^2\hat{w}}{\partial\eta^2}\right)^2+\frac{2}{ab}(1-\nu+2k^2)\left(\frac{\partial^2\hat{w}}{\partial\xi\partial\eta}\right)^2\right.$$

$$+2\frac{\nu+k^2}{ab}\frac{\partial^2\hat{w}}{\partial\xi^2}\frac{\partial^2\hat{w}}{\partial\eta^2}-4\frac{k(1+k^2)}{a^2}\frac{\partial^2\hat{w}}{\partial\xi^2}\frac{\partial^2\hat{w}}{\partial\xi\partial\eta}$$

$$-4\frac{k}{b^2}\frac{\partial^2\hat{w}}{\partial\eta^2}\frac{\partial^2\hat{w}}{\partial\xi\partial\eta}\Big\}d\xi d\eta-ab\iint_{\hat{\Omega}}q\hat{w}d\xi d\eta$$

此处$\hat{w}(\xi,\ \eta)=w(a\xi+kb\eta,\ b\eta)$．边界条件的形式保持不变，即

$$\hat{w}=\frac{\partial\hat{w}}{\partial\hat{n}}=0\qquad 于(\partial\hat{\Omega})_1 \tag{3.3}$$

$$\hat{w}=0\qquad 于(\partial\hat{\Omega})_2 \tag{3.4}$$

此处$(\partial\hat{\Omega})_j$是$(\partial\Omega)_j$ ($j=1,\ 2,\ 3$)在变换(3.1)下之原像．

依照有限条方法，将域$\hat{\Omega}$剖分成若干个条形域（图 2）：

$$\hat{\Omega}=\bigcup_{j=0}^{n-1}\hat{\Omega}_j,\quad \hat{\Omega}_j=[\xi_j,\ \xi_{j+1}]\times[0,\ 1],\quad \xi_j=\frac{j}{n}$$

在$\hat{\Omega}$内，$\hat{w}(\xi,\ \eta)$沿ξ方向取Hermite三次插值多项式，沿η方向取三次样条多项式，即

$$\hat{w}(\xi,\ \eta)=[N_1(\bar{\xi}),\ \hat{h}N_2(\bar{\xi}),\ N_3(\bar{\xi}),\hat{h}N_4(\bar{\xi})]\begin{bmatrix}\Phi(\eta) & & & 0\\ & \Phi(\eta) & & \\ & & \Phi(\eta) & \\ 0 & & & \Phi(\eta)\end{bmatrix}\begin{bmatrix}\{\alpha\}_j\\ \{\beta\}_j\\ \{\alpha\}_{j+1}\\ \{\beta\}_{j+1}\end{bmatrix} \tag{3.5}$$

此处 $\bar{\xi}=\frac{\xi-\xi_j}{\hat{h}},\ \hat{h}=\frac{1}{n}$,

$\{\alpha\}_j=[\alpha_{j,-1},\ \cdots,\ \alpha_{j,m+1}]^T,\ \{\beta\}_j=[\beta_{j,-1},\ \cdots,\ \beta_{j,m+1}]^T$;

$N_1(t)=1-3t^2+2t^3,\ N_2(t)=t(1-2t+t^2)$,

$N_3(t)=3t^2-2t^3,\ N_4(t)=t(t^2-t)$;

$\Phi(\eta)=[\varphi_{-1}(\eta),\ \varphi_0(\eta),\ \cdots,\ \varphi_{m+1}(\eta)]$,

$$\varphi_i(\eta)=\frac{1}{6\hat{l}^3}\begin{cases}0 & \eta\leqslant\eta_{i-2}\\ (\eta-\eta_{i-2})^3 & \eta_{i-2}\leqslant\eta\leqslant\eta_{i-1}\\ \hat{l}^3+3\hat{l}^2(\eta-\eta_{i-1})+3\hat{l}(\eta-\eta_{i-1})^2-3(\eta-\eta_{i-1})^3 & \eta_{i-1}\leqslant\eta\leqslant\eta_i\\ \hat{l}^3+3\hat{l}^2(\eta_{i+1}-\eta)+3\hat{l}(\eta_{i+1}-\eta)^2-3(\eta_{i+1}-\eta)^3 & \eta_i\leqslant\eta\leqslant\eta_{i+1}\\ (\eta_{i+2}-\eta)^3 & \eta_{i+1}\leqslant\eta\leqslant\eta_{i+2}\\ 0 & \eta_{i+2}\leqslant\eta\end{cases}$$

此处$\hat{l}=\frac{1}{m},\ \eta_i=i\hat{l}$.

为使$\hat{w}$在上下边缘满足所论的边界条件，须将Φ的前面三个和后面三个样条基函数$\varphi_j(\eta)$，$j=-1,\ 0,\ 1,\ m-1,\ m,\ m+1$，按照表 1 作相应的修正，修改后的$\Phi$写成：

$$\Phi(\eta)=[\tilde{\varphi}_{-1},\ \tilde{\varphi}_0,\ \tilde{\varphi}_1,\ \varphi_2,\ \cdots,\ \varphi_{m-2},\ \tilde{\varphi}_{m-1},\ \tilde{\varphi}_m,\ \tilde{\varphi}_{m+1}]$$

亦须对左右两侧的边界条件作相应的处理，兹不详述．

剩下的做法，如形成总刚度方程组等，与有限条的做法相同，亦不详述．

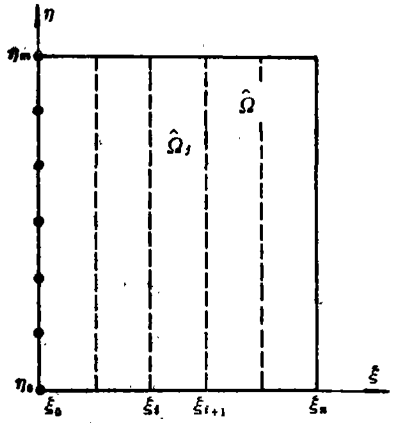

图2　$\hat{\Omega}$: $0\leqslant\xi\leqslant1$, $0\leqslant\eta\leqslant1$, $\xi_j=j\hat{h}$, $\eta_i=i\hat{l}$, $\hat{h}=\frac{1}{n}$, $\hat{l}=\frac{1}{m}$

表1　局部地修改样条基函数，使$\eta=\eta_0$的边界条件成立

边界条件 \ 修改后的样条基函数	$\tilde{\varphi}_{-1}$	$\tilde{\varphi}_0$	$\tilde{\varphi}_1$
自由端	φ_{-1}	φ_0	φ_1
简支端 $\hat{w}(\eta_0)=0$	删掉	$\varphi_0-4\varphi_{-1}$	$\varphi_1-\varphi_{-1}$
固定端 $\hat{w}(\eta_0)=\hat{w}'(\eta_0)=0$	删掉	删掉	$\varphi_1-\frac{1}{2}\varphi_0+\varphi_{-1}$

四、方法的误差估计

1. 误差的能量模估计

我们只讨论整个边界$\partial\Omega$均固支的情形，在其他的边界条件情形下，问题的分析方法与结果都是类似的。另外，为避免引用抽象的Sobolev插值空间理论，将在较强的条件下给出误差估计。

假设真正的挠度函数$w(x,y)\in C^4$，用$w_{n,m}(x,y)$表示以前述方法求得的近似挠度函数。我们将证明，存在常数$c>0$使

$$\left|w-w_{n,m}\right|_{2,\Omega}\leqslant c\left(\left|\frac{\partial^4 w}{\partial x^4}\right|_{0,\Omega}h^2+\left|\frac{\partial^4 w}{\partial x^2\partial y^2}\right|_{0,\Omega}hl+\left|\frac{\partial^4 w}{\partial y^4}\right|_{0,\Omega}l^2\right) \tag{4.1a}$$

$$\left|w-w_{n,m}\right|_{1,\Omega}\leqslant c\left(\left|\frac{\partial^4 w}{\partial x^4}\right|_{0,\Omega}h^2+\left|\frac{\partial^4 w}{\partial x^2\partial y^2}\right|_{0,\Omega}hl+\left|\frac{\partial^4 w}{\partial y^4}\right|_{0,\Omega}l^2\right)(h+l) \tag{4.1b}$$

$$\left|w-w_{n,m}\right|_{0,\Omega}\leqslant c\left(\left|\frac{\partial^4 w}{\partial x^4}\right|_{0,\Omega}h^2+\left|\frac{\partial^4 w}{\partial x^2\partial y^2}\right|_{0,\Omega}hl+\left|\frac{\partial^4 w}{\partial y^4}\right|_{0,\Omega}l^2\right)(h+l)^2 \tag{4.1c}$$

此处$h=a\hat{h}$，$l=\hat{l}$，$|\cdot|_{j,\Omega}$的含意为：

$$\forall\varphi,\quad |\varphi|^2_{j,\Omega}=\iint_\Omega\sum_{n_1+n_2=j}\left(\frac{\partial^j\varphi}{\partial x^{n_1}\partial y^{n_2}}\right)^2dx\,dy$$

在证明(4.1a)，(4.1b)，(4.1c)前，我们指出，$w(x,\ y)$满足边界条件

$$w=\frac{\partial w}{\partial n}=0 \qquad \text{于}\partial\Omega \tag{4.2}$$

和(2.4)式．$w_{n,m}\in V_{n,m}$满足同样的边界条件并且使

$$D\iint_{\Omega}\left\{\frac{\partial^2 w_{n,m}}{\partial x^2}\frac{\partial^2 w^*_{n,m}}{\partial x^2}+\frac{\partial^2 w_{n,m}}{\partial y^2}\frac{\partial^2 w^*_{n,m}}{\partial y^2}+\nu\left(\frac{\partial^2 w_{n,m}}{\partial x^2}\frac{\partial^2 w^*_{n,m}}{\partial y^2}\right.\right.$$
$$\left.\left.+\frac{\partial^2 w^*_{n,m}}{\partial x^2}\frac{\partial^2 w_{n,m}}{\partial y^2}\right)+2(1-\nu)\frac{\partial^2 w_{n,m}}{\partial x\partial y}\frac{\partial^2 w^*_{n,m}}{\partial x\partial y}\right\}dx\,dy$$
$$=\iint_{\Omega} q w^*_{n,m}dx\,dy \qquad \forall w_{n,m}\in V_{n,m} \tag{4.3}$$

$V_{n,m}$表示一个有限维空间，其中之每一元素$v(x,y)$如下述。令

$$\Omega_j=\{(x,y)\mid kbt+jh\leqslant x\leqslant kbt+(j+1)h,\ \ y=bt,\ \ 0\leqslant t\leqslant 1\}$$

（见图1）．作变换

$$x=h\xi+kb\eta+jh,\ \ y=b\eta,\ \ (\xi,\eta)\in\hat{\Omega} \tag{4.4}$$

其逆为

$$\xi=\frac{x-ky-jh}{h},\ \ \eta=\frac{y}{b},\ \ (x,y)\in\Omega_j \tag{4.5}$$

变换(4.5)将Ω_j变为$\hat{\Omega}$．对任何$v(x,y)\in V_{n,m}$，记

$$\hat{v}_j(\xi,\eta)=v(h\xi+kb\eta+jh,\ \ b\eta)$$

函数$\hat{v}_j(\xi,\eta)$可写成：

$$\hat{v}_j(\xi,\eta)=\sum_{i=-1}^{m+1}[\alpha_{j,i}N_1(\xi)+\beta_{j,i}N_2(\xi)+\alpha_{j+1,i}N_3(\xi)+\beta_{j+1,i}N_4(\xi)]\varphi_i(\eta) \tag{4.6}$$

这表$v(x,y)$是分条（分片）的三次Hermite与三次样条的乘积型多项式，但从整体看$v\in C^1(\Omega)$．此外，v还满足边界条件：

$$v=\frac{\partial v}{\partial n}=0 \qquad \text{于}\partial\Omega \tag{4.7}$$

定义三次样条插值算子θ_S与三次Hermite插值算子θ_H：

$\forall\varphi(\xi)$，$(\theta_H\varphi)(\xi)$为三次多项式，满足条件：

$$(\theta_H\varphi)(0)=\varphi(0),\ \ (\theta_H\varphi)(1)=\varphi(1),\ \ (\theta_H\varphi)'(0)=\varphi'(0),\ \ (\theta_H\varphi)'(1)=\varphi'(1)$$

$\forall\psi(\eta)$，$(\theta_S\psi)(\eta)$为三次样条函数，满足条件：

$$(\theta_S\psi)(\eta_j)=\psi(\eta_j) \qquad (j=0,1,\cdots,m)$$
$$(\theta_S\psi)'(\eta_i)=\psi'(\eta_i) \qquad (i=0,m)$$

定义$w(x,\ y)$到$V_{n,m}$的插值投影$\tilde{w}(x,\ y)$：当$(x,\ y)\in\Omega_j$时

$$\tilde{w}_{n,m}(x,\ y)=(\hat{\Pi}\hat{w}_j)(\xi,\ \eta) \tag{4.8}$$

其中$\hat{w}_j(\xi,\ \eta)=w(h\xi+kb\eta+jh,\ b\eta)$，$\xi$，$\eta$由(4.5)表示，$\hat{\Pi}=\theta_H\theta_S=\theta_S\theta_H$．

有了以上准备，现在可着手证明(4.1)诸式．

根据熟知的有限元法的基本估计[6]，我们有

$$|w-w_{n,m}|_{2,\Omega}\leqslant c_0|w-\tilde{w}_{n,m}|_{2,\Omega} \tag{4.9}$$

其中 $c_0=\dfrac{1+\nu}{1-\nu}$.

为证(4.1a)，只须估计(4.9)式的右端．按照$|\cdot|_{2,\Omega_j}$的定义，利用Cauchy不等式，我们有

$$|w-\tilde{w}_{n,m}|^2_{2,\Omega_j}=|w-\hat{\Pi}\hat{w}_j|^2_{2,\Omega},$$

$$=bh\iint_{\hat{\Omega}}\Big\{\Big[\frac{\partial^2(\hat{w}_j-\hat{\Pi}\hat{w}_j)}{h^2\partial\xi^2}\Big]^2+\Big[k^2\frac{\partial^2(\hat{w}_j-\hat{\Pi}\hat{w}_j)}{h^2\partial\xi^2}+2k\frac{\partial^2(\hat{w}_j-\hat{\Pi}\hat{w}_j)}{bh\partial\xi\partial\eta}$$

$$+\frac{\partial^2(\hat{w}_j-\hat{\Pi}\hat{w}_j)}{b^2\partial\eta^2}\Big]^2+2\Big[k\frac{\partial^2(\hat{w}_j-\hat{\Pi}\hat{w}_j)}{h^2\partial\xi^2}+\frac{\partial^2(\hat{w}_j-\hat{\Pi}\hat{w}_j)}{bh\partial\xi\partial\eta}\Big]^2\Big\}d\xi d\eta$$

$$\leqslant bh\iint_{\hat{\Omega}}\Big\{(k^4+2k^3+3k^2+2k+1)\Big[\frac{\partial^2(\hat{w}_j-\hat{\Pi}\hat{w}_j)}{h^2\partial\xi^2}\Big]^2$$

$$+(2k^3+4k^2+4k+2)\Big[\frac{\partial^2(\hat{w}_j-\hat{\Pi}\hat{w}_j)}{bh\,\partial\xi\,\partial\eta}\Big]^2$$

$$+(k^2+2k+1)\Big[\frac{\partial^2(\hat{w}_j-\hat{\Pi}\hat{w}_j)}{b^2\,\partial\xi\,\partial\eta}\Big]^2\Big\}d\xi d\eta$$

$$\leqslant c(k,\ b)\,h\iint_{\hat{\Omega}}\Big\{\Big[\frac{\partial^2(\hat{w}_j-\hat{\Pi}\hat{w}_j)}{h^2\,\partial\xi^2}\Big]^2+\Big[\frac{\partial^2(\hat{w}_j-\hat{\Pi}\hat{w}_j)}{h\,\partial\xi\,\partial\eta}\Big]^2$$

$$+\Big[\frac{\partial^2(\hat{w}_j-\hat{\Pi}\hat{w}_j)}{\partial\eta^2}\Big]^2\Big\}d\xi\,d\eta \tag{4.10}$$

此处$c(k,\ b)$是与k，b有关的常数．以下出现的与k，b有关的常数，与这里的$c(k,\ b)$只差一个常因子，故仍用相同的符号表之．注意$\hat{\Pi}=\theta_S\theta_H=\theta_H\theta_S$，利用熟知的Hermite插值与三次样条插值的误差估计理论[4],[5]，由(4.10)便得：

$$|w-\tilde{w}_{n,m}|^2_{2,\Omega_j}\leqslant c(k,\ b)\Big(\Big|\frac{\partial^4 w}{\partial x^4}\Big|^2_{0,\Omega_j}h^4+\Big|\frac{\partial^4 w}{\partial x^2\partial y^2}\Big|^2_{0,\Omega_j}h^2l^2+\Big|\frac{\partial^4 w}{\partial y^4}\Big|^2_{0,\Omega_j}l^4\Big)$$

因而

$$|w-\tilde{w}_{n,m}|^2_{2,\Omega}\leqslant c(k,b)\Big(\Big|\frac{\partial^4 w}{\partial x^4}\Big|^2_{0,\Omega}h^4+\Big|\frac{\partial^4 w}{\partial x^2\partial y^2}\Big|^2_{0,\Omega}h^2l^2+\Big|\frac{\partial^4 w}{\partial y^4}\Big|^2_{0,\Omega}l^4\Big) \tag{4.11}$$

由此与(4.9)便得(4.1a).

为证(4.1c)，我们考虑薄板Ω在以$(w-w_{n,m})$为外荷载作用下的挠度函数，记为$u(x,y)$，相应的近似挠度函数记为$u_{n,m}(x,\ y)$，按照$u(x,\ y)$的定义（参见(2.4)式），有

$$|w-w_{n,m}|^2_{0,\Omega}=\iint_{\Omega}\Big\{\frac{\partial^2 u}{\partial x^2}\,\frac{\partial^2(w-w_{n,m})}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\,\frac{\partial^2(w-w_{n,m})}{\partial y^2}$$

$$+\nu\Big[\frac{\partial^2 u}{\partial x^2}\,\frac{\partial^2(w-w_{n,m})}{\partial y^2}+\frac{\partial^2 u}{\partial y^2}\,\frac{\partial^2(w-w_{n,m})}{\partial x^2}\Big]$$

$$+2(1-\nu)\frac{\partial^2 u}{\partial x\partial y}\,\frac{\partial^2(w-w_{n,m})}{\partial x\,\partial y}\Big\}\,dx\,dy \tag{4.12}$$

又按w与$w_{n,m}$的定义((2.4)，(4.3))有：

$$\iint_{\Omega}\left\{\frac{\partial^2(w-w_{n,m})}{\partial x^2}\frac{\partial^2 u_{n,m}}{\partial x^2}+\frac{\partial^2(w-w_{n,m})}{\partial y^2}\frac{\partial^2 u_{n,m}}{\partial y^2}\right.$$

$$+\nu\left[\frac{\partial^2(w-w_{n,m})}{\partial x^2}\frac{\partial^2 u_{n,m}}{\partial y^2}+\frac{\partial^2(w-w_{n,m})}{\partial y^2}\frac{\partial^2 u_{n,m}}{\partial x^2}\right]$$

$$\left.+2(1-\nu)\frac{\partial^2(w-w_{n,m})}{\partial x\partial y}\frac{\partial^2 u_{n,m}}{\partial x\partial y}\right\}dx\,dy=0 \tag{4.13}$$

(4.12)与(4.13)结合起来，再利用Cauchy不等式得

$$|w-w_{n,m}|_{0,\Omega}^2=\iint_{\Omega}\left\{\frac{\partial^2(u-u_{n,m})}{\partial x^2}\frac{\partial^2(w-w_{n,m})}{\partial x^2}+\frac{\partial^2(u-u_{n,m})}{\partial y^2}\frac{\partial^2(w-w_{n,m})}{\partial y^2}\right.$$

$$+\nu\left[\frac{\partial^2(u-u_{n,m})}{\partial x^2}\frac{\partial^2(w-w_{n,m})}{\partial y^2}+\frac{\partial^2(u-u_{n,m})}{\partial y^2}\frac{\partial^2(w-w_{n,m})}{\partial x^2}\right]$$

$$\left.+2(1-\nu)\frac{\partial^2(u-u_{n,m})}{\partial x\partial y}\frac{\partial^2(w-w_{n,m})}{\partial x\partial y}\right\}dx\,dy$$

$$\leqslant(1+\nu)|w-w_{n,m}|_{2,\Omega}|u-u_{n,m}|_{2,\Omega} \tag{4.14}$$

对$|u-u_{n,m}|_{2,\Omega}$的估计类似于(4.11)或(4.1a)，同时注意到u是在以$w-w_{n,m}$为外荷载作用下的挠度函数，故有

$$|u-u_{n,m}|_{2,\Omega}\leqslant c^{\frac{1}{2}}(k,b)\left(\left|\frac{\partial^4 u}{\partial x^4}\right|_{0,\Omega}^2 h^4+\left|\frac{\partial^4 u}{\partial x^2\partial y^2}\right|_{0,\Omega}^2 h^2l^2+\left|\frac{\partial^4 u}{\partial y^4}\right|_{0,\Omega}^2 l^4\right)^{\frac{1}{2}}$$

$$\leqslant c^{\frac{1}{2}}(k,b)|w-w_{n,m}|_{0,\Omega}(h+l)^2$$

将此式代入(4.14)式，两端消去公因子$|w-w_{n,m}|_{0,\Omega}$便得(4.1c)。

据Green公式和Cauchy不等式，有

$$|w-w_{n,m}|_{1,\Omega}^2=\iint_{\Omega}\left\{\left[\frac{\partial(w-w_{n,m})}{\partial x}\right]^2+\left[\frac{\partial(w-w_{n,m})}{\partial y}\right]^2\right\}dx\,dy$$

$$\leqslant\sqrt{2}\,|w-w_{n,m}|_{0,\Omega}\cdot|w-w_{n,m}|_{2,\Omega}$$

$$\leqslant\sqrt{2}\,c^2(k,b)|w-w_{n,m}|_{0,\Omega}^2(h+l)^2$$

再利用已证的(4.1a)，(4.1c)便得(4.1b)。

2．**最大模估计**

在前述假设下，有下述的估计式：

$$\|w-w_{n,m}\|_{\Omega}\leqslant ch^{-\frac{1}{2}}(h+l^4)\left(\sum_{n_1+n_2=4}\left\|\frac{\partial^4 w}{\partial x^{n_1}\partial y^{n_2}}\right\|_{\Omega}\right) \tag{4.15}$$

此处$c>0$为常数，$\|\cdot\|_{\Omega}$表示在Ω上的最大模，即

$$\forall v,\quad \|v\|_{\Omega}=\sup_{(x,y)\in\Omega}|v(x,y)|$$

证明　首先利用三角不等式有

$$\|w-w_{n,m}\|_{\Omega}\leqslant\|w-\tilde{w}_{n,m}\|_{\Omega}+\|\tilde{w}_{n,m}-w_{n,m}\|_{\Omega} \tag{4.16}$$

按最大模的定义知必有i_0 $(0\leqslant i_0\leqslant n-1)$使

$$\|w-\tilde{w}_{n,m}\|_{\Omega}=\|w-\tilde{w}_{n,m}\|_{\Omega_{i_0}}\leqslant\|\hat{w}_{i_0}-\theta_B\theta_H\hat{w}_{i_0}\|_{\hat{\Omega}}$$

$$\leqslant \|\hat{w}_{i_0}-\theta_S\hat{w}_{i_0}\|_{\hat{\Omega}}+\|\hat{w}_{i_0}-\theta_H\hat{w}_{i_0}\|_{\hat{\Omega}}+\|\hat{w}_{i_0}-\theta_H\hat{w}_{i_0}-\theta_S(\hat{w}_{i_0}-\theta_H\hat{w}_{i_0})\|_{\hat{\Omega}}$$

利用熟知的Hermite和三次样条插值误差估计于上式右端各项，经整理，我们便得到[4],[5]：

$$\|w-\tilde{w}_{n,m}\|_\Omega \leqslant c\left(\sum_{n_1+n_2=4}\left\|\frac{\partial^4 w}{\partial x^{n_1}\partial y^{n_2}}\right\|_\Omega\right)(k^2+l^2)^2 \tag{4.17}$$

类似地存在 j_0 $(0\leqslant j_0\leqslant n-1)$ 使

$$\|\tilde{w}_{n,m}-w_{n,m}\|_\Omega=\|\tilde{w}_{n,m}-w_{n,m}\|_{\Omega j_0}=\|\hat{\Pi}\hat{w}_{j_0}-\hat{w}_{n,m}\|_{\hat{\Omega}} \tag{4.18}$$

此处 $\hat{w}_{n,m}(\xi,\eta)=w_{n,m}(h\xi+kb\eta+j_0h,\ b\eta)$ 又必存在 k_0 $(0\leqslant k_0\leqslant m-1)$ 使

$$\|\hat{\Pi}\hat{w}_{j_0}-\hat{w}_{n,m}\|_{\hat{\Omega}}=\|\Pi\hat{w}_{j_0}-\hat{w}_{n,m}\|_{\hat{K}_{k_0}} \tag{4.19}$$

此处 $\hat{K}_{k_0}=\{(\xi,\eta)\,|\,0\leqslant\xi\leqslant1,\ \eta_{k_0}\leqslant\eta\leqslant\eta_{k_0+1}\}$

不难得到

$$\begin{aligned}\|\hat{\Pi}\hat{w}_{j_0}-\hat{w}_{n,m}\|_{\hat{K}_{k_0}} &\leqslant c\,|\hat{\Pi}\hat{w}_{j_0}-\hat{w}_{n,m}|_{0,\hat{K}_{k_0}}{}^{1)}\\ &\leqslant ch^{-\frac{1}{2}}|\tilde{w}_{n,m}-w_{n,m}|_{0,\Omega}\\ &\leqslant ch^{-\frac{1}{2}}(|\tilde{w}_{n,m}-w|_{0,\Omega}+|w-w_{n,m}|_{0,\Omega})\end{aligned}$$

利用(4.1c)与(4.11)于上式右端得：

$$\|\hat{\Pi}\hat{w}_{j_0}-\hat{w}_{n,m}\|_{\hat{K}_{k_0}}\leqslant ch^{-\frac{1}{2}}\left(\left|\frac{\partial^4 w}{\partial x^4}\right|_{0,\Omega}h^2+\left|\frac{\partial^4 w}{\partial x^2\partial y^2}\right|_{0,\Omega}hl+\left|\frac{\partial^4 w}{\partial y^4}\right|_{0,\Omega}l^2\right)(h+l)^2 \tag{4.20}$$

(4.16)，(4.17)，(4.18)，(4.19)，(4.20)结合起来即能得到(4.15)。

五、数值例子

我们计算的例子是一块受均匀荷载作用的平行四边形薄板，板的上、下边缘是简支，左右两侧是自由．斜角θ从零度（即矩形板）变化到60°，泊松比取0.3．计算时"沿y方向"取9个节点($m=8$)，沿x方向取7个节点($n=7$)，见图3．我们将图3所示的两个点（A和B）的计算结果和Ramstad[7]的结果进行对比，可以看到用我们的方法计算挠度和力矩均有较高的精度，见图4～5．

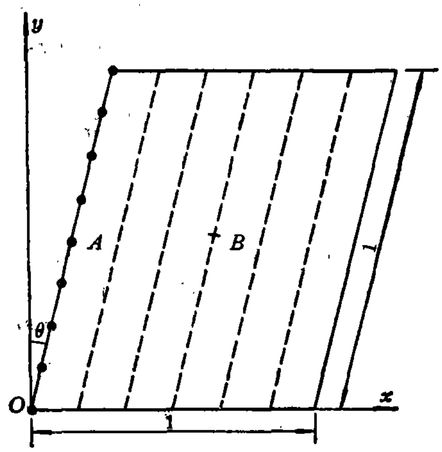

图3 数值例子的网格

1) 有限维空间中的所有模都是等价的．

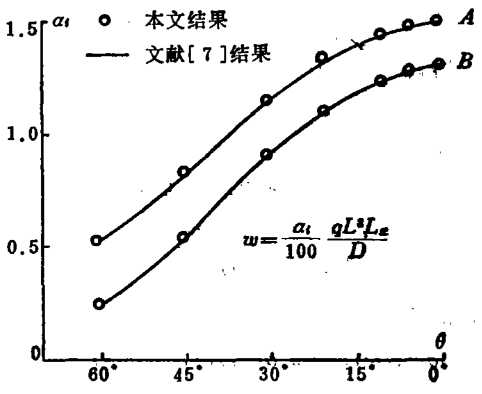

图4 *A*, *B*点挠度计算结果

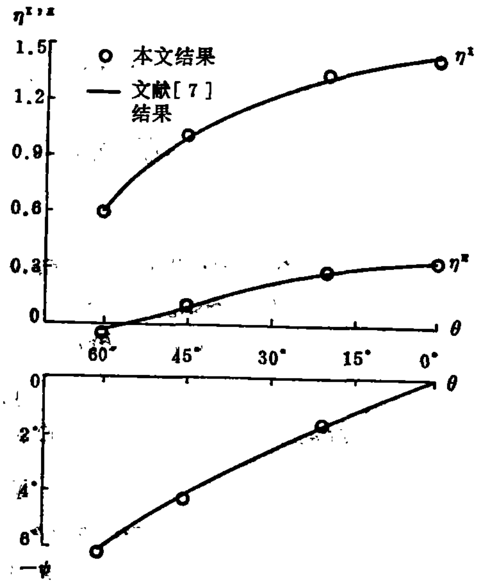

图5 主弯矩系数及其方向计算结果(*B*点)

六、结　论

不论在理论方面还是在数值计算方面，样条有限条法均能用于分析平行四边形的斜板(如前指出亦能用于分析曲边平行四边形板等)．为构造属于C^1类的形函数，样条有限条实际上只涉及w——挠度和θ——转角．这方面它优于标准的协调有限元法，后者须涉及四个未知量(w，θ_x，θ_y，θ_{xy})．然而，两者的误差阶是相同的．

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Analysis of Thin Parallelogram Plates' Bending by Spline-Finite-Strip Method

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Abstract

Spline finite strip has been successfully applied in solving right plates and shells by Cheung et al in 1982. In this paper, the method is extended to the analysis of parallelogram plate. This extension still retains the banded nature of the spline finite strip and only small amount of extra computing effort is required. Furthermore, the discretisation error of the above method is established theoretically as a general case for the spline finite strip method.