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环形和圆形薄板在各种支承条件下的非对称非线性弯曲问题（Ⅰ）

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摘　要

本文研究环形和圆形薄板在各种支承条件下的非对称非线性弯曲问题．应用［7］中提出的摄动方法导出一致有效的渐近解．

一、引　言

早在1948年，钱伟长[1]就应用摄动方法研究周边固支的圆形薄板在均匀法向载荷下的非线性弯曲问题，导出理论与实验相一致的渐近解．奠定了依据薄膜解来研究薄板大挠度问题的基础．1954年，钱伟长和叶开沅[2]又研究圆形薄板在其它支承条件下的非线性弯曲问题．1956年，Bromberg[3]又研究了同样的问题．他们都是研究轴对称的情形．

1961年，Fife[4]开始研究非轴对称的情形；研究周边固支的薄板，当周边作用有均匀的法向力和切向力为零的情形．1964年，Срубщик[5]又研究同样的问题，只是假设周边作用有非均匀的法向力和切向力．

本文应用作者在［6］和［7］中提出的方法，研究环形和圆形的薄板在各种支承条件下的非线性弯曲问题．依据薄膜解构造出弯曲问题的N阶形式渐近解．

二、环形薄板

引进极坐标系(r,θ)，我们知道薄板弯曲的挠度$w(r,\theta)$和应力函数$F(r,\theta)$确定于下面的von Kármán方程[8]

$$\left.\begin{aligned}\Delta^2 w&=\frac{h}{D}L(w,F)+\frac{q}{D}\\ \Delta^2 F&=-\frac{E}{2}L(w,w)\end{aligned}\right\}\tag{2.1}$$

其中E是材料的弹性模数，h是板的厚度，$D=\dfrac{Eh^3}{12(1-\nu^2)}$是板的弯曲刚度，$\nu$是泊松比，

和

$$\Delta \equiv \frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \tag{2.2}$$

$$L(w,F)\equiv\frac{\partial^2 w}{\partial r^2}\left(\frac{1}{r}\frac{\partial F}{\partial r}+\frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2}\right)+\frac{\partial^2 F}{\partial r^2}\left(\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)$$

$$-2\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial F}{\partial \theta}\right)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial \theta}\right) \tag{2.3}$$

再以 r_0 表示环形板的内缘的半径，r_1 表示外缘的半径，引进无量纲量：

$$\tilde{w}=\frac{w}{r_1},\qquad \tilde{r}=\frac{r}{r_1},\qquad \tilde{F}=\frac{F}{Er_1^2},\qquad \tilde{q}=\frac{r_1 q}{hE}$$

方程（2.1）化为（略去字母上的"～"号）

$$\left.\begin{aligned}&\Pi_\varepsilon(w,F)\equiv\varepsilon^2\Delta^2 w-L(w,F)=q\\&\Pi(w,F)\equiv\Delta^2 F+\frac{1}{2}L(w,w)=0\end{aligned}\right\} \tag{2.4}$$

其中 $\varepsilon^2=\dfrac{h^2}{12(1-\nu^2)r_1^3}$．为了讨论确定起见，假设给出的边界条件是

$$w\Big|_{r=b}=f_0(\theta),\qquad \frac{\partial w}{\partial r}\Big|_{r=t}=g_0(\theta) \tag{2.5}$$

$$w\Big|_{r=1}=f_1(\theta),\quad \left[\frac{\partial^2 w}{\partial r^2}+\nu\left(\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)\right]\Big|_{r=1}=g_1(\theta) \tag{2.6}$$

$$\left(\frac{1}{r}\frac{\partial F}{\partial r}+\frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2}\right)\Big|_{r=b}=T_0(\theta),\ \frac{\partial^2 F}{\partial r^2}\Big|_{r=b}=S_0(\theta) \tag{2.7}$$

$$\left(\frac{1}{r}\frac{\partial F}{\partial r}+\frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2}\right)\Big|_{r=1}=T_1(\theta),\ \frac{\partial^2 F}{\partial r^2}\Big|_{r=1}=S_1(\theta) \tag{2.8}$$

其中 $b=\dfrac{r}{r_1}$．对于其它形式的边界条件可以类似地处理．

1．微分算子的展开式

同[7]一样地，在 $r=b$ 的邻域引进两变量 ξ 和 η：

$$\xi=\frac{u(r,\theta)}{\varepsilon},\quad \eta=r \tag{2.9}$$

将关于 r 的偏导数代替以关于 ξ 和 η 的偏导数：

$$\frac{\partial}{\partial r}=\varepsilon^{-i}(\delta_{i,0}+\varepsilon\delta_{i,1}+\cdots+\varepsilon^i\delta_{i,i}),\ (i=1,2,3,4) \tag{2.10}$$

其中

$$\delta_{1,0}=u_r\frac{\partial}{\partial \xi},\quad \delta_{1,1}=\frac{\partial}{\partial \eta};$$

$$\delta_{2,0}=u_r^2\frac{\partial^2}{\partial \xi^2},\ \delta_{2,1}=2u_r\frac{\partial^2}{\partial \xi\partial \eta}+u_{rr}\frac{\partial}{\partial \xi},\quad \delta_{2,2}=\frac{\partial^2}{\partial \eta^2}$$

$$\delta_{3,0}=u_r^3\frac{\partial^3}{\partial\xi^3},\quad \delta_{3,1}=3u_r^2\frac{\partial^3}{\partial\xi^2\partial\eta}+3u_ru_{rr}\frac{\partial^2}{\partial\xi^2}$$

$$\delta_{3,2}=3u_r\frac{\partial^3}{\partial\xi\partial\eta^2}+3u_{rr}\frac{\partial^2}{\partial\xi\partial\eta}+u_{rrr}\frac{\partial}{\partial\xi},\quad \delta_{3,3}=\frac{\partial^3}{\partial\eta^3}$$

$$\delta_{4,0}=u_r^4\frac{\partial^4}{\partial\xi^4},\quad \delta_{4,1}=4u_r^3\frac{\partial^4}{\partial\xi^3\partial\eta}+6u_r^2u_{rr}\frac{\partial^3}{\partial\xi^3}$$

$$\delta_{4,2}=6u_r^2\frac{\partial^4}{\partial\xi^2\partial\eta^2}+12u_ru_{rr}\frac{\partial^3}{\partial\xi^2\partial\eta}+4u_ru_{rrr}\frac{\partial^2}{\partial\xi^2}+3u_{rr}^2\frac{\partial^2}{\partial\xi^2}$$

$$\delta_{4,3}=4u_r\frac{\partial^4}{\partial\xi\partial\eta^3}+6u_{rr}\frac{\partial^3}{\partial\xi\partial\eta^2}+4u_{rrr}\frac{\partial^2}{\partial\xi\partial\eta}+u_{rrrr}\frac{\partial}{\partial\xi}$$

$$\delta_{4,4}=\frac{\partial^4}{\partial\eta^4}$$

得到算子Δ^2的展开式：

$$\Delta^2\equiv\varepsilon^{-4}(D_0+\varepsilon D_1+\varepsilon^2D_2+\varepsilon^3D_3+\varepsilon^4D_4)\tag{2.11}$$

其中

$$D_0\equiv\delta_{4,0},\quad D_1\equiv\delta_{4,1}+\frac{2}{\eta}\delta_{3,0}$$

$$D_2\equiv\delta_{4,2}+\frac{2}{\eta^2}\delta_{2,0}\frac{\partial^2}{\partial\theta^2}+\frac{2}{\eta}\delta_{3,1}-\frac{1}{\eta^2}\delta_{2,0}$$

$$D_3\equiv\delta_{4,3}+\frac{2}{\eta^2}\delta_{2,1}\frac{\partial^2}{\partial\theta^2}-\frac{2}{\eta^3}\delta_{1,0}\frac{\partial^2}{\partial\theta^2}+\frac{2}{\eta}\delta_{3,2}-\frac{1}{\eta^2}\delta_{2,1}+\frac{1}{\eta^3}\delta_{1,0}$$

$$D_4\equiv\delta_{4,4}+\frac{2}{\eta^2}\delta_{2,2}\frac{\partial^2}{\partial\theta^2}+\frac{1}{\eta^4}\frac{\partial^4}{\partial\theta^4}-\frac{2}{\eta^3}\delta_{1,1}\frac{\partial^2}{\partial\theta^2}+\frac{2}{\eta}\delta_{3,3}-\frac{1}{\eta^2}\delta_{2,2}$$
$$+\frac{4}{\eta^2}\frac{\partial^2}{\partial\theta^2}+\frac{1}{\eta^3}\delta_{1,1}$$

又从 (2.3) 式可以得到 $L(w(r,\theta),\ v(\xi,\eta,\theta))$ 和 $L(h(\xi,\eta,\theta),\ v(\xi,\eta,\theta))$的展开式：

$$L(w(r,\theta),\ v(\xi,\eta,\theta))=\varepsilon^{-2}\sum_{i=0}^{2}\varepsilon^iM_i(w,v)$$

$$L(h(\xi,\eta,\theta),v(\xi,\eta,\theta))=\varepsilon^{-3}\sum_{i=0}^{3}\varepsilon^iN_i(h,v)$$

其中

$$M_0(w,v)\equiv\left(\frac{w_r}{\eta}+\frac{w_{\theta\theta}}{\eta^2}\right)\delta_{2,0}v$$

$$M_1(w,v)\equiv\left(\frac{w_r}{\eta}+\frac{w_{\theta\theta}}{\eta^2}\right)\delta_{2,1}v+\frac{w_{rr}}{\eta}\delta_{1,0}v-2\left(\frac{-w_\theta}{\eta^2}+\frac{w_{r\theta}}{\eta}\right)\frac{\delta_{1,0}v_\theta}{\eta}$$

$$M_2(w,v)\equiv\left(\frac{w_r}{\eta}+\frac{w_{\theta\theta}}{\eta^2}\right)\delta_{2,2}v+w_{rr}\left(\frac{\delta_{1,1}v}{\eta}+\frac{v_{\theta\theta}}{\eta^2}\right)$$

$$-2\left(\frac{-w_\theta}{\eta^2}+\frac{w_{r\theta}}{\eta}\right)\left(\frac{-v_\theta}{\eta^2}+\frac{\delta_{1,1}v_\theta}{\eta}\right)$$

$$N_0(h,v)\equiv\delta_{2,0}h\frac{\delta_{1,0}v}{\eta}+\delta_{2,0}v\frac{\delta_{1,0}h}{\eta}$$

$$N_1(h,v)\equiv\delta_{2,0}h\left(\frac{\delta_{1,1}v}{\eta}+\frac{v_{\theta\theta}}{\eta^2}\right)+\delta_{2,1}h\frac{\delta_{1,0}v}{\eta}+\delta_{2,0}v\left(\frac{\delta_{1,1}h}{\eta}+\frac{h_{\theta\theta}}{\eta^2}\right)$$

$$+\delta_{2,1}v\frac{\delta_{1,0}h}{\eta}-2\frac{\delta_{1,0}h_\theta}{\eta}\frac{\delta_{1,0}v_\theta}{\eta}$$

$$N_2(h,v)\equiv\delta_{2,1}h\left(\frac{\delta_{1,1}v}{\eta}+\frac{v_{\theta\theta}}{\eta^2}\right)+\delta_{2,2}h\frac{\delta_{1,0}v}{\eta}+\delta_{2,1}v\left(\frac{\delta_{1,1}h}{\eta}+\frac{h_{\theta\theta}}{\eta^2}\right)$$

$$+\delta_{2,2}v\frac{\delta_{1,0}h}{\eta}-2\left[\frac{\delta_{1,0}h_\theta}{\eta}\left(\frac{\delta_{1,1}v_\theta}{\eta}-\frac{v_\theta}{\eta^2}\right)+\frac{\delta_{1,0}v_\theta}{\eta}\left(\frac{\delta_{1,1}h_\theta}{\eta}-\frac{h_\theta}{\eta^2}\right)\right]$$

$$N_3(h,v)\equiv\delta_{2,2}h\left(\frac{\delta_{1,1}v}{\eta}+\frac{v_{\theta\theta}}{\eta^2}\right)+\delta_{2,2}v\left(\frac{\delta_{1,1}h}{\eta}+\frac{h_{\theta\theta}}{\eta^2}\right)$$

$$-2\left(\frac{\delta_{1,1}h_\theta}{\eta}-\frac{h_\theta}{\eta^2}\right)\left(\frac{\delta_{1,1}v_\theta}{\eta}-\frac{v_\theta}{\eta^2}\right)$$

类似地在 $r=1$ 的邻域引进两变量：

$$\tilde{\xi}=\frac{\tilde{u}(r,\theta)}{\varepsilon},\qquad \tilde{\eta}=r$$

可以得到相应的展开式：

$$\Delta^2=\varepsilon^{-4}(\tilde{D}_0+\varepsilon\tilde{D}_1+\cdots+\varepsilon^4\tilde{D}_4)$$

$$L(w(r,\theta),\ \tilde{v}(\tilde{\xi},\tilde{\eta},\theta))=\varepsilon^{-2}\sum_{i=0}^{2}\varepsilon^i\tilde{M}_i(w,\tilde{v})$$

$$L(h(\xi,\eta,\theta),\tilde{v}(\tilde{\xi},\tilde{\eta},\theta))=\varepsilon^{-3}\sum_{i=0}^{3}\varepsilon^i\tilde{N}_i(h,\tilde{v})$$

$$L(\tilde{h}(\tilde{\xi},\tilde{\eta},\theta),\tilde{v}(\tilde{\xi},\tilde{\eta},\theta))=\varepsilon^{-3}\sum_{i=0}^{3}\varepsilon^i\tilde{\tilde{N}}_i(\tilde{h},\tilde{v})$$

其中 $\tilde{\delta}_{1,0}=\tilde{u}_r\dfrac{\partial}{\partial\tilde{\xi}}$, $\tilde{D}_1=\tilde{\delta}_{4,1}+\dfrac{2}{\tilde{\eta}}\tilde{\delta}_{3,0}$, $\tilde{M}_0(w,\tilde{v})=\left(\dfrac{w_r}{\tilde{\eta}}+\dfrac{w_{\theta\theta}}{\tilde{\eta}^2}\right)\tilde{\delta}_{2,0}\tilde{v}$,

$\tilde{N}_0(h,\tilde{v})=\delta_{2,0}h\dfrac{\tilde{\delta}_{1,0}\tilde{v}}{\tilde{\eta}}+\tilde{\delta}_{2,0}\tilde{v}\dfrac{\delta_{1,0}h}{\eta}$, $\tilde{\tilde{N}}_0(\tilde{h},\tilde{v})=\tilde{\delta}_{2,0}\tilde{h}\dfrac{\tilde{\delta}_{1,0}\tilde{v}}{\tilde{\eta}}+\tilde{\delta}_{2,0}\tilde{v}\dfrac{\tilde{\delta}_{1,0}\tilde{h}}{\tilde{\eta}}$, …等

2. 递推方程和边界条件

假设挠度和应力函数的N阶近似式是

$$W_N(r,\theta;\varepsilon)=\sum_{n=0}^{N}\varepsilon^n w_n(r,\theta)+\sum_{n=0}^{N}\varepsilon^{n+\alpha_1}v_n(\xi,\eta,\theta)+\sum_{n=0}^{N}\varepsilon^{n+\alpha_2}\tilde{v}_n(\tilde{\xi},\tilde{\eta},\theta) \quad (2.12)$$

$$F_N(r,\theta;\varepsilon)=\sum_{n=0}^{N}\varepsilon^n f_n(r,\theta)+\sum_{n=0}^{N}\varepsilon^{n+\beta_1}h_n(\xi,\eta,\theta)+\sum_{n=0}^{N}\varepsilon^{n+\beta_2}\tilde{h}_n(\tilde{\xi},\tilde{\eta},\theta) \quad (2.13)$$

其中α_1,α_2，β_1，β_2是待定常数，v_n和h_n是待求的在$r=b$的邻域是指数型减小的函数（称这类函数为边界层型函数），$\tilde{v}_n$和$\tilde{h}_n$是$r=1$的邻域的边界层型函数.

将(2.12)和(2.13)式代入方程(2.4)的左端，考虑到边界层型函数的性质得

$$\begin{aligned}
\Pi_\varepsilon(W_N,F_N)\equiv&\left\{\varepsilon^2\Delta^2\left(\sum_{n=0}^{N}\varepsilon^n w_n\right)-L\left(\sum_{n=0}^{N}\varepsilon^n w_n,\ \sum_{n=0}^{N}\varepsilon^n f_n\right)\right\}\\
&+\left\{\varepsilon^{-2}\sum_{i=0}^{4}\varepsilon^i D_i\left(\sum_{n=0}^{N}\varepsilon^{n+\alpha_1}v_n\right)-\varepsilon^{-2}\sum_{i=0}^{2}\varepsilon^i\left[M_i\left(\sum_{n=0}^{N}\varepsilon^n w_n,\ \sum_{n=0}^{N}\varepsilon^{n+\beta_1}h_n\right)\right.\right.\\
&\left.\left.+M_i\left(\sum_{n=0}^{N}\varepsilon^n f_n,\ \sum_{n=0}^{N}\varepsilon^{n+\alpha_1}v_n\right)\right]-\varepsilon^{-3}\sum_{i=0}^{3}\varepsilon^i N_i\left(\sum_{n=0}^{N}\varepsilon^{n+\alpha_1}v_n,\ \sum_{n=0}^{N}\varepsilon^{n+\beta_1}h_n\right)\right\}\\
&+\left\{\varepsilon^{-2}\sum_{i=0}^{4}\varepsilon^i\tilde{D}_i\left(\sum_{n=0}^{N}\varepsilon^{n+\alpha_2}\tilde{v}_n\right)-\varepsilon^{-2}\sum_{i=0}^{2}\varepsilon^i\left[\tilde{M}_i\left(\sum_{n=0}^{N}\varepsilon^n w_n,\ \sum_{n=0}^{N}\varepsilon^{n+\beta_2}\tilde{h}_n\right)\right.\right.\\
&\left.\left.+\tilde{M}_i\left(\sum_{n=0}^{N}\varepsilon^n f_n,\ \sum_{n=0}^{N}\varepsilon^{n+\alpha_2}\tilde{v}_n\right)\right]-\varepsilon^{-3}\sum_{i=0}^{3}\varepsilon^i\tilde{\tilde{N}}_i\left(\sum_{n=0}^{N}\varepsilon^{n+\alpha_2}\tilde{v}_n,\ \sum_{n=0}^{N}\varepsilon^{n+\beta_2}\tilde{h}_n\right)\right\}\\
&-\varepsilon^{-3}\sum_{i=0}^{3}\varepsilon^i\left[\tilde{N}_i\left(\sum_{n=1}^{N}\varepsilon^{n+\alpha_1}v_n,\ \sum_{n=0}^{N}\varepsilon^{n+\beta_2}\tilde{h}_n\right)+\tilde{N}_i\left(\sum_{n=0}^{N}\varepsilon^{n+\alpha_2}\tilde{v}_n,\ \sum_{n=0}^{N}\varepsilon^{n+\beta_1}h_n\right)\right]
\end{aligned} \quad (2.14)$$

$$\begin{aligned}
\Pi(W_N,F_N)=&\left\{\Delta^2\left(\sum_{n=0}^{N}\varepsilon^n f_n\right)+\frac{1}{2}L\left(\sum_{n=0}^{N}\varepsilon^n w_n,\ \sum_{n=0}^{N}\varepsilon^n w_n\right)\right\}\\
&+\left\{\varepsilon^{-4}\sum_{i=0}^{4}\varepsilon^i D_i\left(\sum_{n=0}^{N}\varepsilon^{n+\beta_1}h_n\right)+\frac{1}{2}\varepsilon^{-3}\sum_{i=0}^{3}\varepsilon^i N_i\left(\sum_{n=0}^{N}\varepsilon^{n+\alpha_1}v_n,\ \sum_{n=0}^{N}\varepsilon^{n+\alpha_1}v_n\right)\right.\\
&\left.+\varepsilon^{-2}\sum_{i=0}^{2}\varepsilon^i M_i\left(\sum_{n=0}^{N}\varepsilon^n w_n,\ \sum_{n=0}^{N}\varepsilon^{n+\alpha_1}v_n\right)\right\}\\
&+\left\{\varepsilon^{-4}\sum_{i=0}^{4}\varepsilon^i\tilde{D}_i\left(\sum_{n=0}^{N}\varepsilon^{n+\beta_2}\tilde{h}_n\right)+\frac{1}{2}\varepsilon^{-3}\sum_{i=0}^{3}\varepsilon^i\tilde{\tilde{N}}_i\left(\sum_{n=0}^{N}\varepsilon^{n+\alpha_2}\tilde{v}_n,\ \sum_{n=0}^{N}\varepsilon^{n+\alpha_2}\tilde{v}_n\right)\right.
\end{aligned}$$

$$+\varepsilon^{-2}\sum_{i=0}^{2}\varepsilon^{i}\widetilde{M}_{i}\Big(\sum_{n=0}^{N}\varepsilon^{n}w_{n},\ \sum_{n=0}^{N}\varepsilon^{n+\alpha_2}\tilde{v}_{n}\Big)\Big\}-\varepsilon^{-3}\sum_{i=0}^{3}\varepsilon^{i}\widetilde{N}_{i}\Big(\sum_{n=0}^{N}\varepsilon^{n+\alpha_1}v_{n},\ \sum_{n=0}^{N}\varepsilon^{n+\alpha_2}\tilde{v}_{n}\Big) \tag{2.15}$$

为了使W_N，F_N近似地满足方程(2.4)，应要求(2.14)、(2.15)中ε的较低次幂项与方程(2.4)的右端相同．在(2.14)的第一大括号中，令ε^0的系数等于q，令ε的其它幂的系数等于零；在(2.15)的第一大括号中，令ε的各次幂的系数等于零，得到关于$w_n, f_n(n=0, 1,\cdots,N)$的递推方程：

$$L(w_0,f_0)=-q,\qquad \Delta^2 f_0+\frac{1}{2}L(w_0,w_0)=0 \tag{2.16}$$

$$\left.\begin{aligned}&L(w_0,f_n)+L(w_n,f_0)=\Delta^2 w_{n-2}-\sum_{i=1}^{n-1}L(w_i,f_{n-i})\\&\Delta^2 f_n+L(w_0,w_n)=-\frac{1}{2}\sum_{i=1}^{n-1}L(w_i,w_{n-i})\end{aligned}\right\} \tag{2.17}$$

$$(n=1,2,\cdots,N)$$

在上式以及以后各式中，都将负下标的量取作零．

再确定$\alpha_1,\alpha_2,\beta_1,\beta_2$．将(2.12)式代入边界条件(2.5)和(2.6)，考虑到v_n和$\tilde{v}_n$是边界层型函数得

$$\sum_{n=0}^{N}\varepsilon^n w_n|_{r=b}+\varepsilon^{\alpha_1}\sum_{n=0}^{N}v_n|_{\eta=b}=f_0(\theta) \tag{2.18}$$

$$\sum_{n=0}^{N}\varepsilon^n w_{n,r}|_{r=b}+\varepsilon^{\alpha_1}\varepsilon^{-1}(\delta_{1,0}+\varepsilon\delta_{1,1})\sum_{n=0}^{N}\varepsilon^n v_n|_{\eta=b}=g_0(\theta) \tag{2.19}$$

$$\sum_{n=0}^{N}\varepsilon^n w_n|_{r=1}+\varepsilon^{\alpha_2}\sum_{n=0}^{N}\tilde{v}_n|_{\bar{\eta}=1}=f_1(\theta) \tag{2.20}$$

$$\sum_{n=0}^{N}\varepsilon^n[w_{n,rr}+\nu(w_{n,r}+w_{n,\theta\theta})]|_{r=1}+\varepsilon^{\alpha_2}\varepsilon^{-2}\Big[\tilde{\delta}_{2,0}+\varepsilon(\tilde{\delta}_{2,1}+\nu\tilde{\delta}_{1,0})$$

$$+\varepsilon^2\Big(\tilde{\delta}_{2,2}+\nu\tilde{\delta}_{1,1}+\nu\frac{\partial^2}{\partial\theta^2}\Big)\Big]\sum_{n=0}^{N}\varepsilon^n\tilde{v}_n|_{\bar{\eta}=1}=g_1(\theta) \tag{2.21}$$

从(2.19)和(2.21)式看出应取$\alpha_1=1$，$\alpha_2=2$．逐次地比较ε的较低次幂的系数，得到关于w_n, v_n和$\tilde{v}_n$的边界条件：

$$w_0|_{r=b}=f_0(\theta),\quad w_n|_{r=b}+v_{n-1}|_{\eta=b}=0\quad(n=1,2,\cdots,N) \tag{2.22}$$

$$w_{0,r}|_{r=b}+\delta_{1,0}v_0|_{\eta=b}=g_0(\theta),\quad w_{n,r}|_{r=b}+(\delta_{1,0}v_n+\delta_{1,1}v_{n-1})|_{\eta=b}=0 \tag{2.23}$$

$$(n=1,2,\cdots,N)$$

$$w_0|_{r=1}=f_1(\theta),\quad w_n|_{r=1}+\tilde{v}_{n-2}|_{\bar{\eta}=1}=0\quad(n=1,2,\cdots,N) \tag{2.24}$$

$$[w_{0,rr}+\nu(w_{0,r}+w_{0,\theta\theta})]|_{r=1}+\tilde{\delta}_{2,0}\tilde{v}_0|_{\tilde{\eta}=1}=g_1(\theta)$$

$$[w_{n,rr}+\nu(w_{n,r}+w_{n,\theta\theta})]|_{r=1}+\Big[\tilde{\delta}_{2,0}\tilde{v}_n+(\tilde{\delta}_{2,1}+\nu\tilde{\delta}_{1,0})\tilde{v}_{n-1}$$

$$+\Big(\tilde{\delta}_{2,2}+\nu\tilde{\delta}_{1,1}+\nu\frac{\partial^2}{\partial\theta^2}\Big)\tilde{v}_{n-2}\Big]\Big|_{\tilde{\eta}=1}=0$$

$$(n=1,2,\cdots,N) \tag{2.25}$$

又在 (2.14) 和(2.15)式的第二和第三大括弧中比较 ε 的最低次幂的系数，知道应取$\beta_1=3$和$\beta_2=4$．逐次地令 ε 的较低次幂的系数为零，得到关于 v_n, h_n, $\tilde{v}_n$, $\tilde{h}_n$ 的递推方程:

$$D_0v_0-M_0(f_0,v_0)=0 \tag{2.26}$$

$$D_0v_n-M_0(f_0,v_0)=\sum_{\substack{j+k=n\\(j\neq0)}}M_0(f_j,v_k)+\sum_{i=1}^{2}\sum_{j+k=n-i}M_i(f_j,v_k)$$

$$+\sum_{i=0}^{2}\sum_{j+k=n-2-i}M_i(w_j,h_k)+\sum_{i=0}^{3}\sum_{j+k=n-2-i}N_i(v_j,h_k)-\sum_{i=1}^{4}D_iv_{n-i} \tag{2.27}$$

$$D_0h_0=-M_0(w_0,v_0)-\frac{1}{2}N_0(v_0,v_0) \tag{2.28}$$

$$D_0h_n=-\sum_{i=0}^{2}\sum_{j+k=n-i}M_i(w_j,v_k)-\frac{1}{2}\sum_{i=0}^{3}\sum_{j+k=n-i}N_i(v_j,v_k)-\sum_{i=1}^{4}D_ih_{n-i} \tag{2.29}$$

$$\tilde{D}_0\tilde{v}_0-\tilde{M}_0(f_0,\tilde{v}_0)=0 \tag{2.30}$$

$$\tilde{D}_0\tilde{v}_n-\tilde{M}_0(f_0,\tilde{v}_n)=\sum_{\substack{j+k=n\\(j\neq0)}}\tilde{M}_0(f_j,\tilde{v}_k)+\sum_{i=1}^{2}\sum_{j+k=n-i}\tilde{M}_i(f_j,\tilde{v}_k)$$

$$+\sum_{i=0}^{2}\sum_{j+k=n-2-i}\tilde{M}_i(w_j,\tilde{h}_k)+\sum_{i=0}^{3}\sum_{j+k=n-3-i}\tilde{N}_i(\tilde{v}_j,\tilde{h}_k)-\sum_{i=1}^{4}\tilde{D}_i\tilde{v}_{n-i} \tag{2.31}$$

$$\tilde{D}_0\tilde{h}_0=-\tilde{M}_0(w_0,\tilde{v}_0) \tag{2.32}$$

$$\tilde{D}_0\tilde{h}_n=-\sum_{i=0}^{2}\sum_{j+k=n-i}\tilde{M}_i(w_j,\tilde{v}_k)-\frac{1}{2}\sum_{i=0}^{3}\sum_{j+k=n-1-i}\tilde{N}_i(\tilde{v}_j,\tilde{v}_k)-\sum_{i=1}^{4}\tilde{D}_i\tilde{h}_{n-i} \tag{2.33}$$

$$(n=1,2,\cdots,N)$$

再将 (2.13) 式代入边界条件 (2.7) 和 (2.8)，逐次地比较 ε 的较低次幂的系数，得到关于f_n, h_n 和 $\tilde{h}_n$ 的边界条件:

$$\Big(\frac{f_{0,r}}{r}+\frac{f_{0,\theta\theta}}{r^2}\Big)\Big|_{r=b}=T_0(\theta),$$

$$\Big(\frac{f_{n,r}}{r}+\frac{f_{n,\theta\theta}}{r^2}\Big)\Big|_{r=b}+\Big[\frac{\delta_{1,0}h_{n-2}}{\eta}+\Big(\frac{\delta_{1,1}}{\eta}+\frac{1}{\eta^2}\frac{\partial^2}{\partial\theta^2}\Big)h_{n-3}\Big]\Big|_{\eta=b}=0$$

$$f_{0,rr}|_{r=b}=S_0(\theta),\quad f_{n,rr}|_{r=b}+(\delta_{2,0}h_{n-1}+\delta_{2,1}h_{n-2}+\delta_{2,2}h_{n-3})|_{\eta=b}=0$$

$$(f_{0,r}+f_{0,\theta\theta})\Big|_{r=1}=T_1(\theta),(f_{n,r}+f_{n,\theta\theta})\Big|_{r=1}+\Big[\tilde{\delta}_{1,0}\tilde{h}_{n-3}+\Big(\tilde{\delta}_{1,1}+\frac{\partial^2}{\partial\theta^2}\Big)\tilde{h}_{n-4}\Big]\Big|_{\tilde{\eta}=1}=0$$

$$f_{0,rr}|_{r=1}=S_1(\theta),f_{n,rr}|_{r=1}+(\tilde{\delta}_{2,0}\tilde{h}_{n-2}+\tilde{\delta}_{2,1}\tilde{h}_{n-3}+\tilde{\delta}_{2,2}\tilde{h}_{n-4})|_{\tilde{\eta}=1}=0$$

$$(n=1,2,\cdots,N)$$

3. 形式渐近解

综合2. 中各式知道 w_0 和 f_0 应是薄膜理论的解:

$$\left.\begin{aligned}&L(w_0,f_0)=-q,\quad \Delta^2 f_0+\frac{1}{2}L(w_0,w_0)=0\\&w_0\Big|_{r=b}=f_0(\theta),\quad \Big(\frac{f_{0,r}}{r}+\frac{f_{0,\theta\theta}}{r^2}\Big)\Big|_{r=b}=T_0(\theta),\quad f_{0,rr}\Big|_{r=b}=S_0(\theta)\\&w_0\Big|_{r=1}=f_1(\theta),\quad (f_{0,r}+f_{0,\theta\theta})\Big|_{r=1}=T_1(\theta),\quad f_{0,rr}\Big|_{r=1}=S_1(\theta)\end{aligned}\right\}\tag{2.34}$$

求得w_0和f_0后，代入（2.26）和（2.30）式得

$$u_r^4\frac{\partial^4 v_0}{\partial\xi^4}-\Big(\frac{f_{0,r}}{\eta}+\frac{f_{0,\theta\theta}}{\eta^2}\Big)u_r^2\frac{\partial^2 v_0}{\partial\xi^2}=0$$

$$\tilde{u}_r^4\frac{\partial^4\tilde{v}_0}{\partial\tilde{\xi}^4}-\Big(\frac{f_{0,r}}{\tilde{\eta}}+\frac{f_{0,\theta\theta}}{\tilde{\eta}^2}\Big)\tilde{u}_r^2\frac{\partial^2\tilde{v}_0}{\partial\tilde{\xi}^2}=0$$

因假设$T_0(\theta)>0$，$T_1(\theta)>0$，若取

$$u=\int_b^r\sqrt{\frac{f_{0,r}(r,\theta)}{r}+\frac{f_{0,\theta\theta}(r,\theta)}{r^2}}dr,\quad \tilde{u}=\int_r^1\sqrt{\frac{f_{0,r}(r,\theta)}{r}+\frac{f_{0,\theta\theta}(r,\theta)}{r^2}}dr\tag{2.35}$$

则得到常系数方程

$$\frac{\partial^4 v_0}{\partial\xi^4}-\frac{\partial^2 v_0}{\partial\xi^2}=0,\quad \frac{\partial^4\tilde{v}_0}{\partial\tilde{\xi}^4}-\frac{\partial^2\tilde{v}_0}{\partial\tilde{\xi}^2}=0\tag{2.36}$$

可以求得边界层型的解为

$$v_0=C_0(\eta,\theta)e^{-\xi}=C_0(r,\theta)\exp\Big[\frac{-1}{\varepsilon}\int_b^r\sqrt{\frac{f_{0,r}}{r}+\frac{f_{0,\theta\theta}}{r^2}}dr\Big]\tag{2.37}$$

$$\tilde{v}_0=\tilde{C}_0(\eta,\theta)e^{-\tilde{\xi}}=\tilde{C}_0(r,\theta)\exp\Big[\frac{-1}{\varepsilon}\int_r^1\sqrt{\frac{f_{0,r}}{r}+\frac{f_{0,\theta\theta}}{r^2}}dr\Big]\tag{2.38}$$

又从（2.23）和（2.25）式得到C_0和$\tilde{C}_0$的边界条件

$$C_0(\eta,\theta)_{\eta=b}=-\frac{g_0(\theta)-w_{0,r}(b,\theta)}{\sqrt{T_0(\theta)}}\tag{2.39}$$

$$\tilde{C}_0(\tilde{\eta},\theta)\Big|_{\tilde{\eta}=1}=\frac{g_1(\theta)-[w_{0,rr}(1,\theta)+\nu w_{0,r}(1,\theta)+\nu w_{0,\theta\theta}(1,\theta)]}{T_1(\theta)}\tag{2.40}$$

将v_0和$\tilde{v}_0$再分别代入（2.28）和（2.32）式，可以求得边界层型的解

$$h_0=-T^{-1}(\eta,\theta)\Big(\frac{w_{0,r}}{\eta}+\frac{w_{0,\theta\theta}}{\eta^2}\Big)C_0(\eta,\theta)e^{-\xi}+\frac{1}{16\eta}T^{-\frac{1}{2}}(\eta,\theta)C_0^2(\eta,\theta)e^{-2\xi}\tag{2.41}$$

$$\tilde{h}_0=-T^{-1}(\tilde{\eta},\theta)\left(\frac{w_{0,r}}{\tilde{\eta}}+\frac{w_{0,\theta\theta}}{\tilde{\eta}^2}\right)\tilde{C}_0(\tilde{\eta},\theta)e^{-\tilde{\xi}} \tag{2.42}$$

其中$T(r,\theta)\equiv\frac{f_{0,r}}{r}+\frac{f_{0,\theta\theta}}{r^2}$，$C_0$和$\tilde{C}_0$是待定函数.

将求得的w_0，f_0，v_0，…，$\tilde{h}_0$，再代入方程(2.17)和边界条件(2.22)(取$n=1$)，得到w_1和f_1的线性边值问题：

$$L(w_0,f_1)+L(w_1,f_0)=0,\qquad \Delta^2 f_1+L(w_0,w_1)=0$$

$$w_1\Big|_{r=b}=-v_0\Big|_{\eta=b}=-C_0(b,\theta),\qquad \left(\frac{f_{1,r}}{r}+\frac{f_{1,\theta\theta}}{r^2}\right)\Big|_{r=b}=0,$$

$$f_{1,rr}\Big|_{r=b}=-\delta_{2,0}h_0\Big|_{\eta=b}$$

$$w_1|_{r=1}=0,\ (f_{1,r}+f_{1,\theta\theta})|_{r=1}=0,\ f_{1,rr}|_{r=1}=0$$

求得w_1和f_1后再代入(2.27)和(2.31)式(取$n=1$)，令其右端为零，又得到关于C_0和$\tilde{C}_0$的一阶线性方程：

$$2T\frac{\partial C_0}{\partial\eta}+\frac{2}{\eta}\left(\frac{-f_{0,\theta}}{\eta^2}+\frac{f_{0,r\theta}}{\eta}\right)\frac{\partial C_0}{\partial\theta}+\left[\left(\frac{f_{1,r}}{\eta}+\frac{f_{1,\theta\theta}}{\eta^2}\right)T^{\frac{1}{2}}\right.$$

$$\left.+\frac{5}{2}T_r-\frac{f_{0,rr}}{\eta}+\frac{2}{\eta}T\right]C_0=0$$

$$2T\frac{\partial\tilde{C}_0}{\partial\tilde{\eta}}+\frac{2}{\tilde{\eta}}\left(\frac{-f_{0,\theta}}{\tilde{\eta}^2}+\frac{f_{0,r\theta}}{\tilde{\eta}}\right)\frac{\partial\tilde{C}_0}{\partial\theta}+\left[-\left(\frac{f_{1,r}}{\tilde{\eta}}+\frac{f_{1,\theta\theta}}{\tilde{\eta}^2}\right)T^{\frac{1}{2}}\right.$$

$$\left.+\frac{5}{2}T_r-\frac{f_{0,rr}}{\tilde{\eta}}+\frac{2}{\tilde{\eta}}T\right]\tilde{C}_0=0$$

在条件$T(r,\theta)>0$下，可以根据柯西条件(2.39)和(2.40)唯一地解得C_0和$\tilde{C}_0$，随之完全确定了边界层型函数v_0，…，$\tilde{h}_0$. 这时方程(2.27)和(2.31)($n=1$)化成齐次方程：

$$\frac{\partial^4 v_1}{\partial\xi^4}-\frac{\partial^2 v_1}{\partial\xi^2}=0,\quad \frac{\partial^4\tilde{v}_1}{\partial\tilde{\xi}^4}-\frac{\partial^2\tilde{v}_1}{\partial\tilde{\xi}^2}=0$$

又可以求得v_1和$\tilde{v}_1$：

$$v_1(\xi,\eta,\theta)=C_1(\eta,\theta)e^{-\xi}\qquad \tilde{v}_1(\tilde{\xi},\tilde{\eta},\theta)=\tilde{C}_1(\tilde{\eta},\theta)e^{-\tilde{\xi}}$$

从(2.23)、(2.25)式(取$n=1$)又得到C_1和$\tilde{C}_1$的边界条件

$$C_1(\eta,\theta)|_{\eta=b}=-T_0^{-\frac{1}{2}}[w_{1,r}(b,\theta)+C_{0,\eta}(b,\theta)]$$

$$\tilde{C}_1(\tilde{\eta},\theta)|_{\tilde{\eta}=1}=-T_1^{-1}[(w_{1,rr}+\nu w_{1,r}+\nu w_{1,\theta\theta})|_{r=1}$$

$$+(2T_1^{\frac{1}{2}}\tilde{C}_{0,\eta}+\frac{1}{2}T_1^{-\frac{1}{2}}T_r\tilde{C}_0+\nu T_1^{\frac{1}{2}}\tilde{C}_0)|_{\tilde{\eta}=1}]$$

重复前面的步骤，可以逐次地求得w_n，f_n，…，$\tilde{h}_n$ $(n=1,2,\cdots,N)$等等.

三、圆形薄板及其它

对于圆形板，例如周边简支的情形，只需将边界条件(2.5)、(2.7)代替以

在$r=0$点，w，$\dfrac{\partial w}{\partial r}$，$\dfrac{1}{r}\dfrac{\partial F}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2 F}{\partial\theta^2}$，$\dfrac{\partial^2 F}{\partial r^2}$取有限值，将挠度和应力函数的展开式（2.12）和（2.13）代替以

$$W_N=\sum_{n=0}^{N}\varepsilon^n w_n(r,\theta)+\psi(r)\sum_{n=0}^{N}\varepsilon^{n+2}v_n(\xi,\eta,\theta)$$

$$F_N=\sum_{n=0}^{N}\varepsilon^n f_n(r,\theta)+\psi(r)\sum_{n=0}^{N}\varepsilon^{n+4}h_n(\xi,\eta,\theta)$$

其中$\psi(r)$是在$\dfrac{2}{3}\leqslant r\leqslant 1$取值1，在$0\leqslant r\leqslant\dfrac{1}{3}$取值零的截断函数，和$\xi=\dfrac{1}{\varepsilon}\int_r^1\sqrt{\dfrac{f_{0,r}}{r}+\dfrac{f_{0,\theta\theta}}{r^2}}\,dr$，$\eta=r$.

特别地，对于轴对称的情形问题可以简化．这时 von Kármán 方程（2.4）具有形式：

$$\left.\begin{aligned}&\varepsilon^2\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right]=\frac{1}{r}\left(\frac{dw}{dr}\frac{dF}{dr}\right)+\frac{qr}{2}\\&\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right]=\frac{-1}{2}\frac{1}{r}\left(\frac{dw}{dr}\right)^2\end{aligned}\right\}\tag{3.1}$$

作为一个例子，例如考察周边作用有均匀径向力T_0的固支圆形板的弯曲问题，给出边界条件

$$w\Big|_{r=1}=0,\quad \frac{dw}{dr}\Big|_{r=1}=0,\quad \frac{dF}{dr}\Big|_{r=1}=T_0\tag{3.2}$$

$$\text{在}r=0\text{点，}\frac{dw}{dr}\text{和}\frac{1}{r}\frac{dF}{dr}\text{取有限值，}\tag{3.3}$$

若只要求准确到ε^2量级，可假设

$$w=w_0+\varepsilon w_1+\varepsilon\psi(r)v_0(\xi,\eta,\theta),\qquad F=f_0+\varepsilon f_1+\varepsilon^3\psi(r)h_0(\xi,\eta,\theta)\tag{3.4}$$

其中$\xi=\dfrac{u(r,\theta)}{\varepsilon}$，$\eta=r$，$\psi(r)$是截断函数．将（3.4）式代入(3.1)—(3.3)式[在边界$r=1$的邻域，应采用导数展开式(2.10)]，比较等式两端ε的同次幂的系数，可知w_0和f_0应是薄膜理论的解：

$$\frac{dw_0}{dr}\frac{df_0}{dr}=-\frac{qr^2}{2},\qquad \frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dw_0}{dr}\right)\right]=\frac{-1}{2}\frac{1}{r}\left(\frac{dw_0}{dr}\right)^2\tag{3.5}$$

$$w_0\Big|_{r=1}=0,\qquad \frac{df_0}{dr}\Big|_{r=1}=T_0\tag{3.6}$$

$$\text{在}r=0\text{，}\frac{dw_0}{dr}\text{和}\frac{1}{r}\frac{df_0}{dr}\text{取有限值．}\tag{3.7}$$

在边值问题(3.5)—(3.7)中，若令$y=r^2$，$z=\sqrt[3]{\dfrac{256}{q^2}}\,y\dfrac{df_0}{dy}$，则化为

$$\frac{dw_0}{dy}=\frac{-y}{z}\sqrt[3]{\frac{q}{2}},\quad \frac{d^2z}{dy^2}=-\frac{y^2}{z^2} \tag{3.8}$$

$$w_0\Big|_{y=1}=0,\quad z\Big|_{y=1}=\sqrt[3]{\frac{32}{q^2}}T_0,\quad \frac{dw_0}{dy}\Big|_{y=0}<\infty,\quad z\Big|_{y=0}=0 \tag{3.9}$$

可以应用[1]或[9]的方法求其幂级数形式的近似解.

又v_0和h_0确定于定解问题

$$u_r^3\frac{d^3v_0}{d\xi^3}-\frac{f_{0,r}}{\eta}\frac{dv_0}{d\xi}=0,\quad \frac{dv_0}{d\xi}\Big|_{r=1}=\frac{-w_{0,r}(1)}{T_0},\quad \lim_{\xi\to\infty}v_0=0$$

$$u_r^3\frac{d^3h_0}{d\xi^3}=\frac{-u_rw_{0,r}}{r}\frac{dv_0}{d\xi}-\frac{u_r^2}{2r}\left(\frac{dv_0}{d\xi}\right)^2,\quad \lim_{\xi\to\infty}h_0=0$$

所以取 $u=\int_r^1\sqrt{\frac{f_{0,r}}{r}}dr$，解得

$$v_0=C_0(\eta)e^{-\xi}\equiv C_0(r)\exp\left[\frac{-1}{\varepsilon}\int_r^1\sqrt{\frac{f_{0,r}}{r}}dr\right] \tag{3.10}$$

$$h_0=\frac{-w_{0,r}}{f_{0,r}}C_0(\eta)e^{-\xi}+\frac{1}{16}\sqrt{\frac{1}{\eta f_{0,r}}}C_0^2(\eta)e^{-2\xi} \tag{3.11}$$

其中C_0是满足边界条件:

$$C_0(\eta)\Big|_{\eta=1}=\frac{w_{0,r}(1)}{T_0} \tag{3.12}$$

的待定函数．又w_1和f_1确定于边值问题:

$$w_{0,r}\frac{df_1}{dr}+f_{0,r}\frac{dw_1}{dr}=0,\quad \frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{df_1}{dr}\right)\right]=\frac{-w_{0,r}}{r}\frac{dw_1}{dr} \tag{3.13}$$

$$w_1\Big|_{r=1}=\frac{-w_{0,r}(1)}{T_0},\quad \frac{df_1}{dr}\Big|_{r=1}=0 \tag{3.14}$$

在$r=0$，$\frac{dw_1}{dr}$和$\frac{1}{r}\frac{df_1}{dr}$取有限值. (3.15)

在边值问题(3.13)—(3.15)中，若令$y=r^2$，$z_1=y\frac{df_1}{dy}$，则化为

$$\frac{dw_1}{dy}=-\left(\frac{w_{0,y}}{f_{0,y}}\right)\frac{z_1}{y},\quad \frac{d^2z_1}{dy^2}=\frac{1}{2}\left(\frac{w_{0,y}}{f_{0,y}}\right)\frac{z_1}{y}$$

$$w_1\Big|_{y=1}=\frac{-w_{0,r}(1)}{T_0},\quad z_1\Big|_{y=1}=0,\quad \frac{dw_1}{dy}\Big|_{y=0}<\infty,\quad z_1\Big|_{y=0}=0$$

可以应用幂级数解法求其近似解．又确定$C_0(\eta)$的一阶线性方程是$\left(\text{记 } T(r)\equiv\frac{f_{0,r}(r)}{r}\right)$

$$\frac{dC_0}{d\eta}=\frac{1}{2}\left[\frac{f_{1,r}(\eta)}{\eta}T^{-\frac{1}{2}}(\eta)-\frac{3}{2}T_r(\eta)T^{-1}(\eta)-\frac{1}{\eta}\right]C_0$$

根据边界条件(3.12)可以解得

$$C_0(\eta)=\frac{w_{0},r(1)}{T_0}\left(\frac{T_0}{T(\eta)}\right)^{\frac{3}{4}}\frac{1}{\sqrt{\eta}}\exp\left[\frac{-1}{2}\int_{\eta}^{1}\frac{f_{1},r(t)}{t}T^{-\frac{1}{2}}(t)dt\right]$$

将上式代入（3.10）和(3.11)式就求得v_0和h_0. 将求得的w_0, f_0, $\cdots$, v_0, h_0一并代入（3.4）式就求得w和F.

附注1：对于非圆型的其它光滑周界的弹性薄板，上面方法仍然适用，只需在周界的邻域建立局部坐标系（ρ,φ）（参见[5]）.

附注2：从挠度和应力函数的近似式(2.12)、(2.13)可以看出，边缘效应的影响是随边界条件而异. 例如在简支的情形（与固支情形相比较），边缘效应将在更高阶项出现.

附注3：从边界层项的表示式例如（2.37）式可以看出，边界层的厚度是与作用于周边的径向力T_0有关；当T_0增大时，厚度将减小.

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Unsymmetrical Bending Problems for the Annular and Circular Thin Plates under Various Supports (I)

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Abstract

In this paper we consider the non-linear and unsymmetrical bendings of thin annular and circular plates under various supports. The uniformly valid asymptotic solutions are obtained by means of the perturbation method offered in [7].