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奇异摄动问题的有限元方法

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摘 要

本文利用不同次数的多项式构造了新的有限元子空间，建立了新的有限元格式。证明了格式的收敛性和退化差分方程的稳定性。

一、微分方程问题

考虑在区域 $R+\Gamma$：$(0\leqslant x,\ y\leqslant 1)$ 内的下列微分方程问题：

$$\left.\begin{aligned}&\varepsilon\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)-a_1\frac{\partial u}{\partial x}-a_2\frac{\partial u}{\partial y}-a_3u=f \qquad (x,\ y)\in R\\ &u|_{\Gamma}=0\end{aligned}\right\} \tag{1.1}$$

其中Γ是R的边界，ε是一个小参数满足 $0<\varepsilon\ll 1$，a_1，a_2，a_3 是正的常数，f 是一个给定的函数。

当 $\varepsilon=0$ 时问题(1.1)退化为

$$\left.\begin{aligned}&-a_1\frac{\partial u}{\partial x}-a_2\frac{\partial u}{\partial y}-a_3u=f\\ &u|_{\Gamma_-}=0\end{aligned}\right\} \tag{1.2}$$

其中 $\Gamma_-=\Gamma\cap\{x=0,\ y=0\}$。

二、子空间的构造和有限元格式的建立

定义连续双线性型和线性型如下：

$$\left.\begin{aligned}&a(v,\ w)=\iint_R\{-\varepsilon v_x w_x-\varepsilon v_y w_y-a_1 v_x w-a_2 v_y w-a_3 vw\}dxdy \qquad \forall v,w\in V\\ &f(v)=\iint_R fvdxdy \qquad \forall\ v\in V\end{aligned}\right\} \tag{2.1}$$

于是(1.1)的变分公式为：寻找 $u\in V$ 使得

$$a(u,\ v)=f(v) \qquad \forall v\in V \tag{2.2}$$

成立。

取步长h，τ 和自然数M和N分别使 $Mh=1$，$N\tau=1$。

令 $I_i=[x_{i-1}, x_i]$, $J_j=[y_{j-1}, y_j]$, $K_{ij}=I_i\times J_j$ $(i=1(1)M, j=1(1)N)$

引进参数 θ_1, θ_2, 满足: $0\leqslant\theta_1, \theta_2\leqslant 1$

$$\left.\begin{aligned}&\lim_{\varepsilon\to 0}\theta_1=0, \qquad \lim_{\varepsilon\to 0}\frac{a_1\theta_1 h}{2\varepsilon}=1\\ &\lim_{\varepsilon\to 0}\theta_2=0, \qquad \lim_{\varepsilon\to 0}\frac{a_2\theta_2\tau}{2\varepsilon}=1\end{aligned}\right\} \tag{2.3}$$

令 $I_i^-=[x_{i-1}, x_{i-\theta_1}]$, $I_i^+=[x_{i-\theta_1}, x_i]$

$J_j^-=[y_{j-1}, y_{j-\theta_2}]$, $J_j^+=[y_{j-\theta_2}, y_j]$

则 $K_{ij}=K_{ij}^{--}\cup K_{ij}^{+-}\cup K_{ij}^{-+}\cup K_{ij}^{++}$

其中 $K_{ij}^{--}=I_i^-\times J_j^-$, $K_{ij}^{-+}=I_i^-\times J_j^+$

$K_{ij}^{+-}=I_i^+\times J_j^-$, $K_{ij}^{++}=I_i^+\times J_j^+$

我们按下面的方法构造有限元子空间 $V^h\subset V$。

$$V^h=\begin{cases}V^h|_{K_{ij}^{--}}\in Q_{0,0} & V^h|_{K_{ij}^{-+}}\in Q_{0,1}\\ V^h|_{K_{ij}^{+-}}\in Q_{1,0} & V^h|_{K_{ij}^{++}}\in Q_{1,1}\\ V^h|_\Gamma=0 & (i=1(1)M,\ j=1(1)N)\end{cases} \tag{2.4}$$

其中 $Q_{k,l}$ 表示 x 是 k 次, y 是 l 次的二个变量 x, y 的多项式空间。

基函数 φ_{ij} 由

$$\varphi_{ij}=\begin{cases}\dfrac{(x-x_{i-\theta_1})(y-y_{j-\theta_2})}{(x_i-x_{i-\theta_1})(y_j-y_{j-\theta_2})} & (x, y)\in K_{ij}^{++}\\ (y-y_{j-\theta_2})/(y_j-y_{j-\theta_2}) & (x, y)\in K_{i+1,j}^{-+}\\ \dfrac{(x-x_{i+1})(y-y_{j-\theta_2})}{(x_{i+1-\theta_1}-x_{i+1})(y_j-y_{j-\theta_2})} & (x, y)\in K_{i+1,j}^{++}\\ (x-x_{i-\theta_1})/(x_i-x_{i-\theta_1}) & (x, y)\in K_{i,j+1}^{+-}\\ \dfrac{(x-x_{i-\theta_1})(y-y_{j+1})}{(x_i-x_{i-\theta_1})(y_{j+1-\theta_2}-y_{j+1})} & (x, y)\in K_{i,j+1}^{++}\\ (x-x_{i+1})/(x_{i+1-\theta_1}-x_{i+1}) & (x, y)\in K_{i+1,j+1}^{+-}\\ (y-y_{j+1})/(y_{j+1-\theta_2}-y_{j+1}) & (x, y)\in K_{i+1,j+1}^{-+}\\ \dfrac{(x-x_{i+1})(y-y_{j+1})}{(x_{i+1-\theta_1}-x_{i+1})(y_{j+1-\theta_2}-y_{j+1})} & (x, y)\in K_{i+1,j+1}^{++}\\ 1 & (x, y)\in K_{i+1,j+1}^{--}\\ 0 & \text{其它}\end{cases} \tag{2.5}$$

给出，于是 (2.2) 的离散形式取为:

寻找 $u^h\in V^h$ 使得

$$a(u^h, v^h)=f(v^h) \qquad \forall v^h\in V^h \tag{2.6}$$

成立或者等价地:

寻找 $u^h\in V^h$ 使得

$$a(u^h, \varphi_{ij})=f(\varphi_{ij}) \qquad (i=1(1)M,\ j=1(1)N) \tag{2.7}$$

成立

从基函数 φ_{ij} 的表达式(2.5)可知：

(1) 在$K_{i,j}^{++}$上，　$\varphi_{ij}(x_i, y_j)=1$

(2) 在$K_{i+1,j}^{-+}$上，　$\varphi_{ij}(x_i, y_j)=1$，$\varphi_{ij}(x_{i+1-\theta_1}, y_j)=1$

(3) 在$K_{i+1,j}^{++}$上，　$\varphi_{ij}(x_{i+1-\theta_1}, y_j)=1$

(4) 在$K_{i,j+1}^{+-}$上，　$\varphi_{ij}(x_i, y_j)=1$，$\varphi_{ij}(x_i, y_{j+1-\theta_2})=1$

(5) 在$K_{i,j+1}^{++}$上，　$\varphi_{ij}(x_i, y_{j+1-\theta_2})=1$

(6) 在$K_{i+1,j+1}^{+-}$上，　$\varphi_{ij}(x_{i+1-\theta_1}, y_j)=1$，$\varphi_{ij}(x_{i+1-\theta_1}, y_{j+1-\theta_2})=1$

(7) 在$K_{i+1,j+1}^{-+}$上，　$\varphi_{ij}(x_i, y_{j+1-\theta_2})=1$，$\varphi_{ij}(x_{i+1-\theta_1}, y_{j+1-\theta_2})=1$

(8) 在$K_{i+1,j+1}^{++}$上，　$\varphi_{ij}(x_{i+1-\theta_1}, y_{j+1-\theta_2})=1$

(9) 在$K_{i+1,j+1}^{--}$上，　$\varphi_{ij}(x_i, y_j)=1$，$\varphi_{ij}(x_{i+1-\theta_1}, y_j)=1$，

$\varphi_{ij}(x_i, y_{j+1-\theta_2})=1$，$\varphi_{ij}(x_{i+1-\theta_1}, y_{j+1-\theta_2})=1$

$$(\varphi_{ij})_x=\begin{cases}(y-y_{j-\theta_2})/(x_i-x_{i-\theta_1})(y_j-y_{j-\theta_2}) & (x,y)\in K_{i,j}^{++}\\ 0 & (x,y)\in K_{i+1,j}^{-+}\\ (y-y_{j-\theta_2})/(x_{i+1-\theta_1}-x_{i+1})(y_j-y_{j-\theta_2}) & (x,y)\in K_{i+1,j}^{++}\\ 1/(x_i-x_{i-\theta_1}) & (x,y)\in K_{i,j+1}^{+-}\\ (y-y_{j+1})/(x_i-x_{i-\theta_1})(y_{j+1-\theta_2}-y_{j+1}) & (x,y)\in K_{i,j+1}^{++}\\ 1/(x_{i+1-\theta_1}-x_{i+1}) & (x,y)\in K_{i+1,j+1}^{+-}\\ 0 & (x,y)\in K_{i+1,j+1}^{-+}\\ (y-y_{j+1})/(x_{i+1-\theta_1}-x_{i+1})(y_{j+1-\theta_2}-y_{j+1}) & (x,y)\in K_{i+1,j+1}^{++}\\ 0 & (x,y)\in K_{i+1,j+1}^{--}\\ 0 & \text{其它}\end{cases}\qquad(2.8)$$

不难知道

(1) 在$K_{i,j}^{++}$上，　$\left.\dfrac{\partial\varphi_{ij}}{\partial x}\right|_{(x_i,y_j)}=\dfrac{1}{\theta_1 h}$，$\left.\dfrac{\partial\varphi_{ij}}{\partial x}\right|_{(x_{i-\theta_1},y_j)}=\dfrac{1}{\theta_1 h}$

(2) 在$K_{i+1,j}^{++}$上，　$\left.\dfrac{\partial\varphi_{ij}}{\partial x}\right|_{(x_{i+1},y_j)}=\dfrac{-1}{\theta_1 h}$，$\left.\dfrac{\partial\varphi_{ij}}{\partial x}\right|_{(x_{i+1-\theta_2},y_i)}=\dfrac{-1}{\theta_1 h}$

(3) 在$K_{i,j+1}^{+-}$上，　$\left.\dfrac{\partial\varphi_{ij}}{\partial x}\right|_{(x_i,y_j)}=\dfrac{1}{\theta_1 h}$，$\left.\dfrac{\partial\varphi_{ij}}{\partial x}\right|_{(x_{i-\theta_1},y_j)}=\dfrac{1}{\theta_1 h}$

$\left.\dfrac{\partial\varphi_{ij}}{\partial x}\right|_{(x_i,y_{j+1-\theta_2})}=\dfrac{1}{\theta_1 h}$，$\left.\dfrac{\partial\varphi_{ij}}{\partial x}\right|_{(x_{i-\theta_1},y_{j+1-\theta_2})}=\dfrac{1}{\theta_1 h}$

(4) 在$K_{i,j+1}^{++}$上，　$\left.\dfrac{\partial\varphi_{ij}}{\partial x}\right|_{(x_{i-\theta_1},y_{j+1-\theta_2})}=\dfrac{1}{\theta_1 h}$，$\left.\dfrac{\partial\varphi_{ij}}{\partial x}\right|_{(x_i,y_{j+1-\theta_2})}=\dfrac{1}{\theta_1 h}$

(5) 在$K_{i+1,j+1}^{+-}$上，　$\left.\dfrac{\partial\varphi_{ij}}{\partial x}\right|_{(x_{i+1},y_j)}=\dfrac{-1}{\theta_1 h}$，$\left.\dfrac{\partial\varphi_{ij}}{\partial x}\right|_{(x_{i+1-\theta_1},y_j)}=-\dfrac{1}{\theta_1 h}$

$\left.\dfrac{\partial\varphi_{ij}}{\partial x}\right|_{(x_{i+1},y_{j+1-\theta_2})}=\dfrac{-1}{\theta_1 h}$，$\left.\dfrac{\partial\varphi_{ij}}{\partial x}\right|_{(x_{i+1-\theta_1},y_{j+1-\theta_2})}=\dfrac{-1}{\theta_1 h}$

（6）在$K^{++}_{i+1,j+1}$上，$\left.\frac{\partial\varphi_{ij}}{\partial x}\right|_{(x_{i+1-\theta_1},y_{j+1-\theta_2})}=\frac{-1}{\theta_1 h}$，$\left.\frac{\partial\varphi_{ij}}{\partial x}\right|_{(x_{i+1},y_{j+1-\theta_2})}=\frac{-1}{\theta_1 h}$

另一方面我们有：

（1）在$K^{++}_{i,j}$上，$\left.\frac{\partial\varphi_{i-1,j}}{\partial x}\right|_{(x_i,y_j)}=-\frac{1}{\theta_1 h}$，$\left.\frac{\partial\varphi_{i-1,j}}{\partial x}\right|_{(x_{i-\theta_1},y_j)}=-\frac{1}{\theta_1 h}$

（2）在$K^{++}_{i+1,j}$上，$\left.\frac{\partial\varphi_{i+1,j}}{\partial x}\right|_{(x_{i+1},y_j)}=\frac{1}{\theta_1 h}$，$\left.\frac{\partial\varphi_{i+1,j}}{\partial x}\right|_{(x_{i+1-\theta_1},y_j)}=\frac{1}{\theta_1 h}$

（3）在$K^{+-}_{i,j+1}$上，$\left.\frac{\partial\varphi_{i-1,j}}{\partial x}\right|_{(x_{i-\theta_1},y_{j+1-\theta_2})}=\frac{-1}{\theta_1 h}$，$\left.\frac{\partial\varphi_{i-1,j}}{\partial x}\right|_{(x_i,y_j)}=\frac{-1}{\theta_1 h}$

$$\left.\frac{\partial\varphi_{i-1,j}}{\partial x}\right|_{(x_{i-\theta_1},y_j)}=\frac{-1}{\theta_1 h},\quad \left.\frac{\partial\varphi_{i-1,j}}{\partial x}\right|_{(x_i,y_{j+1-\theta_2})}=\frac{-1}{\theta_1 h}$$

（4）在$K^{++}_{i,j+1}$上，$\left.\frac{\partial\varphi_{i-1,j}}{\partial x}\right|_{(x_{i-\theta_1},y_{j+1-\theta_2})}=\frac{-1}{\theta_1 h}$，$\left.\frac{\partial\varphi_{i-1,j}}{\partial x}\right|_{(x_i,y_{j+1-\theta_2})}=\frac{-1}{\theta_1 h}$

（5）在$K^{--}_{i+1,j+1}$上，$\left.\frac{\partial\varphi_{i+1,j}}{\partial x}\right|_{(x_{i+1-\theta_1},y_{j+1-\theta_2})}=\frac{1}{\theta_1 h}$

$$\left.\frac{\partial\varphi_{i+1,j}}{\partial x}\right|_{(x_{i+1},y_{j+1-\theta_2})}=\frac{1}{\theta_1 h}$$

$$\left.\frac{\partial\varphi_{i+1,j}}{\partial x}\right|_{(x_{i+1-\theta_1},y_j)}=\frac{1}{\theta_1 h},\quad \left.\frac{\partial\varphi_{i+1,j}}{\partial x}\right|_{(x_i,y_j)}=\frac{1}{\theta_1 h}$$

（6）在$K^{++}_{i+1,j+1}$上，$\left.\frac{\partial\varphi_{i+1,j}}{\partial x}\right|_{(x_{i+1-\theta_1},y_{j+1-\theta_2})}=\frac{1}{\theta_1 h}$

$$\left.\frac{\partial\varphi_{i+1,j}}{\partial x}\right|_{(x_{i+1},y_{j+1-\theta_2})}=\frac{1}{\theta_1 h}$$

首先，考虑积分$-\iint_R \varepsilon(u^h)_x(\varphi_{ij})_x dxdy$*，

因为

$$\iint_{K^{++}_{ij}}=-\frac{\varepsilon\theta_1\theta_2 h\tau}{4}\cdot\frac{1}{\theta_1^2 h^2}(2u_{ij}-2u_{i-1,j})$$

$$\iint_{K^{-+}_{i+1,j}}=0,\quad \iint_{K^{-+}_{i+1,j+1}}=0,\quad \iint_{K^{--}_{i+1,j+1}}=0$$

$$\iint_{K^{++}_{i+1,j}}=-\frac{\varepsilon\theta_1\theta_2 h\tau}{4}\cdot\frac{1}{\theta_1^2 h^2}(2u_{i,j}-2u_{i+1,j})$$

$$\iint_{K^{+-}_{i,j+1}}=-\frac{\varepsilon(1-\theta_2)\theta_1 h\tau}{4}\cdot\frac{1}{\theta_1^2 h^2}(4u_{ij}-4u_{i-1,j})$$

$$\iint_{K^{++}_{i,j+1}}=-\frac{\varepsilon\theta_1\theta_2 h\tau}{4}\cdot\frac{1}{\theta_1^2 h^2}(2u_{ij}-2u_{i-1,j})$$

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* 其中积分用数值积分的梯形公式逼近，$u^h(x,y)=\sum_{i=1}^{M-1}\sum_{j=1}^{N-1}u_{ij}\varphi_{ij}(x,y)$。

$$\iint_{K^{+-}_{i+1,j+1}}=-\frac{\varepsilon(1-\theta_2)\theta_1 h\tau}{4}\cdot\frac{1}{\theta_1^2h^2}(4u_{ij}-4u_{i+1,j})$$

$$\iint_{K^{++}_{i+1,j+1}}=-\frac{\varepsilon\theta_1\theta_2 h\tau}{4}\cdot\frac{1}{\theta_1^2h^2}(2u_{ij}-2u_{i+1,j})$$

所以
$$-\iint_B\varepsilon(u^h)_x(\varphi_{ij})_x dxdy=\varepsilon h\tau\frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{\theta_1h^2} \tag{2.9}$$

类似地可得

$$-\iint_B\varepsilon(u^h)_y(\varphi_{ij})_y dxdy=\varepsilon h\tau\frac{u_{i,j+1}-2u_{i,j}+u_{i,j-1}}{\theta_2\tau^2} \tag{2.10}$$

$$-\iint_B a_1(u^h)_x\varphi_{ij}dxdy=-\frac{a_1\tau}{2}(u_{i+1,j}-u_{i-1,j}) \tag{2.11}$$

$$-\iint_B a_2(u^h)_y\varphi_{ij}dxdy=-\frac{a_2h}{2}(u_{i,j+1}-u_{i,j-1}) \tag{2.12}$$

$$-\iint_B a_3u^h\varphi_{ij}dxdy=-a_3h\tau u_{ij} \tag{2.13}$$

$$\begin{aligned}\iint_B f\varphi_{ij}dxdy=\frac{h\tau}{4}\{&f(x_i,y_j)+f(x_{i+1-\theta_1},y_j)\\&+f(x_i,y_{j+1-\theta_2})+f(x_{i+1-\theta_1},y_{j+1-\theta_2})\}\end{aligned} \tag{2.14}$$

综合上面结果可得

$$\left.\begin{aligned}&\varepsilon\frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{\theta_1h^2}+\varepsilon\frac{u_{i,j+1}-2u_{i,j}+u_{i,j-1}}{\theta_2\tau^2}\\&-a_1\frac{u_{i+1,j}-u_{i-1,j}}{2h}-a_2\frac{u_{i,j+1}-u_{i,j-1}}{2\tau}-a_3u_{i,j}\\&=\frac{1}{4}\{f_{i,j}+f_{i+1-\theta_1,j}+f_{i,j+1-\theta_2}+f_{i+1-\theta_1,j+1-\theta_2}\}\\&\qquad(i=1(1)M-1,\ \ j=1(1)N-1)\\&u_{i,0}=u_{i,N}=0\qquad(i=0(1)M)\\&u_{0,j}=u_{M,j}=0\qquad(j=0(1)N)\end{aligned}\right\} \tag{2.15}$$

容易验证当 $\varepsilon\to0$ 时方程 (2.15) 趋向于

$$\begin{aligned}&-a_1\frac{u_{i,j}-u_{i-1,j}}{h}-a_2\frac{u_{i,j}-u_{i,j-1}}{\tau}-a_3u_{i,j}\\&=\frac{1}{4}\{f_{ij}+f_{i+1,j}+f_{i,j+1}+f_{i+1,j+1}\}\end{aligned} \tag{2.16}$$

其中$f_{i,j}=f(x_i,y_j)$，当 $\theta_1=1$，$\theta_2=1$ 时方程(2.15)就是通常的 5 点差分格式.

三、收敛性和退化差分方程的稳定性

如果选取

$$\theta_1=\text{th}\left(\frac{a_1h}{2\varepsilon}\right)\Big/\frac{a_1h}{2\varepsilon},\quad \theta_2=\text{th}\left(\frac{a_2\tau}{2\varepsilon}\right)\Big/\frac{a_2\tau}{2\varepsilon} \tag{3.1}$$

易知$\theta_i(i=1,\ 2)$满足(2.3)且方程(2.15)化为：

$$\frac{a_1h}{2}\mathrm{ctg}\left(\frac{a_1h}{2\varepsilon}\right)\frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{h^2}+\frac{a_2\tau}{a}\mathrm{cth}\left(\frac{a_2\tau}{2\varepsilon}\right)\frac{u_{i,j+1}-2u_{i,j}+u_{i,j-1}}{\tau^2}$$

$$-a_1\frac{u_{i+1,j}-u_{i-1,j}}{2h}-a_2\frac{u_{i,j+1}-u_{i,j-1}}{2\tau}-a_3u_{i,j}$$

$$=\frac{1}{4}\{f_{i,j}+f_{i+1-\theta_1,j}+f_{i,j+1-\theta_2}+f_{i+1-\theta_1,j+1-\theta_2}\}^{(*)} \tag{3.2}$$

令

$$L_\varepsilon u\equiv\varepsilon\left(\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}\right)-a_1\frac{\partial u}{\partial x}-a_2\frac{\partial u}{\partial y}-a_3u \tag{3.3}$$

$$L_\varepsilon^{h,\tau}u\equiv\frac{a_1h}{2}\mathrm{cth}\left(\frac{a_1h}{2\varepsilon}\right)u_{x\bar{x}}+\frac{a_2\tau}{2}\mathrm{cth}\left(\frac{a_2\tau}{2\varepsilon}\right)u_{y\bar{y}}-a_1u_{\hat{x}}-a_2u_{\hat{y}}-a_3u \tag{3.4}$$

首先让我们证明下列引理。

引理 假设$v(x,\ y)$是在网格区域$R_{h,\tau}$上的函数，则

$$|v(x,\ y)|\leqslant\max_{\Gamma_{h,\tau}}|v(x,\ y)|+\frac{3K}{2a}\max_{R_{h,\tau}}\left|L_\varepsilon^{h,\tau}v(x,\ y)\right| \tag{3.5}$$

其中K，a为常数。

证明：令 $a=\min(a_1,\ a_2)$，构造辅助函数

$$W(x,\ y)=\frac{1}{K}\left\{\left(1-\frac{x}{3}\right)x+\left(1-\frac{y}{3}\right)y\right\} \tag{3.6}$$

其中

$$K=\max_{x,y}\left\{\left(1-\frac{x}{3}\right)x+\left(1-\frac{y}{3}\right)y\right\} \tag{3.7}$$

显然

$$0\leqslant W(x,\ y)\leqslant1 \tag{3.8}$$

$$L_\varepsilon^{(h,\tau)}W(x,\ y)=L_\varepsilon W(x,\ y)$$

$$=\varepsilon\frac{\partial^2W}{\partial x^2}+\varepsilon\frac{\partial^2W}{\partial y^2}-a_1\frac{\partial W}{\partial x}-a_2\frac{\partial W}{\partial y}-a_3W$$

$$=-\frac{2\varepsilon}{3K}-\frac{2\varepsilon}{3K}-\frac{a_1}{K}\left(1-\frac{2x}{3}\right)-\frac{a_2}{K}\left(1-\frac{2}{3}y\right)$$

$$-a_3\frac{1}{K}\left\{\left(1-\frac{x}{3}\right)x+\left(1-\frac{y}{3}\right)y\right\}$$

$$<-\frac{1}{3K}(a_1+a_2)\leqslant-\frac{2a}{3K} \tag{3.9}$$

假设不等式(3.5)右端的第一项为m_1，第二项为m_2，$z=m_1+m_2W(x,\ y)\pm v(x,\ y)$则

$$L_\varepsilon^{(h,\tau)}z(x,\ y)=L_\varepsilon^{(h,\tau)}[m_1+m_2W(x,\ y)]\pm L_\varepsilon^{(h,\tau)}v(x,\ y)$$

$$<-\frac{2a}{3K}\cdot\frac{3K}{2a}\max_{R_{h,\tau}}\left|L_\varepsilon^{(h,\tau)}v(x,\ y)\right|\pm L_\varepsilon^{(h,\tau)}v(x,\ y)\leqslant0 \tag{3.10}$$

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* 当h，τ适当小时，差分格式(3.2)是正型格式。

因为 $z(x, y)$ 在 $\Gamma_{h,\tau}$ 上非负，所以在 $R_{h,\tau}$ 上 $z(x, y)\geqslant 0$，即 $|v(x, y)|\leqslant m_1+m_2$。

定理 1 假设右端函数 $f(x, y)$ 有直到二阶为止的连续偏导数，微分方程 (1.1) 的解 $u(x, y)$ 有直到四阶为止的连续偏导数，则对于固定的 $\varepsilon>0$，在 $R_{h,\tau}$ 上差分方程(2.15)的解 $u^{(h,\tau)}(x, y)$ 与微分方程(1.1)的解 $u(x, y)$ 有下面的估计式

$$|u^{(h,\tau)}(x, y)-u(x, y)|\leqslant C(h^2+\tau^2) \tag{3.11}$$

其中 C 是与 x，y 无关的常数。

证明：令 $v^{(h,\tau)}(x, y)=u^{(h,\tau)}(x, y)-u(x, y)$，则

$$\begin{aligned}
L_\varepsilon^{(h,\tau)}v^{(h,\tau)}(x, y)&=\frac{1}{4}\{f(x_i,y_j)+f(x_i+(1-\theta_1)h,y_j)+f(x_i,y_j+(1-\theta_2)\tau)\\
&\quad+f(x_i+(1-\theta_1)h,\ y_j+(1-\theta_2)\tau)\}-L_\varepsilon^{(h,\tau)}u(x, y)\\
&=\frac{1}{4}\{4f(x_i,\ y_j)+f(x_i+(1-\theta_1)h,\ y_j)\\
&\quad+f(x_i,\ y_j+(1-\theta_2)\tau)+f(x_i+(1-\theta_1)h,\ y_j+(1-\theta_2)\tau)\\
&\quad-3f(x_i,\ y_j)\}-L_\varepsilon^{(h,\tau)}u(x, y)\\
&=\varepsilon\left(\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}\right)-a_1\frac{\partial u}{\partial x}-a_2\frac{\partial u}{\partial y}-a_3u\\
&\quad+\frac{1}{4}\{f(x_i+(1-\theta_1)h,\ y_j)+f(x_i,\ y_j+(1-\theta_2)\tau)\\
&\quad+f(x_i+(1-\theta_1)h,\ y_j+(1-\theta_2)\tau)-3f(x_i,\ y_j)\}\\
&\quad-\frac{a_1h}{2}\operatorname{cth}\left(\frac{a_1h}{2\varepsilon}\right)u_{x\bar{x}}-\frac{a_2\tau}{2}\operatorname{cth}\left(\frac{a_2\tau}{2\varepsilon}\right)u_{y\bar{y}}+a_1u_{\hat{x}}+a_2u_{\hat{y}}+a_3u\\
&=\varepsilon\left(\frac{\partial^2u}{\partial x^2}-u_{x\bar{x}}\right)+\left(\varepsilon-\frac{a_1h}{2}\operatorname{cth}\left(\frac{a_1h}{2\varepsilon}\right)\right)u_{x\bar{x}}\\
&\quad+\varepsilon\left(\frac{\partial^2u}{\partial y^2}-u_{y\bar{y}}\right)+\left(\varepsilon-\frac{a_2\tau}{2}\operatorname{cth}\left(\frac{a_2\tau}{2\varepsilon}\right)\right)u_{y\bar{y}}\\
&\quad+a_1\left(u_{\hat{x}}-\frac{\partial u}{\partial x}\right)+a_2\left(u_{\hat{y}}-\frac{\partial u}{\partial y}\right)+\frac{1}{4}\{f(x_i+(1-\theta_1)h,\ y_j)\\
&\quad-f(x_i,\ y_j)+f(x_i,\ y_j+(1-\theta_2)\tau)-f(x_i,\ y_j)\\
&\quad+f(x_i+(1-\theta_1)h,\ y_j+(1-\theta_2)\tau)-f(x_i,\ y_j)\}
\end{aligned} \tag{3.12}$$

按照定理假设的条件有：

$$\left.\begin{aligned}
&\frac{\partial^2u}{\partial x^2}-u_{x\bar{x}}=O(h^2),\qquad \frac{\partial u}{\partial x}-u_{\hat{x}}=O(h^2)\\
&\frac{\partial^2u}{\partial y^2}-u_{y\bar{y}}=O(\tau^2),\qquad \frac{\partial u}{\partial y}-u_{\hat{y}}=O(\tau^2)
\end{aligned}\right\} \tag{3.13}$$

当 ε 固定时易知

$$\left.\begin{aligned}
&\varepsilon-\frac{a_1h}{2}\operatorname{cth}\left(\frac{a_1h}{2\varepsilon}\right)=O(h^2)\\
&\varepsilon-\frac{a_2\tau}{2}\operatorname{cth}\left(\frac{a_2\tau}{2\varepsilon}\right)=O(\tau^2)
\end{aligned}\right\} \tag{3.14}$$

因为参数θ_1，θ_2满足(3.1)，不难证明

$$\left.\begin{aligned} h(1-\theta_1)&=O(h^3)\\ \tau(1-\theta_2)&=O(\tau^3)\end{aligned}\right\} \tag{3.15}$$

则

$$|L_s^{(h,\tau)}v^{(h,\tau)}(x, y)|\leqslant C(h^2+\tau^2)$$

因为

$$\max_{\Gamma_{h,\tau}}|v^{(h,\tau)}(x, y)|=0$$

所以由上述引理得到定理的结论.

利用分离变量法不难证明

定理 2 半显式格式(2.16)是无条件稳定的.

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The Finite Element Method of Singular Perturbation Problem

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Abstract

In this paper, we construct new finite element subspace using polynomials of different degrees and the new finite element scheme is established. The convergence of the scheme and the stability of the reduced difference equation are proved.