应用数学和力学，第5卷第1期（1984年1月）
Applied Mathematics and Mechanics

应用数学和力学编委会编
四川人民出版社出版

Whittaker 方程对非完整力学系统的推广*

梅 凤 翔

(北京工业学院，1983年1月24日收到)

摘　　要

1904年 Whittaker 利用能量积分将一个完整保守力学系统问题降阶为一个带有较少自由度系统问题，并得到了 Whittaker 方程[1].

本文推导对于非完整力学系统的这类方程，并称之为广义 Whittaker 方程；然后把这些方程变换为 Nielsen 形式；最后举例说明新方程的应用.

一、广义 Whittaker 方程的推导

设某力学系统的位形由 n 个广义坐标 $q_s(s=1,2,\cdots,n)$ 确定，系统的运动受到 Четаев 型理想非完整约束[2]

$$\dot q_{\varepsilon+\beta}=\dot q_{\varepsilon+\beta}(\dot q_\sigma,q_s,t)\quad\left(\begin{array}{l}\sigma=1,\ 2,\ \cdots,\ \varepsilon;\ \ \varepsilon=n-g;\\ \beta=1,\ 2,\ \cdots,\ g;\ \ s=1,2,\cdots,n.\end{array}\right)\tag{1.1}$$

如果广义力是有势的，广义坐标下的广义 Чаплыгин 方程为[3][4]

$$\frac{d}{dt}\frac{\partial\widetilde L}{\partial\dot q_\sigma}-\frac{\partial\widetilde L}{\partial q_\sigma}-\sum_{\beta=1}^{g}\frac{\partial L}{\partial\dot q_{\varepsilon+\beta}}\left(\frac{d}{dt}\frac{\partial\dot q_{\varepsilon+\beta}}{\partial\dot q_\sigma}-\frac{\partial\dot q_{\varepsilon+\beta}}{\partial q_\sigma}-\sum_{\gamma=1}^{g}\frac{\partial\dot q_{\varepsilon+\beta}}{\partial q_{\varepsilon+\gamma}}\frac{\partial\dot q_{\varepsilon+\gamma}}{\partial\dot q_\sigma}\right)$$

$$-\sum_{\beta=1}^{g}\frac{\partial\widetilde L}{\partial q_{\varepsilon+\beta}}\frac{\partial\dot q_{\varepsilon+\beta}}{\partial\dot q_\sigma}=0\qquad(\sigma=1,2,\cdots,\varepsilon)\tag{1.2}$$

其中 L 为 Lagrange 函数，而

$$\widetilde L(q_s,\dot q_\sigma,t)=L(q_s,\dot q_\sigma,\dot q_{\varepsilon+\beta}(q_s,\dot q_\sigma,t),t)$$

假定系统属于 Чаплыгин 型：L 和 $\dot q_{\varepsilon+\beta}$ 都不依赖于 $q_{\varepsilon+\beta}$，并且 L 和 $\dot q_{\varepsilon+\beta}$ 都不依赖于时间 t. 在此情形下，约束方程和运动微分方程成为

$$\dot q_{\varepsilon+\beta}=\dot q_{\varepsilon+\beta}(\dot q_\sigma,\ q_\sigma)\tag{1.3}$$

和

$$\frac{d}{dt}\frac{\partial\widetilde L}{\partial\dot q_\sigma}-\frac{\partial\widetilde L}{\partial q_\sigma}-\sum_{\beta=1}^{g}\frac{\partial L}{\partial\dot q_{\varepsilon+\beta}}\left(\frac{d}{dt}\frac{\partial\dot q_{\varepsilon+\beta}}{\partial\dot q_\sigma}-\frac{\partial\dot q_{\varepsilon+\beta}}{\partial q_\sigma}\right)=0\tag{1.4}$$

---

* 薛大为推荐.

令

$$\dot{q}_\nu=\dot{q}_\varepsilon q'_\nu,\ q'_\nu=\frac{dq_\nu}{dq_\varepsilon}\qquad(\nu=1,\ 2,\ \cdots,\ \varepsilon-1)\tag{1.5}$$

以及

$$\Omega(q'_\nu,\ \dot{q}_\varepsilon,\ q_\sigma)=\widetilde{L}(q'_\nu\dot{q}_\varepsilon,\ \dot{q}_\varepsilon,\ q_\sigma)\tag{1.6}$$

我们有

$$\frac{\partial\Omega}{\partial\dot{q}_\varepsilon}=\frac{\partial\widetilde{L}}{\partial\dot{q}_\varepsilon}+\sum_{\nu=1}^{\varepsilon-1}\frac{\partial\widetilde{L}}{\partial\dot{q}_\nu}\frac{\dot{q}_\nu}{\dot{q}_\varepsilon}=\sum_{\sigma=1}^{\varepsilon}\frac{\partial\widetilde{L}}{\partial\dot{q}_\sigma}\frac{\dot{q}_\sigma}{\dot{q}_\varepsilon}\tag{1.7}$$

$$\frac{\partial\Omega}{\partial q'_\nu}=\frac{\partial\widetilde{L}}{\partial\dot{q}_\nu}\dot{q}_\varepsilon\tag{1.8}$$

$$\frac{\partial\Omega}{\partial q_\sigma}=\frac{\partial\widetilde{L}}{\partial q_\sigma}\tag{1.9}$$

如果 $\dot{q}_{\varepsilon+\beta}$ 相对 $\dot{q}_\sigma$ 是一阶齐次的，即

$$\sum_{\sigma=1}^{\varepsilon}\frac{\partial\dot{q}_{\varepsilon+\beta}}{\partial\dot{q}_\sigma}\dot{q}_\sigma-\dot{q}_{\varepsilon+\beta}=0\tag{1.10}$$

那么存在能量积分[5]

$$\sum_{\sigma=1}^{\varepsilon}\dot{q}_\sigma\frac{\partial\widetilde{L}}{\partial\dot{q}_\sigma}-\widetilde{L}=h\tag{1.11}$$

其中 h 为一常数。

由(1.5)和(1.11)可求出

$$\dot{q}_\varepsilon=\dot{q}_\varepsilon(q'_\nu,\ q_\sigma)\tag{1.12}$$

据(1.7)，能量积分可写成

$$\dot{q}_\varepsilon\frac{\partial\Omega}{\partial\dot{q}_\varepsilon}-\Omega=h\tag{1.13}$$

其中 $\dot{q}_\varepsilon$ 由(1.12)确定。

对(1.13)求微分，得到

$$\dot{q}_\varepsilon\frac{\partial^2\Omega}{\partial\dot{q}_\varepsilon^2}\frac{\partial\dot{q}_\varepsilon}{\partial q'_\nu}+\dot{q}_\varepsilon\frac{\partial^2\Omega}{\partial\dot{q}_\varepsilon\partial q'_\nu}-\frac{\partial\Omega}{\partial q'_\nu}=0\tag{1.14}$$

$$\dot{q}_\varepsilon\frac{\partial^2\Omega}{\partial\dot{q}_\varepsilon^2}\frac{\partial\dot{q}_\varepsilon}{\partial q_\sigma}+\dot{q}_\varepsilon\frac{\partial^2\Omega}{\partial\dot{q}_\varepsilon\partial q_\sigma}-\frac{\partial\Omega}{\partial q_\sigma}=0\tag{1.15}$$

令

$$L^*(q'_\nu,\ q_\sigma)=\frac{\partial\Omega}{\partial\dot{q}_\varepsilon}\tag{1.16}$$

对(1.16)求微分，得到

$$\frac{\partial L^*}{\partial q'_\nu}=\frac{\partial^2\Omega}{\partial\dot{q}_\varepsilon^2}\frac{\partial\dot{q}_\varepsilon}{\partial q'_\nu}+\frac{\partial^2\Omega}{\partial\dot{q}_\varepsilon\partial q'_\nu}\tag{1.17}$$

$$\frac{\partial L^*}{\partial q_\sigma}=\frac{\partial^2\Omega}{\partial\dot{q}_\varepsilon^2}\frac{\partial\dot{q}_\varepsilon}{\partial q_\sigma}+\frac{\partial^2\Omega}{\partial\dot{q}_\varepsilon\partial q_\sigma}\tag{1.18}$$

据(1.14)和(1.17)，我们有

$$\frac{\partial L^*}{\partial q'_\nu}=\frac{1}{\dot{q}_\varepsilon}\frac{\partial \Omega}{\partial q'_\nu} \tag{1.19}$$

据(1.15)和(1.18)，我们有

$$\frac{\partial L^*}{\partial q_\sigma}=\frac{1}{\dot{q}_\varepsilon}\frac{\partial \Omega}{\partial q_\sigma} \tag{1.20}$$

将(1.19)与(1.20)和方程(1.8)与(1.9)联合，得到

$$\frac{\partial L^*}{\partial q'_\nu}=\frac{\partial \widetilde{L}}{\partial \dot{q}_\nu},\quad \frac{\partial L^*}{\partial q_\sigma}=\frac{1}{\dot{q}_\varepsilon}\frac{\partial \widetilde{L}}{\partial q_\sigma} \tag{1.21}$$

令

$$\omega_{\varepsilon+\beta}(q'_\nu,\dot{q}_\varepsilon,q_\sigma)=\dot{q}_{\varepsilon+\beta}(q'_\nu\dot{q}_\varepsilon,\dot{q}_\varepsilon,q_\sigma) \tag{1.22}$$

$$\dot{q}^*_{\varepsilon+\beta}(q'_\nu,\ q_\sigma)=\frac{\partial \omega_{\varepsilon+\beta}}{\partial \dot{q}_\varepsilon} \tag{1.23}$$

关系(1.10)可写成

$$\dot{q}_\varepsilon\frac{\partial \omega_{\varepsilon+\beta}}{\partial \dot{q}_\varepsilon}-\omega_{\varepsilon+\beta}=0 \tag{1.24}$$

将(1.23)和(1.24)微分并比较所得关系，我们得到

$$\frac{\partial \dot{q}^*_{\varepsilon+\beta}}{\partial q'_\nu}=\frac{\partial \dot{q}_{\varepsilon+\beta}}{\partial \dot{q}_\nu},\quad \frac{\partial \dot{q}^*_{\varepsilon+\beta}}{\partial q_\sigma}=\frac{1}{\dot{q}_\varepsilon}\frac{\partial \dot{q}_{\varepsilon+\beta}}{\partial q_\sigma} \tag{1.25}$$

将关系(1.21)和(1.25)代入方程(1.4)，得到

$$\frac{d}{dq_\varepsilon}\frac{\partial L^*}{\partial q'_\nu}-\frac{\partial L^*}{\partial q_\nu}-\sum_{\beta=1}^{g}\left(\frac{\partial L}{\partial \dot{q}_{\varepsilon+\beta}}\right)^*\left[\frac{d}{dq_\varepsilon}\left(\frac{\partial \dot{q}^*_{\varepsilon+\beta}}{\partial q'_\nu}\right)-\frac{\partial \dot{q}^*_{\varepsilon+\beta}}{\partial q_\nu}\right]=0 \qquad (\nu=1,2,\cdots,\varepsilon-1) \tag{1.26}$$

其中$\left(\frac{\partial L}{\partial \dot{q}_{\varepsilon+\beta}}\right)^*$为用 q'_ν，q_σ表达 的$\frac{\partial L}{\partial \dot{q}_{\varepsilon+\beta}}$.

我们称方程(1.26)为广义 Whittaker 方程．如果系统是完整的，方程(1.26)成为著名的 Whittaker 方程[1]

$$\frac{d}{dq_n}\frac{\partial L^*}{\partial q'_\nu}-\frac{\partial L^*}{\partial q_\nu}=0 \qquad (\nu=1,\ 2,\ \cdots,\ n-1) \tag{1.27}$$

二、广义 Whittaker 方程变换为 Nielsen 形式的方程

据 $L^*=L^*(q'_\nu,\ q_\sigma)$，我们有

$$\frac{dL^*}{dq_\varepsilon}=\sum_{\nu=1}^{\varepsilon-1}\frac{\partial L^*}{\partial q'_\nu}\frac{dq'_\nu}{dq_\varepsilon}+\sum_{\sigma=1}^{\varepsilon}\frac{\partial L^*}{\partial q_\sigma}\frac{dq_\sigma}{dq_\varepsilon}=\sum_{\nu=1}^{\varepsilon-1}\frac{\partial L^*}{\partial q'_\nu}q''_\nu+\sum_{\nu=1}^{\varepsilon-1}\frac{\partial L^*}{\partial q_\nu}q'_\nu+\frac{\partial L^*}{\partial q_\varepsilon} \tag{2.1}$$

$$\frac{\partial}{\partial q'_\nu}\left(\frac{dL^*}{dq_\varepsilon}\right)=\sum_{\mu=1}^{\varepsilon-1}\frac{\partial^2 L^*}{\partial q'_\nu\partial q'_\mu}q''_\mu+\sum_{\mu=1}^{\varepsilon-1}\frac{\partial^2 L^*}{\partial q'_\nu\partial q_\mu}q'_\mu+\frac{\partial^2 L^*}{\partial q'_\nu\partial q_\varepsilon}+\frac{\partial L^*}{\partial q_\nu}$$

$$=\frac{d}{dq_\epsilon}\left(\frac{\partial L^*}{\partial q'_\nu}\right)+\frac{\partial L^*}{\partial q_\nu} \tag{2.2}$$

类似地有

$$\frac{\partial}{\partial q'_\nu}\left(\frac{d\dot{q}^*_{\epsilon+\beta}}{dq_\epsilon}\right)=\frac{d}{dq_\epsilon}\ \frac{\partial\dot{q}^*_{\epsilon+\beta}}{\partial q'_\nu}+\frac{\partial\dot{q}^*_{\epsilon+\beta}}{\partial q_\nu} \tag{2.3}$$

将(2.2)和(2.3)代入方程(1.26)，我们得到 Nielsen 形式的方程

$$\frac{\partial}{\partial q'_\nu}\left(\frac{dL^*}{dq_\epsilon}\right)-2\ \frac{\partial L^*}{\partial q_\nu}-\sum_{\beta=1}^{g}\left(\frac{\partial L}{\partial\dot{q}_{\epsilon+\beta}}\right)^*\left[\frac{\partial}{\partial q'_\nu}\left(\frac{d\dot{q}^*_{\epsilon+\beta}}{dq_\epsilon}\right)-2\ \frac{\partial\dot{q}^*_{\epsilon+\beta}}{\partial q_\nu}\right]=0$$

$$(\nu=1,\ 2,\ \cdots,\ \epsilon-1) \tag{2.4}$$

方程(2.4)实际上是利用能量积分将非完整保守系统的广义 Nielsen 方程[6] 降阶的结果.

三、实　　例

一雪撬在水平面上运动. 中心C在水平面上投影与雪撬和平面的接触点相重合. 设m和J分别为雪撬质量和它相对其中心的惯性矩. 系统的位置由三个参数确定：接触点的坐标x，y以及雪撬对称轴与Ox的夹角θ. 力函数为$U=-\frac{1}{2}k^2\theta^2$，$k=\text{const}$[8].

Lagrange 函数是

$$L=\frac{1}{2}m(\dot{x}^2+\dot{y}^2)+\frac{1}{2}J\dot{\theta}^2-\frac{1}{2}k^2\theta^2 \tag{3.1}$$

约束方程为

$$\dot{y}=\dot{x}\,\mathrm{tg}\theta \tag{3.2}$$

于是

$$\tilde{L}=\frac{1}{2}m\dot{x}^2(1+\mathrm{tg}^2\theta)+\frac{1}{2}J\dot{\theta}^2-\frac{1}{2}k^2\theta^2 \tag{3.3}$$

能量积分为

$$\frac{1}{2}m\dot{x}^2(1+\mathrm{tg}^2\theta)+\frac{1}{2}J\dot{\theta}^2+\frac{1}{2}k^2\theta^2=h \tag{3.4}$$

由此得出

$$\dot{x}^2=\frac{2h-k^2\theta^2}{m(1+\mathrm{tg}^2\theta)+J\theta'^2},\quad \theta'=\frac{d\theta}{dx} \tag{3.5}$$

据(1.6)和(3.3)，得到

$$\Omega=\frac{1}{2}\dot{x}^2[m(1+\mathrm{tg}^2\theta)+J\theta'^2]-\frac{1}{2}k^2\theta^2 \tag{3.6}$$

据(1.16)、(3.5)及(3.6)，我们有

$$L^*=\frac{\partial\Omega}{\partial\dot{x}}=\pm\sqrt{2h-k^2\theta^2}\cdot\sqrt{m(1+\mathrm{tg}^2\theta)+J\theta'^2} \tag{3.7}$$

由(1.23)及(3.2)，得到

$$\dot{y}^*=\frac{\partial(\dot{x}\,\mathrm{tg}\theta)}{\partial\dot{x}}=\mathrm{tg}\theta \tag{3.8}$$

方程(1.26)给出

$$\frac{d}{dx}\frac{\partial L^*}{\partial\theta'}-\frac{\partial L^*}{\partial\theta}-\left(\frac{\partial L}{\partial\dot{y}}\right)^*\left(\frac{d}{dx}\frac{\partial\dot{y}^*}{\partial\theta'}-\frac{\partial\dot{y}^*}{\partial\theta}\right)=0 \tag{3.9}$$

将(3.7)和(3.8)代入(3.9)，我们得到

$$\frac{d}{dx}\left\{\pm(2h-k^2\theta^2)^{\frac{1}{2}}J\theta'[J\theta'^2+m(1+\mathrm{tg}^2\theta)]^{-\frac{1}{2}}\right\}-\left\{\pm(2h-k^2\theta^2)^{-\frac{1}{2}}(-k^2\theta)[J\theta'^2\right.$$

$$\left.+m(1+\mathrm{tg}^2\theta)]^{\frac{1}{2}}\pm(2h-k^2\theta^2)^{\frac{1}{2}}[J\theta'^2+m(1+\mathrm{tg}^2\theta)]^{-\frac{1}{2}}\frac{m\mathrm{tg}\theta}{\cos^2\theta}\right\}$$

$$+m\mathrm{tg}\theta\{\pm(2h-k^2\theta^2)[J\theta'^2+m(1+\mathrm{tg}^2\theta)]^{-\frac{1}{2}}\}=0$$

简化后得

$$J(2h-k^2\theta^2)\theta''+J[-(2h-k^2\theta^2)\mathrm{tg}\theta+k^2\theta]\theta'^2+mk\theta(1+\mathrm{tg}^2\theta)=0 \tag{3.10}$$

此方程的积分为

$$\theta'=\frac{1}{C\cos\theta}\sqrt{\frac{-k^2\theta^2+2h-mC^2}{J}} \tag{3.11}$$

其中 C 是一常数。

将(3.11)代入(3.5)，我们得到

$$\dot{x}=C\cos\theta \tag{3.12}$$

由(3.2)及(3.12)，得

$$\dot{y}=C\sin\theta \tag{3.13}$$

由(3.11)及(3.12)，得

$$\dot{\theta}=\sqrt{\frac{-k^2\theta^2+2h-mC^2}{J}}$$

于是

$$\ddot{\theta}=-\frac{k^2}{J}\theta \tag{3.14}$$

积分(3.14)两次，得

$$\theta=a\cos\left(\frac{k}{\sqrt{J}}t+\varphi\right) \tag{3.15}$$

其中 a 和 φ 是常数。

参 考 文 献

[1] Whittaker, E. T., *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies*, Combrige University Press (1965)

[2] Четаев, Н. Г., *О Принципе Гаусса*, Изв. Казан Физ. Мат., Сос. 6(1933), 68—71.

[3] Новоселов, В. С., *Вариационные Методы в Механике*, Ленинград(1966).

[4] 梅凤翔，非完整系统力学中的交换关系，力学与实践，1，3(1979)

[5] Mei Feng-xiang, Nouvelles équations du mouvement des systèmes mécaniques non-holonomes, Thèse d'Etat, Mai(1982), Nantes, France

[6] Losco, L., M. Langlois, Seminaires de l'Université de Zürich, Mai-Juin(1976).

Extension of the Whittaker Equations to Non-Holonomic Mechanical Systems

Mei Feng-xiang

(Beijing Institute of Technology, Beijing)

Abstract

In 1904, using the energy integral Whittaker studied the reduction of a dynamical problem to a problem with fewer degrees of freedom for the holonomic conservative systems and obtained the Whittaker equations[1].

In this articfe, the Whittaker equations are extended to non-holonomic systems and the generalized Whittaker equations are obtained And then these equations are transformed to Nielsen s form. Finally an example is given.