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非常量导热系数几例微分方程的推立

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摘 要

在本文里，曾先后假设物体的导热系数是依直线和指数函数空间地起改变，就这样来建立了六个二阶热传导微分方程；又对于变密度、变比热、变导热系数这样的更一般的情况也推立了六个二阶热传导的微分方程.

一、引 言

众所熟知，便于分析的热传导问题应属于常量导热系数这一例，对于这一领域迄今确实已积累了丰硕的成果．非常量导热系数问题虽然经常存在于实际之中，但由于分析上的困难，所以成果较差．作者近来曾基于组合固体一例，有过近似的设想和探索[1]，也得出了一个通解．在那里曾只限于一维笛卡儿直角坐标系中的非常量导热系数的导热问题．本文的目的是想扩大这个尝试，例如把它引伸到圆柱坐标和球体坐标，并再引用一个别的导热系数式值的结构.

二、导热系数结构的拟设

在本文里，我们拟取用两种导热系数结构．一种是线性变化结构 $k=k_0(1+\alpha\zeta)$，其中 ζ 可以是笛卡儿直角一维坐标 x，也可以代表圆柱一维径距及球体一维径距坐标 r.

据此，写的更清楚些，一种结构是

$$\left.\begin{aligned} &k=k_0(1+\alpha x),\quad \alpha\gtreqless 0,\quad k_0>0\\ &k=k_0(1+\alpha r),\quad r=\sqrt{x^2+y^2}\\ &k=k_0(1+\alpha r),\quad r=\sqrt{x^2+y^2+z^2} \end{aligned}\right\}\tag{2.1}$$

其中 k_0 和 α 都是常量.

另一种有用的可分析结构是

$$\left.\begin{aligned} &k=k_0e^{\alpha x},\quad \alpha\gtreqless 0,\quad k_0>0\\ &k=k_0e^{\alpha r},\ r=\sqrt{x^2+y^2}\\ &k=k_0e^{\alpha r},\ r=\sqrt{x^2+y^2+z^2} \end{aligned}\right\}\tag{2.2}$$

以上这两种结构在实际中都可以出现或近似的出现．在实际问题中，我们应当弄清它们的影响，哪怕只是侧重于近似的分析呢，况且有时也已经与实际甚相符合．

三、用线性函数建立导热微分方程

本节里的推导是基于这样的拟设：有关材料的密度 ρ 和比热 c_0 都是常数或近似常数．下面的推导系自熟知的微分方程[2]

$$\rho c_0 \frac{\partial v}{\partial t}=\frac{\partial}{\partial x}\left(k\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial v}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial v}{\partial z}\right) \tag{3.1}$$

出发．其中 $v=v(x,y,z,t)$ 是温度函数，t 是时间．

如图 1 所示，对于一维笛卡儿坐标，取 $k=k(x)=k_0(1+\alpha x)$．则 (3.1) 式蜕化成

$$\rho c_0 \frac{\partial v}{\partial t}=k_0\alpha\frac{\partial v}{\partial x}+k_0(1+\alpha x)\frac{\partial^2 v}{\partial x^2} \tag{3.2}$$

如图 2 所示，对圆柱坐标，则以 $k=k_0(1+\alpha r)$，$r=\sqrt{x^2+y^2}$，先推得 $\frac{\partial k}{\partial x}=\alpha k_0\frac{x}{r}$，

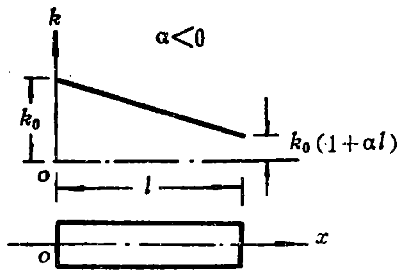

图1． 导热系数 $k=k_0(1+\alpha x)$．$\alpha \gtreqless 0$．

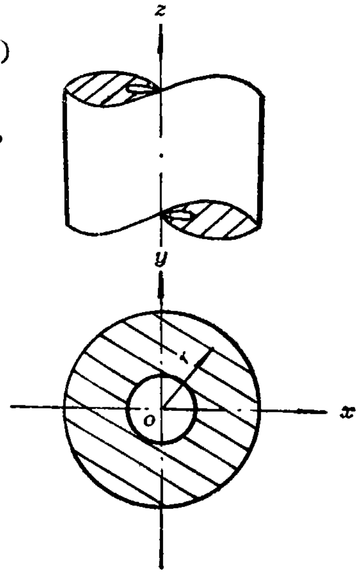

图2． 圆柱坐标，导热系数 $k=k_0(1+\alpha r)$，$r=\sqrt{x^2+y^2}$．$\alpha \gtreqless 0$．

$$\frac{\partial k}{\partial y}=\alpha k_0\frac{y}{r},\quad x\frac{\partial k}{\partial x}+y\frac{\partial k}{\partial y}=\alpha k_0 r．$$ 从而可自变式

$$\rho c_0\frac{\partial v}{\partial t}=\frac{1}{r}\left(x\frac{\partial k}{\partial x}+y\frac{\partial k}{\partial y}\right)\frac{\partial v}{\partial r}+k\left(\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}\right)$$

导得：

$$\rho c_0\frac{\partial v}{\partial t}=k_0(1+\alpha r)\left\{\frac{\partial^2 v}{\partial r^2}+\left(\frac{1}{r}+\frac{\alpha}{1+\alpha r}\right)\frac{\partial v}{\partial r}\right\} \tag{3.3}$$

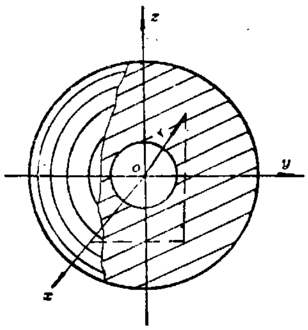

图3． 球体坐标，导热系数 $k=k_0(1+\alpha r)$，$r=\sqrt{x^2+y^2+z^2}$．$\alpha \gtreqless 0$．

如图 3 所示，对于球体坐标，则以 $k=k_0(1+\alpha r)$，$r=\sqrt{x^2+y^2+z^2}$ 自 (3.1) 式导出变式

$$\rho c_0 \frac{\partial v}{\partial t}=\frac{1}{r}\left(x\frac{\partial k}{\partial x}+y\frac{\partial k}{\partial y}+z\frac{\partial k}{\partial z}\right)\frac{\partial v}{\partial r}+k\left\{\frac{\partial^2 v}{\partial r^2}+\frac{2}{r}\frac{\partial v}{\partial r}\right\}$$

再利用关系$\frac{\partial k}{\partial x}=\alpha k_0\frac{x}{r}$，$\frac{\partial k}{\partial y}=\alpha k_0\frac{y}{r}$，$\frac{\partial k}{\partial z}=\alpha k_0\frac{z}{r}$导得

$$\rho c_0 \frac{\partial v}{\partial t}=k_0(1+\alpha r)\left\{\frac{\partial^2 v}{\partial r^2}+\left(\frac{\alpha}{1+\alpha r}+\frac{2}{r}\right)\frac{\partial v}{\partial r}\right\} \tag{3.4}$$

四、用指数函数来推立导热微分方程

几何安排可仍用图 1、图 2 和图 3，而改用指数函数的导热系数来变换热传导的微分方程.

设 $k=k_0e^{\alpha x}$，则$\frac{\partial k}{\partial x}=\alpha k_0 e^{\alpha x}$，于是可自变式

$$\rho c_0 \frac{\partial v}{\partial t}=\frac{\partial k}{\partial x}\frac{\partial v}{\partial x}+k\frac{\partial^2 v}{\partial x^2}$$

导出

$$\rho c_0 \frac{\partial v}{\partial t}=k_0e^{\alpha x}\left\{\alpha\frac{\partial v}{\partial x}+\frac{\partial^2 v}{\partial x^2}\right\} \tag{4.1}$$

对于圆柱坐标，则以 $k=k_0e^{\alpha r}$，$r=\sqrt{x^2+y^2}$，$\frac{\partial r}{\partial x}=\frac{x}{r}$，$\frac{\partial r}{\partial y}=\frac{y}{r}$，$\frac{\partial k}{\partial x}=\alpha k_0e^{\alpha r}\frac{x}{r}$，$\frac{\partial k}{\partial y}=\alpha k_0e^{\alpha r}\frac{y}{r}$，$x\frac{\partial k}{\partial x}=\alpha k_0e^{\alpha r}\frac{x^2}{r}$，$y\frac{\partial k}{\partial y}=\alpha k_0e^{\alpha r}\frac{y^2}{r}$，从变式

$$\rho c_0 \frac{\partial v}{\partial t}=\frac{1}{r}\left(x\frac{\partial k}{\partial x}+y\frac{\partial k}{\partial y}\right)\frac{\partial v}{\partial r}+k\left\{\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}\right\}$$

导得热传导方程

$$\rho c_0 \frac{\partial v}{\partial t}=k_0e^{\alpha r}\left\{\frac{\partial^2 v}{\partial r^2}+\left(\frac{1}{r}+\alpha\right)\frac{\partial v}{\partial r}\right\} \tag{4.2}$$

对于球体坐标，可设$k=k_0e^{\alpha r}$，$r=\sqrt{x^2+y^2+z^2}$；从而可把$\frac{\partial k}{\partial x}=\alpha k_0e^{\alpha r}\frac{x}{r}$，$\frac{\partial k}{\partial y}=\alpha k_0e^{\alpha r}\frac{y}{r}$，$\frac{\partial k}{\partial z}=\alpha k_0e^{\alpha r}\frac{z}{r}$代进变式

$$\rho c_0 \frac{\partial v}{\partial t}=\frac{1}{r}\left(x\frac{\partial k}{\partial x}+y\frac{\partial k}{\partial y}+z\frac{\partial k}{\partial z}\right)\frac{\partial v}{\partial r}+k\left\{\frac{\partial^2 v}{\partial r^2}+\frac{2}{r}\frac{\partial v}{\partial r}\right\}$$

推得热传导方程

$$\rho c_0 \frac{\partial v}{\partial t}=k_0e^{\alpha r}\left\{\frac{\partial^2 v}{\partial r^2}+\left(\frac{2}{r}+\alpha\right)\frac{\partial v}{\partial r}\right\} \tag{4.3}$$

五、变密度、变比热、变导热系数一例

前面曾经提及，我们把材料密度 ρ 和 c 当作常量或允许的近似常量，才推出了前节里的热传导微分方程；若引进两个微小常值 β 和 γ，就可用 $\rho=\rho_0(1+\gamma x)$，$c=c_0(1+\beta x)$，$\rho=\rho_0 e^{\gamma r}$，$c=c_0 e^{\beta r}$，$r=\sqrt{x^2+y^2}$，以及 $\rho=\rho_0 e^{\gamma r}$，$c=c_0 e^{\beta r}$，$r=\sqrt{x^2+y^2+z^2}$
写出相应的六个微分方程

$$\rho_0 c_0(1+\gamma x)(1+\beta x)\frac{\partial v}{\partial t}=k_0 a\frac{\partial v}{\partial x}+k_0(1+ax)\frac{\partial^2 v}{\partial x^2} \tag{5.1}$$

$$\left.\begin{aligned}&\rho_0 c_0(1+\gamma r)(1+\beta r)\frac{\partial v}{\partial t}=k_0(1+ar)\left\{\frac{\partial^2 v}{\partial r^2}+\left(\frac{1}{r}+\frac{a}{1+ar}\right)\frac{\partial v}{\partial r}\right\}\\&r=\sqrt{x^2+y^2}\end{aligned}\right\} \tag{5.2}$$

$$\left.\begin{aligned}&\rho_0 c_0(1+\gamma r)(1+\beta r)\frac{\partial v}{\partial t}=k_0(1+ar)\left\{\frac{\partial^2 v}{\partial r^2}+\left(\frac{2}{r}+\frac{a}{1+ar}\right)\frac{\partial v}{\partial r}\right\}\\&r=\sqrt{x^2+y^2+z^2}\end{aligned}\right\} \tag{5.3}$$

$$\rho_0 c_0 e^{(\beta+\gamma)x}\frac{\partial v}{\partial t}=k_0 e^{\alpha x}\left\{\frac{\partial^2 v}{\partial x^2}+\alpha\frac{\partial v}{\partial x}\right\} \tag{5.4}$$

$$\left.\begin{aligned}&\rho_0 c_0 e^{(\beta+\gamma)r}\frac{\partial v}{\partial t}=k_0 e^{\alpha r}\left\{\frac{\partial^2 v}{\partial r^2}+\left(\frac{1}{r}+\alpha\right)\frac{\partial v}{\partial r}\right\}\\&r=\sqrt{x^2+y^2}\end{aligned}\right\} \tag{5.5}$$

$$\left.\begin{aligned}&\rho_0 c_0 e^{(\beta+\gamma)r}\frac{\partial v}{\partial t}=k_0 e^{\alpha r}\left\{\frac{\partial^2 v}{\partial r^2}+\left(\frac{2}{r}+\alpha\right)\frac{\partial v}{\partial r}\right\}\\&r=\sqrt{x^2+y^2+z^2}\end{aligned}\right\} \tag{5.6}$$

六、用参数来概括变导热系数的微分方程

在前几节里，对于 $\rho=\text{const}$ 和 $c=\text{const}$，我们曾推立了(3.2)、(3.3)、(3.4)、(4.1)、(4.2)、(4.3)等式所代表的六个微分方程．对于求解这类方程，一般是把两个自变量隔离来处理，例如可设 $v=\Phi\cdot\Psi$，其中 Φ 是空间坐标的函数，Ψ 是时间函数，从而易借助待定常数 K 把前面的六个方程各拆成两式如下．

$$\left.\begin{aligned}&k=k_0(1+ax)\\&\frac{d^2\Phi}{dx^2}+\frac{a}{1+ax}\frac{d\Phi}{dx}+\frac{K}{k_0(1+ax)}\Phi=0\\&\frac{d\Psi}{dt}+\frac{K}{\rho c}\Psi=0\end{aligned}\right\} \tag{6.1}$$

$$\left.\begin{aligned}&k=k_0(1+\alpha r),\quad r=\sqrt{x^2+y^2}\\&\frac{d^2\Phi}{dr^2}+\left(\frac{\alpha}{1+\alpha r}+\frac{1}{r}\right)\frac{d\Phi}{dr}+\frac{K}{k_0(1+\alpha r)}\Phi=0\\&\frac{d\Psi}{dt}+\frac{K}{\rho c}\Psi=0\end{aligned}\right\}\tag{6.2}$$

$$\left.\begin{aligned}&k=k_0(1+\alpha r),\quad r\sqrt{x^2+y^2+z^2}\\&\frac{d^2\Phi}{dr^2}+\left(\frac{\alpha}{1+\alpha r}+\frac{2}{r}\right)\frac{d\Phi}{dr}+\frac{K}{k_0(1+\alpha r)}\Phi=0\\&\frac{d\Psi}{dt}+\frac{K}{\rho c}\Psi=0\end{aligned}\right\}\tag{6.3}$$

$$\left.\begin{aligned}&k=k_0e^{\alpha x},\quad \frac{d\Psi}{dt}+\frac{K}{\rho c}\Psi=0\\&\frac{d^2\Phi}{dx^2}+\alpha\frac{d\Phi}{dx}+\frac{K}{k_0}e^{-\alpha x}\Phi=0\end{aligned}\right\}\tag{6.4}$$

$$\left.\begin{aligned}&k=k_0e^{\alpha r},\quad r=\sqrt{x^2+y^2},\quad \frac{d\Psi}{dt}+\frac{K}{\rho c}\Psi=0\\&\frac{d^2\Phi}{dr^2}+\left(\frac{1}{r}+\alpha\right)\frac{d\Phi}{dr}+\frac{K}{k_0}e^{\alpha r}\Phi=0\end{aligned}\right\}\tag{6.5}$$

$$\left.\begin{aligned}&k=k_0e^{\alpha r},\quad r=\sqrt{x^2+y^2+z^2},\quad \frac{d\Psi}{dt}+\frac{K}{\rho c}\Psi=0\\&\frac{d^2\Phi}{dr^2}+\left(\frac{2}{r}+\alpha\right)\frac{d\Phi}{dr}+\frac{K}{k_0}e^{-\alpha r}\Phi=0\end{aligned}\right\}\tag{6.6}$$

还可把(6.1)、(6.2)、(6.3)等式里的三个二阶微分方程合成

$$\left.\begin{aligned}&\zeta(1+\alpha\zeta)\frac{d^2\Phi}{d\zeta^2}+(\omega_1+\omega_2\zeta)\frac{d\Phi}{d\zeta}+\mu\zeta\Phi=0\\&\omega_1=0,\ \omega_2=\alpha;\ \omega_1=1,\ \omega_2=2\alpha;\ \omega_1=2,\ \omega_2=3\alpha\end{aligned}\right\}\tag{6.7a}$$

或写成

$$\left.\begin{aligned}&\zeta(1+\alpha\zeta)\frac{d^2\Phi}{d\zeta^2}+[\Omega+(\Omega+1)\alpha\zeta]\frac{d\Phi}{d\zeta}+\mu\zeta\Phi=0\\&\Omega=0;\ 1;\ 2.\end{aligned}\right\}\tag{6.7b}$$

也可把(6.4)、(6.5)、(6.6)里的三个二阶微分方程并为一个

$$\left.\begin{aligned}&\frac{d^2\Phi}{d\zeta^2}+\left(\frac{\omega}{\zeta}+\alpha\right)\frac{d\Phi}{d\zeta}+\mu e^{-\alpha\zeta}\Phi=0\\&\mu=K/k_0;\ \omega=0;\ 1;\ 2.\end{aligned}\right\}\tag{6.8a}$$

或者

$$\left.\begin{aligned}&\zeta\frac{d^2\Phi}{d\zeta^2}+(\omega+\alpha\zeta)\frac{d\Phi}{d\zeta}+\mu\zeta e^{-\alpha\zeta}\Phi=0\\&\mu=K/k_0,\ \omega=0;\ 1;\ 2.\end{aligned}\right\}\tag{6.8b}$$

经过概括，就可以只注意筹划去求解(6.7a)和(6.8a)或(6.7b)和(6.8b)．

七、变密度、变比热、变导热系数的微分方程

若照一惯所作，引用 $v=\Phi(\zeta)\cdot\Psi(t)$ 及一待定常数 K，就可把 (5.1) 式至 (5.6) 式拆成六对常微分方程如下：

$$\left.\begin{aligned}&\frac{d^2\Phi}{dx^2}+\frac{\alpha}{1+\alpha x}\frac{d\Phi}{dx}+\frac{K(1+\beta x)(1+\gamma x)}{k_0(1+\alpha x)}\Phi=0\\&\frac{d\Psi}{dt}+\frac{K}{\rho_0c_0}\Psi=0\end{aligned}\right\}\tag{7.1}$$

$$\left.\begin{aligned}&\frac{d^2\Phi}{dr^2}+\left(\frac{1}{r}+\frac{\alpha}{1+\alpha r}\right)\frac{d\Phi}{dr}+\frac{K(1+\beta r)(1+\gamma r)}{k_0(1+\alpha r)}\Phi=0\\&\frac{d\Psi}{dt}+\frac{K}{\rho_0c_0}\Psi=0,\quad r=\sqrt{x^2+y^2}\end{aligned}\right\}\tag{7.2}$$

$$\left.\begin{aligned}&\frac{d^2\Phi}{dr^2}+\left(\frac{\alpha}{1+\alpha r}+\frac{2}{r}\right)\frac{d\Phi}{dr}+\frac{K(1+\beta r)(1+\gamma r)}{k_0(1+\alpha r)}\Phi=0\\&\frac{d\Psi}{dt}+\frac{K}{\rho_0c_0}\Psi=0,\quad r=\sqrt{x^2+y^2+z^2}\end{aligned}\right\}\tag{7.3}$$

$$\left.\begin{aligned}&\frac{d^2\Phi}{dx^2}+\alpha\frac{d\Phi}{dx}+\frac{K}{k_0}e^{-(\alpha-\beta-\gamma)x}\Phi=0\\&\frac{d\Psi}{dt}+\frac{K}{\rho_0c_0}\Psi=0\end{aligned}\right\}\tag{7.4}$$

$$\left.\begin{aligned}&\frac{d^2\Phi}{dr^2}+\left(\frac{1}{r}+\alpha\right)\frac{d\Phi}{dr}+\frac{K}{k_0}e^{-(\alpha-\beta-\gamma)r}\Phi=0\\&\frac{d\Psi}{dt}+\frac{K}{\rho_0c_0}\Psi=0,\quad r=\sqrt{x^2+y^2}\end{aligned}\right\}\tag{7.5}$$

$$\left.\begin{aligned}&\frac{d^2\Phi}{dr^2}+\left(\frac{2}{r}+\alpha\right)\frac{d\Phi}{dr}+\frac{K}{k_0}e^{-(\alpha-\beta-\gamma)r}\Phi=0\\&\frac{d\Psi}{dt}+\frac{K}{\rho_0c_0}\Psi=0,\quad r=\sqrt{x^2+y^2+z^2}\end{aligned}\right\}\tag{7.6}$$

在上列六套公式里，我们有六个二阶微分方程．为了概括地求解，可把前三个二阶微分方程总括为

$$\left.\begin{aligned}&\frac{d^2\Phi}{d\zeta^2}+\left(\frac{\Omega}{\zeta}+\frac{\alpha}{1+\alpha\zeta}\right)\frac{d\Phi}{d\zeta}+\frac{K}{k_0}\quad\frac{(1+\beta\zeta)(1+\gamma\zeta)}{1+\alpha\zeta}\Phi=0\\&\Omega=0;\ 1;\ 2.\end{aligned}\right\}\tag{7.7}$$

把后三个总括成

$$\left.\begin{aligned}&\frac{d^2\Phi}{d\zeta^2}+\left(\frac{\Omega}{\zeta}+\alpha\right)\frac{d\Phi}{d\zeta}+\frac{K}{k_0}e^{-(\alpha-\beta-\gamma)\zeta}\Phi=0\\&\Omega=0;\ 1;\ 2.\end{aligned}\right\}\tag{7.8}$$

八、结　语

在处理热传导问题时，(3.1)式里出现了三个重要的物理量；ρ是有关材料的密度，c是材料的比热，k是热导系数．一般都把这三个量当做常数，但严格地说，他们都是变量．

本文曾考虑它们是符合实际或近似符合某些实际情况的空间函数．为此，我们推得了其相应的导热微分方程．对于求其通解，我们已计划用另文报道．

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The Differential Equations of Heat Transfer for Some Cases of Variable Thermal Conductivities

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Abstract

By managing the heat conduction problem in solids the thermal conductivity is usually taken as a constant, but in reality varying thermal conductivity reveals in every heat transfer process. Therefore, in the present paper, we wish to consider it to vary with the space coordinate according to a linear and an exponential law; basing on this proposal we have been able to set up six second order heat conduction differential equations. By the way, for the case of variable density, specific heat as well as thermal conductivity, we have been successful to deduce other similar six heat transfer differential equations.