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有吸除的层流边界层问题*

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摘 要

本文对于有吸除的普遍层流边界层方程求得其渐近解，然后推导了位移厚度、动量厚度和表面摩擦的计算公式．此外还处理了确定分离点位置的问题．最后以具有恒定吸除的平板均匀绕流情形为例，就某些边界层特征参数作了数值计算，我们所得的结果与 Iglisch 获得的结果很好地符合．

一、普遍的边界层方程及边界条件

众所周知，对于有吸除的不可压缩性定常层流，我们有边界层方程

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=U\frac{dU}{dx}+\nu\frac{\partial^2 u}{\partial y^2} \tag{1.1}$$

$$\frac{\partial}{\partial x}(ur^{\epsilon})+\frac{\partial}{\partial y}(vr^{\epsilon})=0 \tag{1.2}$$

及边界条件

$$\left.\begin{array}{l} y=0:\ u=0,\ v=v_0(x) \\ y=\infty:\ u=U \end{array}\right\} \tag{1.3}$$

式中 x、y 分别沿着和垂直于固体壁面，$x=0$ 为前滞止点；在边界上 $y=0$（对于轴线与未扰流平行的钝头旋成体，x 为离开前滞止点的距离，它是沿着物体表面与子午面的交线量度的 y 为离开表面的法向距离，而 $r(x)$ 则为物体在 $x=x$ 处的横截面的半径）；u、v 为相应的速度分量；$U(x)$ 为边界层外缘速度；ν 为流体的粘滞率；指数 ϵ 取值

$$\epsilon=\begin{cases}0 & \text{（平面流）}\\ 1 & \text{（轴对称流）}\end{cases} \tag{1.4}$$

$v_0(x)$ 为壁面上的法向速度（它是事先规定的），在我们所考察的吸除情形，$v_0(x)<0$．

假如引入下面所定义的流函数 ψ：

$$u=\left(\frac{l}{r}\right)^{\epsilon}\frac{\partial\psi}{\partial y},\quad v=-\left(\frac{l}{r}\right)^{\epsilon}\frac{\partial\psi}{\partial x} \tag{1.5}$$

式中 l 为物体的特征长度．则方程(1.2)是满足的．再作变换

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$$\xi=\int_0^x\left(\frac{r}{l}\right)^{2\epsilon}\frac{U}{U_\infty}dx,\quad \eta=\frac{U}{(2\nu U_\infty\xi)^{\frac{1}{2}}}\left(\frac{r}{l}\right)^{\epsilon}y,\quad \psi=(2\nu U_\infty\xi)^{\frac{1}{2}}f(\xi,\eta) \tag{1.6}$$

式中 U_∞为来流速度，则由 (1.5) 可得

$$\left.\begin{aligned} u&=U\frac{\partial f}{\partial\eta}\\ v&=-U\left(\frac{\nu}{2U_\infty\xi}\right)^{\frac{1}{2}}\left(\frac{r}{l}\right)^{\epsilon}\left[f+2\xi\frac{\partial f}{\partial\xi}-\left(1+\lambda-2\xi\frac{d\ln r^\epsilon}{d\xi}\right)\eta\frac{\partial f}{\partial\eta}\right]\end{aligned}\right\} \tag{1.7}$$

其中

$$\lambda=-2\xi\frac{d\ln U}{d\xi} \tag{1.8}$$

利用 (1.7) 以及它们的微商，方程 (1.1) 经过一番运算后化为

$$\frac{\partial^3 f}{\partial\eta^3}+f\frac{\partial^2 f}{\partial\eta^2}=\lambda\left[1-\left(\frac{\partial f}{\partial\eta}\right)^2\right]+2\xi\left(\frac{\partial f}{\partial\eta}\frac{\partial^2 f}{\partial\xi\partial\eta}-\frac{\partial f}{\partial\xi}\frac{\partial^2 f}{\partial\eta^2}\right) \tag{1.9}$$

这就是对于不可压缩性流体的平面流和轴对称流均适用的普遍的层流边界层方程．假定吸除起始于壁面的头部 ($\xi=0$)，边界条件 (1.3)便化为

$$\left.\begin{aligned} &\eta=0:\ \frac{\partial f}{\partial\eta}=0,\ f=\xi^{-\frac{1}{2}}\int_0^\xi\frac{-v_0}{U}\left(\frac{U_\infty}{2\nu}\right)^{\frac{1}{2}}\left(\frac{l}{r}\right)^{\epsilon}d\xi\equiv k(\xi)\\ &\eta=\infty:\ \frac{\partial f}{\partial\eta}=1\end{aligned}\right\} \tag{1.10}$$

在我们所考察的吸除情形，$k(\xi)>0$.

二、普遍方程的解

现在、让我们用渐近函数法[1,2]来求解普遍的边界层方程(1.9)．令

$$f(\xi,\eta)=k(\xi)+g(\xi,\eta),\qquad g(\xi,\eta)=\sum_{i=2}^{\infty}\frac{a_i(\xi)}{i!}\eta^i \tag{2.1}$$

则在$\eta=0$处的边界条件是满足的．将(2.1)代入(1.9)便得到以 a_2 和k 以及它们对 ξ 的微商来表示的各个 a_i．如果记

$$a\equiv a_2,\quad \kappa\equiv k+2\xi\frac{dk}{d\xi} \tag{2.2}$$

便有

$$\begin{aligned} &a_2=a,\quad a_3=\lambda-\kappa a,\quad a_4=-\kappa(\lambda-\kappa a)\\ &a_5=-(1+2\lambda)a^2+2\xi aa'+\kappa^2(\lambda-\kappa a)\\ &a_6=-(4+6\lambda)\lambda a+2\xi(2\lambda'-2\kappa'a-3\kappa a')a+(5+8\lambda)\kappa a^2-\kappa^3(\lambda-\kappa a)\\ &a_7=-(4+6\lambda)\lambda^2+4\xi\lambda\lambda'+[(19+26\lambda)\lambda-14\xi\lambda']\kappa a-[(16+22\lambda)a\\ &\qquad -12\xi a']\kappa^2 a+\kappa^4(\lambda-\kappa a)-10\xi(\lambda-2\kappa a)\kappa' a\\ &a_8=[(1+2\lambda)(11+10\lambda)-16\xi\lambda']a^3-[(60+76\lambda)\lambda-(42+52\lambda)\kappa a]\kappa^2 a\\ &\qquad +(19+26\lambda)\lambda^2\kappa-\kappa^5(\lambda-\kappa a)+2\xi[8\xi a^2 a''-2\xi a c'^2\end{aligned}$$

$$
\left.\begin{aligned}
&-(6+16\lambda)a^2a'-7\lambda\kappa(\lambda'-4\kappa'a)+16\lambda'\kappa^2a\\
&-(5\lambda^2+32\kappa^2a^2)\kappa'-10\kappa^3aa'\Big]\\
a_9=&2(45+113\lambda+66\lambda^2)\lambda a^2-(117+302\lambda+176\lambda^2)\kappa a^3\\
&+4(4+3\lambda)(\lambda-\kappa a)\kappa^3a-(41+50\lambda)(\lambda-\kappa a)^2\kappa^2\\
&-(19+26\lambda)\lambda^2\kappa^2+10(6+7\lambda)\lambda\kappa^3a-2(21+26\lambda)\kappa^4a^2\\
&-(\lambda-\kappa a)\kappa^6-2\xi[(27+92\lambda)\lambda'a^2+(10+42\lambda)\lambda aa'\\
&-(2+80\lambda)\kappa a^2a'-(8+6\lambda)(4\kappa'a+6\kappa a')a^2\\
&+(98\kappa'a-16\lambda')\lambda\kappa^2-(30\lambda\kappa'+58\lambda'\kappa)a^3\\
&+(30\lambda'-15\kappa a')\kappa^3a-28\lambda^2\kappa\kappa'-84\kappa^3\kappa'a^2]\\
&-4\xi^2[13\lambda'aa'-9\lambda(a'^2+aa'')-10\lambda''a^2\\
&-(9a'^2-28aa'')\kappa a+(12\kappa'a'+10\kappa''a)a^2]
\end{aligned}\right\}\tag{2.3}
$$

式中的撇号表示对 ξ 求微商．我们保留了 κ 的高次幂，为的是使我们的解也可用于强吸除的情形．

未知的参数函数 $a(\xi)$ 可由 $\eta=\infty$ 处的边界条件 $\left(即\left.\frac{\partial f}{\partial\eta}\right|_{\eta=\infty}=1\right)$ 求出，这是问题的关键，其步骤如下：

将方程 (1.9) 写作

$$
\frac{\partial^3 f}{\partial\eta^3}+f\frac{\partial^2 f}{\partial\eta^2}=P(\xi,\eta)
$$

并把它当作 $\frac{\partial^2 f}{\partial\eta^2}$ 的非齐次线性微分方程来求解，其中 $f(\xi,\eta)$ 和 $P(\xi,\eta)$ 被看作是已知的，我们得出

$$
\frac{\partial^2 f}{\partial\eta^2}=\left[a+\int_0^\eta Pe^F d\eta\right]e^{-F}=\left\{\left[a+\int_0^\eta Pe^F d\eta\right]e^{-k\eta}\right\}e^{-G}
$$

其中

$$
a(\xi)=\left.\frac{\partial^2 f}{\partial\eta^2}\right|_{\eta=0},\quad F(\xi,\eta)=\int_0^\eta f(\xi,\eta)d\eta,\quad G(\xi,\eta)=\int_0^\eta g(\xi,\eta)d\eta \tag{2.4}
$$

即解答具有形式

$$
\frac{\partial^2 f}{\partial\eta^2}=e^{-G}\phi(\xi,\eta) \tag{2.5}
$$

令

$$
\phi(\xi,\eta)=\sum_{i=0}^{\infty}\frac{b_i(\xi)}{i!}\eta^i \tag{2.6}
$$

将(2.1)、(2.4)、(2.6)代入(2.5)，两边按 η 的乘幂展开并比较 η 的同次幂．我们有

$$
\left.\begin{aligned}
&b_0=a,\ b_1=a_3,\ b_2=a_4,\ b_3=a^2+a_5,\ b_4=5aa_3+a_6\\
&b_5=11aa_4+5a_3^2+a_7,\ b_6=10a^3+21aa_5+21a_3a_4+a_8
\end{aligned}\right\}\tag{2.7}
$$

为了求出未知的参数函数 $a(\xi)$ 以及边界层内的速度，必须计算积分

$$
\frac{\partial f}{\partial\eta}=\int_0^\eta e^{-G}\phi d\eta \tag{2.8}
$$

这可以用最陡下降法来完成．令[1,2]

$$G(\xi,\eta)=\eta^3\sum_{i=0}^{\infty}c_i(\xi)\eta^i \tag{2.9}$$

由(2.1)、(2.4)、(2.9)可得

$$\left.\begin{aligned}&c_0=\frac{a}{3!},\quad c_1=\frac{a_3}{4!},\quad c_2=\frac{a_4}{5!},\quad c_3=\frac{a_5}{6!}\\&c_4=\frac{a_6}{7!},\quad c_5=\frac{a_7}{8!},\quad c_6=\frac{a_8}{9!}\end{aligned}\right\} \tag{2.10}$$

现在将(2.9)反演，亦即用 G 来表述 η．设

$$\eta=\sum_{i=0}^{\infty}\frac{A_i}{i+1}G^{\frac{1}{3}(i+1)} \tag{2.11}$$

Meksyn[1,2] 给出了反演成 G 的幂级数的普遍方法的细节并且证明 A_i 乃是

$$(c_0+c_1\eta+c_2\eta^2+\cdots)^{-\frac{1}{3}(i+1)}$$

展开后 η^i 项的系数，因此，在我们的问题里，

$$\left.\begin{aligned}&A_0=\left(\frac{6}{a}\right)^{\frac{1}{3}},\quad A_1=-\left(\frac{6}{a}\right)^{\frac{2}{3}}\frac{a_3}{6a},\quad A_2=-\frac{3}{10}\frac{a_4}{a^2}+\frac{3}{8}\frac{a_3^2}{a^3}\\&A_3=-\left(\frac{6}{a}\right)^{\frac{1}{3}}\left(\frac{1}{15}\frac{a_5}{a^2}-\frac{7}{30}\frac{a_3a_4}{a^3}+\frac{35}{216}\frac{a_3^3}{a^4}\right)\\&A_4=-\left(\frac{6}{a}\right)^{\frac{2}{3}}\left(\frac{1}{84}\frac{a_6}{a^2}-\frac{1}{18}\frac{a_3a_5}{a^3}-\frac{1}{30}\frac{a_4^2}{a^3}+\frac{11}{72}\frac{a_3^2a_4}{a^4}-\frac{385}{5184}\frac{a_3^4}{a^5}\right)\\&A_5=-\frac{3}{a^2}\left(\frac{1}{280}\frac{a_7}{a}-\frac{3}{140}\frac{a_3a_6}{a^2}-\frac{3}{100}\frac{a_4a_5}{a^2}+\frac{9}{100}\frac{a_3a_4^2}{a^3}\right.\\&\qquad\left.+\frac{3}{40}\frac{a_3^2a_5}{a^3}-\frac{3}{16}\frac{a_3^3a_4}{a^4}+\frac{9}{128}\frac{a_3^5}{a^5}\right)\end{aligned}\right\} \tag{2.12}$$

他还证明

$$\frac{\partial f}{\partial\eta}=\int_0^G e^{-G}\phi\frac{d\eta}{dG}dG=\int_0^G e^{-G}\sum_{i=0}^{\infty}d_iG^{\frac{1}{3}(i-2)}dG \tag{2.13}$$

其中 d_i 等于

$$(c_0+c_1\eta+c_2\eta^2+c_3\eta^3+\cdots)^{-\frac{1}{3}(i+1)}\cdot\left(b_0+b_1\eta+\frac{b_2}{2!}\eta^2+\frac{b_3}{3!}\eta^3+\cdots\right)$$

展开后 η^i 项的系数的三分之一，于是在我们的情形[1]，

$$\begin{aligned}&d_0=\frac{1}{3}c_0^{-\frac{1}{3}}b_0=\frac{1}{3}(6a^2)^{\frac{1}{3}}\\&d_1=\frac{1}{3}c_0^{-\frac{2}{3}}\left(-\frac{2}{3}c_0^{-1}c_1b_0+b_1\right)=\frac{5}{18}\left(\frac{6}{a}\right)^{\frac{2}{3}}a_3\\&d_2=\frac{1}{3}\left[\left(-c_0^{-2}c_2+c_0^{-3}c_1^2\right)b_0-c_0^{-2}c_1b_1+c_0^{-1}\frac{b^2}{2!}\right]=\frac{9}{10}\frac{a_4}{a}-\frac{3}{8}\left(\frac{a_3}{a}\right)^2\end{aligned}$$

$$
\left.\begin{aligned}
d_3 &= \frac{1}{3}c_0^{-\frac{1}{3}}\Big[\Big(-\frac{4}{3}c_0^{-2}\,c_3+\frac{28}{9}c_0^{-3}\,c_1c_2-\frac{140}{81}c_0^{-4}c_1^3\Big)b_0 \\
&\quad +\Big(-\frac{4}{3}c_0^{-2}\,c_2+\frac{14}{9}c_0^{-3}c_1^2\Big)b_1-\frac{4}{3}c_0^{-2}c_1\frac{b_2}{2!}+c_0^{-1}\frac{b_3}{3!}\Big] \\
&= \Big(\frac{6}{a}\Big)^{\frac{1}{3}}\Big[\frac{1}{3}a+\frac{14}{45}\,\frac{a_5}{a}-\frac{7}{18}\,\frac{a_3a_4}{a^2}+\frac{91}{648}\Big(\frac{a_3}{a}\Big)^3\Big] \\
d_4 &= \frac{1}{3}c_0^{-\frac{2}{3}}\Big\{\Big[-\frac{5}{3}c_0^{-2}\,c_4+\frac{20}{4}c_0^{-3}\,(c_2^2+2c_1c_3)-\frac{220}{27}c_0^{-4}c_1^2c_2 \\
&\quad +\frac{770}{243}c_0^{-5}c_1^4\Big]b_0+\Big(-\frac{5}{3}c_0^{-2}c_3+\frac{40}{9}c_0^{-3}c_1c_2-\frac{220}{81}c_0^{-4}c_1^3\Big)b_1 \\
&\quad +\Big(-\frac{5}{3}c_0^{-2}c_2+\frac{20}{9}c_0^{-3}c_1^2\Big)\frac{b_2}{2!}-\frac{5}{3}c_0^{-2}c_1\frac{b_3}{3!}+c_0^{-1}\frac{b_4}{4!}\Big\} \\
&= \Big(\frac{6}{a}\Big)^{\frac{2}{3}}\Big[\frac{5}{18}a_3+\frac{5}{63}\,\frac{a_6}{a}-\frac{4}{27}\,\frac{a_3a_5}{a^2}-\frac{13}{180}\Big(\frac{a_4}{a}\Big)^2+\frac{43}{216}\,\frac{a_3^2a_4}{a^3} \\
&\quad -\frac{935}{15552}\Big(\frac{a_3}{a}\Big)^4\Big] \\
b_5 &= \frac{1}{3}\Big\{\Big[-2c_0^{-3}\,c_5+3c_0^{-4}\cdot 2(c_1c_4+c_2c_3)-4c_0^{-5}\cdot 3(c_1^2c_3+c_1c_2^2) \\
&\quad +5c_0^{-6}\cdot 4c_1^3c_2-6c_0^{-7}c_1^5\Big]b_0+[-2c_0^{-3}c_4+3c_0^{-4}(c_2^2+2c_1c_3) \\
&\quad -4c_0^{-5}\cdot 3c_1^2\,c_2+5c_0^{-6}c_1^4]\,b_1+[-2c_0^{-3}\,c_3+3c_0^{-4}\cdot 2c_1c_2-4c_0^{-5}c_1^3]\frac{b_2}{2!} \\
&\quad +[-2c_0^{-3}\,c_2+3c_0^{-4}c_1^2]\frac{b_3}{3!}-2c_0^{-3}c_1\frac{b_4}{4!}+c_0^{-2}\frac{b_5}{5!}\Big\} \\
&= \frac{3}{a^2}\Big[\frac{3}{10}aa_4+\frac{9}{280}a_7-\frac{1}{8}a_3^2-\frac{3}{5}\cdot\frac{1}{a}\Big(\frac{1}{7}a_3a_6+\frac{3}{20}a_4a_5\Big) \\
&\quad +\frac{3}{20}\,\frac{a_3}{a^2}(a_4^2+a_3a_5)-\frac{17}{80}\,\frac{a_3{}^3a_4}{a^3}+\frac{7}{128}\,\frac{a_3{}^5}{a^4}\Big]
\end{aligned}\right\}\quad(2.14)
$$

这样，$a(\xi)$便由条件

$$1=\frac{\partial f}{\partial\eta}\Big|_{\eta=\infty}=\int_0^\infty e^{-G}\sum_{i=0}^{\infty}d_iG^{\frac{1}{3}(i-2)}dG$$

或

$$\sum_{i=0}^{\infty}d_i\Gamma\left(\frac{i+1}{3}\right)=1 \qquad (2.15)$$

给出，其中 Γ 是伽马函数．因为 (2.15) 中的级数多半不收敛，必要时得使用 Euler 变换(2)——发散级数的一种求和法．

表达式(2.15)，假如只保留前六项，乃是关于$a(\xi)$的非线性一阶常微分方程，通常可用

(1) 在d_i的诸公式中，除了以a_i来表示的最终表达式外，我们还特意保留了以c_i和b_i来表示的表达式，因为它们将在下节推导位移厚度和动量厚度的公式中要用到．

(2) 关于Euler 变换的扼要叙述，见文献[2]或[1]．

逐步法求解．设a在某点$\xi=\xi_0$的数值已经给定，既经求出$\frac{da}{d\xi}$，则毗邻ξ_0的下一个点的a值即可算出．关于这种步骤，文献[3]中有详尽的说明．

要计算边界层内的速度，就得把(2.13)积分成不完全伽马函数[1)]，亦即

$$\frac{\partial f}{\partial \eta}=\sum_{i=0}^{\infty} d_i \gamma_G\left(\frac{i+1}{3}\right) \tag{2.16}$$

与G相对应的η值可由(2.11)求出

三、位移厚度、动量厚度和表面摩擦

现在，我们能够推导下列各个重要的量的计算公式．

1. 位移厚度

位移厚度的定义是

$$\delta_1=\int_0^{\infty}\left(1-\frac{u}{U}\right)dy \tag{3.1}$$

据此．由(1.6)和(1.7)．

$$\delta_1=\frac{(2\nu U_{\infty}\xi)^{\frac{1}{2}}}{U}\left(\frac{l}{r}\right)^{\epsilon}\int_0^{\infty}\left(1-\frac{\partial f}{\partial \eta}\right)d\eta$$

若在有限的上下限之间作分部积分，然后令上限趋近于无穷大，则

$$\int_0^{\eta}\left(1-\frac{\partial f}{\partial \eta}\right)d\eta=\eta-\eta\frac{\partial f}{\partial \eta}+\int_0^{\eta}\frac{\partial^2 f}{\partial \eta^2}\eta d\eta \rightarrow \int_0^{\eta\rightarrow\infty}\frac{\partial^2 f}{\partial \eta^2}\eta d\eta$$

(在最后一步中已利用了$\eta=\infty$处的边界条件)．由此，

$$\delta_1=\frac{(2\nu U_{\infty}\xi)^{\frac{1}{2}}}{U}\left(\frac{l}{r}\right)^{\epsilon}\int_0^{\infty}\frac{\partial^2 f}{\partial \eta^2}\eta d\eta$$

根据(2.5)和(2.6)．我们有

$$\int_0^{\infty}\frac{\partial^2 f}{\partial \eta^2}\eta d\eta=\int_0^{\infty}e^{-G}\phi\eta d\eta=\int_0^{\infty}e^{-G}\Phi^{(\delta_1)}\frac{d\eta}{dG}dG$$

其中

$$\Phi^{(\delta_1)}=\phi\eta=\sum_{i=0}^{\infty}\frac{b_i(\xi)}{i!}\eta^{i+1} \tag{3.2}$$

若令

$$\Phi^{(\delta_1)}=\sum_{i=0}^{\infty}\frac{b_i^{(\delta_1)}(\xi)}{i!}\eta^{i} \tag{3.3}$$

并比较(3.2)、(3.3)，便得

1) 专著[2]的附录里有不完全伽马函数$\gamma_x(n)=\int_0^x t^{n-1}e^{-t}dt$对应于各种$x$和$n$的数值表．

$$\left.\begin{aligned}&b_0^{(\delta_1)}=0,\ b_1^{(\delta_1)}=a,\ b_2^{(\delta_1)}=2a_3,\ b_3^{(\delta_1)}=3a_4,\ b_4^{(\delta_1)}=4(a^2+a_5)\\&b_5^{(\delta_1)}=5(5aa_3+a_6),\ b_6^{(\delta_1)}=6(11aa_4+a_7+5a_3^2),\cdots,\end{aligned}\right\}\tag{3.4}$$

此处已应用了(2.7).

利用 (3.3) 并注意到(2.13). 我们有

$$\int_0^\infty \frac{\partial^2 f}{\partial\eta^2}\eta d\eta=\int_0^\infty e^{-G}\Phi^{(\delta_1)}\frac{d\eta}{dG}=\int_0^\infty e^{-G}\sum_{i=0}^\infty d_i^{(\delta_1)}G^{\frac{1}{3}(i-2)}dG$$

或

$$\int_0^\infty \frac{\partial^2 f}{\partial\eta^2}\eta d\eta=\sum_{i=0}^\infty d_i^{(\delta_1)}\Gamma\left(\frac{i+1}{3}\right)\tag{3.5}$$

因此，位移厚度的最终表达式为

$$\delta_1=\frac{(2\nu U_\infty\xi)^{\frac{1}{2}}}{U}\left(\frac{l}{r}\right)^\epsilon\sum_{i=0}^\infty d_i^{(\delta_1)}\Gamma\left(\frac{i+1}{3}\right)\tag{3.6}$$

在(2.14)中、如果每个$b_i(i=0,1,2,\cdots)$被相应的$b_i^{(\delta_1)}$所取代，则d_i将被$d_i^{(\delta_1)}$所取代. 所以我们有

$$\left.\begin{aligned}&d_0^{(\delta_1)}=0,\ d_1^{(\delta_1)}=\frac{1}{3}\left(\frac{6}{a}\right)^{\frac{2}{3}}a,\ d_2^{(\delta_1)}=\frac{3}{2}\frac{a_3}{a}\\&d_3^{(\delta_1)}=\left(\frac{6}{a}\right)^{\frac{1}{3}}\left[\frac{13}{15}\frac{a_4}{a}-\frac{17}{36}\left(\frac{a_3}{a}\right)^2\right]\\&d_4^{(\delta_1)}=\left(\frac{6}{a}\right)^{\frac{2}{3}}\left[\frac{1}{3}a+\frac{11}{36}\frac{a_5}{a}-\frac{17}{36}\frac{a_3a_4}{a^2}+\frac{125}{648}\left(\frac{a_3}{a}\right)^3\right]\\&d_5^{(\delta_1)}=\frac{1}{a}\left[\frac{3}{2}a_3+\frac{33}{70}\frac{a_6}{a}-\frac{21}{20}\frac{a_3a_5}{a^2}-\frac{51}{100}\left(\frac{a_4}{a}\right)^2+\frac{63}{40}\frac{a_3^2a_4}{a^3}-\frac{33}{64}\left(\frac{a^3}{a}\right)^4\right]\\&d_6^{(\delta_1)}=\frac{1}{a}\left(\frac{6}{a}\right)^{\frac{1}{3}}\left[\frac{13}{15}a_4-\frac{17}{36}\frac{a_3^2}{a}+\frac{23}{240}\frac{a_7}{a}-\frac{107}{360}\frac{a_3a_6}{a^2}-\frac{14}{45}\frac{a_4a_5}{a^2}\right.\\&\qquad\left.+\frac{413}{720}\frac{a_3a_4^2}{a^3}+\frac{497}{864}\frac{a_3^2a_5}{a^3}-\frac{2275}{2592}\frac{a_3^3a_4}{a^4}+\frac{3731}{15552}\left(\frac{a_3}{a}\right)^5\right]\end{aligned}\right\}\tag{3.7}$$

但是，由于(3.6)中的级数可能是发散的，那就得应用 Euler 变换以求出它的收敛的数值.

2. 动量厚度

动量厚度的定义是

$$\delta_2=\int_0^\infty \frac{u}{U}\left(1-\frac{u}{U}\right)dy\tag{3.8}$$

据此. 由(1.6)和(1.7),

$$\delta_2=\frac{(2\nu U_\infty\xi)^{\frac{1}{2}}}{U}\left(\frac{l}{r}\right)^\epsilon\int_0^\infty\frac{\partial f}{\partial\eta}\left(1-\frac{\partial f}{\partial\eta}\right)d\eta$$

作分部积分

$$\int_0^\eta \frac{\partial f}{\partial \eta}\left(1-\frac{\partial f}{\partial \eta}\right)d\eta=\eta\frac{\partial f}{\partial \eta}-\eta\left(\frac{\partial f}{\partial \eta}\right)^2+\int_0^\eta\left(2\frac{\partial f}{\partial \eta}-1\right)\frac{\partial^2 f}{\partial \eta^2}\eta d\eta$$

由此,

$$\delta_2=\frac{(2\nu U_\infty\xi)^{\frac{1}{2}}}{U}\left(\frac{l}{r}\right)^\epsilon\cdot 2\int_0^\infty\frac{\partial^2 f}{\partial \eta^2}\frac{\partial f}{\partial \eta}\eta d\eta-\delta_1$$

根据(2.5)、(2.6)和(2.1),积分

$$\int_0^\infty\frac{\partial^2 f}{\partial \eta^2}\frac{\partial f}{\partial \eta}\eta d\eta=\int_0^\infty e^{-G}\phi\frac{\partial f}{\partial \eta}\eta d\eta=\int_0^\infty e^{-G}\Phi^{(\delta_2)}\frac{d\eta}{dG}dG$$

其中

$$\Phi^{(\delta_2)}=\phi\frac{\partial f}{\partial \eta}\eta=\left(b_0+b_1\eta+\frac{b_2}{2!}\eta^2+\cdots\right)\left(a\eta^2+\frac{a_3}{2!}\eta^3+\frac{a_4}{3!}\eta^4+\cdots\right) \quad (3.9)$$

令

$$\Phi^{(\delta_2)}=\sum_{i=0}^\infty\frac{b_i^{(\delta_2)}(\xi)}{i!}\eta^i \quad (3.10)$$

我们得

$$\int_0^\infty\frac{\partial^2 f}{\partial \eta^2}\frac{\partial f}{\partial \eta}\eta d\eta=\int_0^\infty e^{-G}\sum_{i=0}^\infty d_i^{(\delta_2)}G^{\frac{1}{3}(i-2)}dG$$

$$=\sum_{i=0}^\infty d_i^{(\delta_2)}\Gamma\left(\frac{i+1}{3}\right) \quad (3.11)$$

于是动量厚度的最终表达式为

$$\delta_2=\frac{(2\nu U_\infty\xi)^{\frac{1}{2}}}{U}\left(\frac{l}{r}\right)^\epsilon\cdot 2\sum_{i=0}^\infty d_i^{(\delta_2)}\Gamma\left(\frac{i+1}{3}\right)-\delta_1$$

或

$$\delta_2=\frac{(2\nu U_\infty\xi)^{\frac{1}{2}}}{U}\left(\frac{l}{r}\right)^\epsilon\sum_{i=0}^\infty\left[\left(2d_i^{(\delta_2)}-d_i^{(\delta_1)}\right)\Gamma\left(\frac{i+1}{3}\right)\right] \quad (3.12)$$

其中 $d_i^{(\delta_1)}$已由(3.7)给出, 而 $d_i^{(\delta_2)}$则有待以同样的办法来确定. 比较(3.9)、(3.10)二式并考虑到(2.7), 得

$$\left.\begin{aligned}&b_0^{(\delta_2)}=0,\quad b_1^{(\delta_2)}=0,\quad b_2^{(\delta_2)}=2a^2,\quad b_3^{(\delta_2)}=9aa_3\\&b_4^{(\delta_2)}=4!\left(\frac{2}{3}aa_4+\frac{1}{2}a_3^2\right),\quad b_5^{(\delta_2)}=5!\left(\frac{1}{6}a^3+\frac{5}{24}aa_5+\frac{5}{12}a_3a_4\right)\\&b_6^{(\delta_2)}=6!\left(\frac{7}{24}a^2a_3+\frac{1}{20}aa_6+\frac{1}{8}a_3a_5+\frac{1}{12}a_4^2\right)\\&b_7^{(\delta_2)}=7!\left(\frac{43}{360}a^2a_4+\frac{7}{720}aa_7+\frac{7}{48}aa_3^2+\frac{7}{240}a_3a_6+\frac{7}{144}a_4a_5\right)\end{aligned}\right\} \quad (3.13)$$

在(2.14)中, 如果 $b_i(i=0,1,2,\cdots)$ 被相应的 $b_i^{(\delta_2)}$ 所取代, 则 d_i 将被 $d_i^{(\delta_2)}$所取代, 因而我们有

$$\left.\begin{aligned}
&d_0^{(\delta_2)}=0,\quad d_1^{(\delta_2)}=0,\quad d_2^{(\delta_2)}=2a,\quad d_3^{(\delta_2)}=\frac{7}{3}\left(\frac{6}{a}\right)^{\frac{1}{3}}a_3\\
&d_4^{(\delta_2)}=\left(\frac{6}{a}\right)^{\frac{2}{3}}\left(\frac{7}{6}a_4+\frac{1}{36}\frac{a_3^2}{a}\right),\quad d_5^{(\delta_2)}=2a+\frac{23}{10}\frac{a_5}{a}+\frac{1}{10}\frac{a_3a_4}{a^2}-\frac{3}{8}\left(\frac{a_3}{a}\right)^3\\
&d_6^{(\delta_2)}=\left(\frac{6}{a}\right)^{\frac{1}{3}}\left[\frac{7}{3}a_3+\frac{17}{30}\frac{a_6}{a}-\frac{41}{360}\frac{a_3a_5}{a^2}+\frac{11}{60}\left(\frac{a_4}{a}\right)^2-\frac{133}{240}\frac{a_3^2a_4}{a^3}\right.\\
&\qquad\left.+\frac{595}{2592}\left(\frac{a_3}{a}\right)^4\right]\\
&d_7^{(\delta_2)}=\left(\frac{6}{a}\right)^{\frac{2}{3}}\left[\frac{7}{6}a_4+\frac{47}{420}\frac{a_7}{a}+\frac{1}{36}\frac{a_3^2}{a}-\frac{91}{1260}\frac{a_3a_6}{a^2}+\frac{109}{900}\frac{a_4a_5}{a^2}\right.\\
&\qquad\left.-\frac{23}{75}\frac{a_3a_4^2}{a^3}-\frac{157}{1080}\frac{a_3^2a_5}{a^3}+\frac{1397}{3240}\frac{a_3^3a_4}{a_4}-\frac{1925}{15552}\left(\frac{a_3}{a}\right)^5\right]
\end{aligned}\right\}\quad(3.14)$$

为了找出级数(3.12)的收敛的数值，必要时仍得使用 Euler 变换.

3. 表面摩擦

表面摩擦的定义是

$$\tau_0=\mu\left(\frac{\partial u}{\partial y}\right)_{y=0}$$

其中 μ 为流体的粘性系数. 根据 (1.6) 和 (1.7) 不难求得

$$\tau_0=\frac{\mu U^2}{(2\nu U_\infty\xi)^{\frac{1}{2}}}\left(\frac{r}{l}\right)^\epsilon\left(\frac{\partial^2 f}{\partial\eta^2}\right)_{\eta=0}=\frac{\mu U^2}{(2\nu U_\infty\xi)^{\frac{1}{2}}}\left(\frac{r}{l}\right)^\epsilon a \tag{3.15}$$

于是当地表面摩擦系数为

$$c_f=\frac{\tau_0}{\frac{1}{2}\rho U_\infty^2}=2\left(\frac{U}{U_\infty}\right)^2\left(\frac{\nu}{2U_\infty\xi}\right)^{\frac{1}{2}}\left(\frac{r}{l}\right)^\epsilon a \tag{3.16}$$

式中 ρ 为流体的密度 ($\rho\nu=\mu$).

四、分 离 点

众所周知，在分离点表面摩擦 τ_0 消失，于是在该点有补充条件

$$a=\left.\frac{\partial^2 f}{\partial\eta^2}\right|_{\eta=0}=0$$

从而(根据 2.1) $g(\xi,\eta)$ 始自 η^3 项，所以我们需要另外一组公式.

我们要找出 $a=0$ 处 (分离点) 的 ξ 值. 由于表达式还将用来计算该点下 游 的慢回流，不妨假定 a 很小并略去其平方及更高次的乘幂.

现在把 (2.8) 写作

$$\frac{\partial f}{\partial\eta}=\int_0^\eta e^{-H}\left(1-\frac{a}{3!}\eta^3\right)\phi\,d\eta=\int_0^\eta e^{-H}\tilde{\phi}\,d\eta \tag{4.1}$$

其中

$$\left.\begin{aligned}&H=G-\frac{a}{3!}\eta^3=\frac{a_3}{4!}\eta^4+\frac{a_4}{5!}\eta^5+\cdots\\&\left(1-\frac{a}{3!}\eta^3\right)\phi(\xi,\eta)=\tilde{\phi}(\xi,\eta)\equiv\sum_{i=0}^{\infty}\frac{\tilde{b}_i(\xi)}{i!}\eta^i\end{aligned}\right\}\tag{4.2}$$

由(2.6)、(2.7)和(4.2)，我们找出

$$\left.\begin{aligned}&\tilde{b}_0=a,\quad \tilde{b}_1=a_3,\quad \tilde{b}_2=a_4,\quad \tilde{b}_3=a_5,\quad \tilde{b}_4=aa_3+a_6\\&\tilde{b}_5=aa_4+a_7+5a_3^2,\quad \tilde{b}_6=aa_5+21a_3a_4+a_8\end{aligned}\right\}\tag{4.3}$$

在 $\tilde{b}_i$ 的表达式(4.3)中，$O(a^2)$项要略去.

为了计算(4.1)，令

$$H(\xi,\eta)=\eta^4\sum_{i=0}^{\infty}\tilde{c}_i(\xi)\eta^i\tag{4.4}$$

比较(4.2)、(4.4)给出

$$\left.\begin{aligned}&\tilde{c}_0=a_3/4!,\quad \tilde{c}_1=a_4/5!,\quad \tilde{c}_2=a_5/6!,\quad \tilde{c}_3=a_6/7!\\&\tilde{c}_4=a_7/8!,\quad \tilde{c}_5=a_8/9!,\quad \tilde{c}_6=a_9/10!\end{aligned}\right\}\tag{4.5}$$

以 H 为自变数．我们得

$$\frac{\partial f}{\partial\eta}=\int_0^H e^{-H}\sum_{i=0}^{\infty}\tilde{d}_iH^{\frac{1}{4}(i-3)}dH=\sum_{i=0}^{\infty}\tilde{d}_i\gamma_H\left(\frac{i+1}{4}\right)\tag{4.6}$$

其中$\tilde{d}_i$ 等于$(\tilde{c}_0+\tilde{c}_1\eta+\tilde{c}_2\eta^2+\tilde{c}_3\eta^3+\cdots)^{-\frac{1}{4}(i+1)}\cdot\left(\tilde{b}_0+\tilde{b}_1\eta+\frac{\tilde{b}_2}{2!}\eta^2+\frac{\tilde{b}_3}{3!}\eta^3+\cdots\right)$展开后 η^i 项的系数的四分之一，即

$$\left.\begin{aligned}&\tilde{d}_0=\frac{1}{4}\left(\frac{24}{\lambda-\kappa a}\right)^{\frac{1}{4}}a,\quad \tilde{d}_1=\frac{1}{4}\left(\frac{24}{\lambda-\kappa a}\right)^{\frac{1}{2}}\left(\lambda-\frac{9}{10}\kappa a\right)\\&\tilde{d}_2=-\frac{1}{80}\left(\frac{24}{\lambda-\kappa a}\right)^{\frac{3}{4}}\left(7\lambda\kappa-\frac{281}{40}\kappa^2a\right)\\&\tilde{d}_3=\frac{2}{5(\lambda-\kappa a)}\left(4\xi aa'+\frac{11}{10}\lambda\kappa^2-\frac{194}{175}\kappa^3a\right)\\&\tilde{d}_4=\frac{1}{4}\left(\frac{24}{\lambda-\kappa a}\right)^{\frac{1}{4}}\left\{a-\frac{1}{28}\left(\frac{48}{\lambda-\kappa a}-\frac{1}{\lambda-2\kappa a}\right)[(2+3\lambda)\lambda-2\xi\lambda']a\right.\\&\qquad\left.-\frac{213}{70}\frac{\xi\kappa aa'}{\lambda-\kappa a}-\frac{37}{224000}\frac{\kappa^4a}{\lambda-\kappa a}-\frac{531}{2800}\kappa^3\right\}\\&\tilde{d}_5=\frac{1}{4}\left(\frac{24}{\lambda-\kappa a}\right)^{\frac{1}{2}}\left\{\lambda-\frac{9}{10}\kappa a+\frac{43}{14000}\kappa^4+\frac{183}{350}\frac{\xi\kappa^2aa'}{\lambda-\kappa a}+\frac{359}{21\times10^5}\frac{\kappa^5a}{\lambda-\kappa a}\right.\\&\qquad-\frac{5}{28}\frac{2[(2+3\lambda)\lambda-2\xi\lambda']\lambda-[(19+26\lambda)\lambda\kappa-2\xi(7\lambda'\kappa+5\lambda\kappa')]a}{\lambda-\kappa a}\\&\qquad-\frac{3}{5}\frac{[(2+3\lambda)\lambda-2\xi\lambda']\kappa a}{\lambda-\kappa a}\\&\qquad\left.-\frac{1}{420}\frac{[(1-\lambda)\lambda\kappa+2\xi(2\lambda'\kappa-5\lambda\kappa')]a}{\lambda-2\kappa a}\right\}\end{aligned}\right\}\tag{4.7}$$

$$\tilde{a}_6=-\frac{1}{4}\left(\frac{24}{\lambda-\kappa a}\right)^{\frac{3}{4}}\left\{\frac{1}{900}\left(\frac{110}{\lambda-\kappa a}+\frac{9}{\lambda-2\kappa a}\right)\xi^2aa'^2+\frac{7}{20}\kappa(\lambda-\kappa a)\right.$$
$$-\frac{1}{7200}(91+100\lambda)\kappa^2a-\frac{40073}{288\times10^5}\kappa^6-\frac{13506747}{24192\times10^6}\frac{\kappa^6a}{\lambda-\kappa a}$$
$$+\frac{3619}{24000}\frac{\xi\kappa^3aa'}{\lambda-\kappa a}-\frac{11}{7200}\frac{50(4+5\lambda)\lambda^2\kappa-(441+494\lambda)\lambda\kappa^2a}{\lambda-\kappa a}$$
$$+\frac{11}{3600}\frac{\xi[50\lambda(\lambda'\kappa+2\lambda\kappa')+(145\lambda\kappa\kappa'-53\lambda'\kappa^2-46\kappa^3a')a]}{\lambda-\kappa a}$$
$$\left.-\frac{1}{28800}\frac{[(406+599\lambda)\lambda\kappa-2\xi(265\lambda'\kappa+204\lambda\kappa')]\kappa a}{\lambda-2\kappa a}\right\}$$

关于 a 的方程现在是

$$1=\frac{\partial f}{\partial\eta}\bigg|_{\eta=\infty}=\sum_{i=0}^{\infty}\tilde{a}_i\,\Gamma\left(\frac{i+1}{4}\right)\tag{4.8}$$

以H来表述的变数η等于

$$\eta=\sum_{i=0}^{\infty}\frac{\tilde{A}_i}{i+1}H^{\frac{1}{4}(i+1)}\tag{4.9}$$

其中$\tilde{A}_i$是$(\tilde{c}_0+\tilde{c}_1\eta+\tilde{c}_2\eta^2+\cdots)^{-\frac{1}{4}(i+1)}$展开后$\eta^i$项的系数，即

$$\tilde{A}_0=\left(\frac{24}{\lambda-\kappa a}\right)^{\frac{1}{4}},\qquad \tilde{A}_1=\frac{1}{10}\left(\frac{24}{\lambda-\kappa a}\right)^{\frac{1}{2}}\kappa$$
$$\tilde{A}_2=-\frac{1}{20}\left(\frac{24}{\lambda-\kappa a}\right)^{\frac{3}{4}}\left(\frac{\xi aa'}{\lambda-\kappa a}-\frac{\kappa^2}{40}\right)$$
$$\tilde{A}_3=\frac{1}{(\lambda-\kappa a)^2}\left\{\frac{8a}{35}[(2+3\lambda)\lambda-\xi(2\lambda'-3\kappa a')]-\frac{12}{875}\kappa^3(\lambda-\kappa a)\right.$$
$$\left.-\frac{16}{25}\xi\kappa aa'\right\}$$
$$\tilde{A}_4=\left(\frac{24}{\lambda-\kappa a}\right)^{\frac{1}{4}}\left\{\frac{1}{56}\frac{(4+6\lambda)\lambda^2-4\xi\lambda\lambda'-[(19+26\lambda)\lambda-14\xi\lambda']\kappa a-12\xi\kappa^2aa'+10\xi\lambda\kappa'a}{\lambda(\lambda-2\kappa a)}\right.$$
$$+\frac{9}{140}\frac{[(4+6\lambda)\lambda-2\xi(2\lambda'-3\kappa a')]\kappa a}{\lambda(\lambda-2\kappa a)}-\frac{57}{400}\frac{\xi\kappa^2aa'}{\lambda(\lambda-2\kappa a)}$$
$$\left.-\frac{259}{1568000}\frac{\kappa^4}{\lambda-\kappa a}\right\}$$
$$\tilde{A}_5=\left(\frac{24}{\lambda-\kappa a}\right)^{\frac{1}{2}}\left\{\frac{1}{105}\frac{\xi^2aa'^2}{\lambda(\lambda-2\kappa a)}+\frac{359}{21\times10^5}\frac{\kappa^5}{\lambda-\kappa a}\right.$$
$$-\frac{51}{3500}\frac{\xi\kappa^3aa'}{\lambda(\lambda-2\kappa a)}+\frac{11}{350}\frac{[(2+3\lambda)\lambda-\xi(2\lambda'-3\kappa a')]\kappa^2a}{\lambda(\lambda-2\kappa a)}$$

$$-\frac{1}{840}\frac{[(51+82\lambda)\lambda-62\xi\lambda']\kappa^2a+2(1-\lambda)\lambda^2\kappa}{\lambda(\lambda-2\kappa a)}$$

$$\left.-\frac{1}{420}\frac{\xi(4\lambda\lambda'\kappa+11\lambda\kappa\kappa'a-10\lambda^2\kappa'+34\kappa^3aa')}{\lambda(\lambda-2\kappa a)}\right\} \tag{4.10}$$

分离点的位置可在方程(4.8)中令 $a=0$ 而求得，不过 $a'=\frac{da}{d\xi}$ 仍属未知．因为分离点的下游有一个很缓慢的回流，我们可在 $\xi=\xi_s+0$ 处令 $a'\equiv 0$，其中 ξ_s 为分离点．这样，方程(4.8)就化为未知量λ的常微分方程．关于求分离点以及该点下游流动情况的步骤的细节，参阅文献[4]及[2]．

五、算 例

现在把我们的解答应用于具有恒定吸除的平板均匀绕流问题，因为这个问题在理论上和实验上都已作过广泛的研究，便于比较．在这一特殊情形：$\epsilon=0$，$U(x)=U_\infty=\text{const}$，$v_0(x)=\text{const}<0$，而且毋须担心会发生分离．仍假定吸除起始于板的前缘．由(1.6)、(1.8)、(1.10)、(2.2)、(3.6)、(3.12)、(3.15)和(3.16)我们可得下列公式：

$$\left.\begin{aligned}
&\xi=x,\quad \eta=\left(\frac{U_\infty}{2\nu x}\right)^{\frac{1}{2}}y,\quad \lambda(\xi)=0\\
&k(\xi)=\frac{-v_0}{U_\infty}\left(\frac{U_\infty x}{2\nu}\right)^{\frac{1}{2}},\quad \kappa(\xi)=\frac{-v_0}{U_\infty}\left(\frac{2U_\infty x}{\nu}\right)^{\frac{1}{2}}\\
&\delta_1=\left(\frac{2\nu x}{U_\infty}\right)^{\frac{1}{2}}\sum_{i=0}^{\infty}d_i^{(\delta_1)}\,\Gamma\left(\frac{i+1}{3}\right),\quad \delta_2=\left(\frac{2\nu x}{U_\infty}\right)^{\frac{1}{2}}\sum_{i=0}^{\infty}\Big[(2d_i^{(\delta_2)}\\
&\qquad -d_i^{(\delta_1)})\,\Gamma\left(\frac{i+1}{3}\right)\Big]\\
&\tau_0=\frac{\mu U_\infty^2 a}{(2\nu U_\infty x)^{\frac{1}{2}}},\quad c_f=\left(\frac{2\nu}{U_\infty x}\right)^{\frac{1}{2}}a
\end{aligned}\right\} \tag{5.1}$$

按照 Iglisch[5]，我们引入两个参数：一个是所谓吸除参数 x^*，其定义为

$$x^*=\left(\frac{-v_0}{U_\infty}\right)^2\frac{U_\infty x}{\nu} \tag{5.2}$$

另一个是σ，它与 x^* 之间有关系式

$$x^*=\frac{1}{2}\sigma^2 \tag{5.3}$$

于是上述有吸除的平板绕流公式(5.1)可以用 x^* 或 σ表述如下：

$$\begin{aligned}
&\lambda=0,\quad k=\left(\frac{x^*}{2}\right)^{\frac{1}{2}}=\frac{\sigma}{2},\quad \kappa=(2x^*)^{\frac{1}{2}}=\sigma\\
&\xi\frac{d\kappa}{d\xi}=\left(\frac{x^*}{2}\right)^{\frac{1}{2}}=\frac{\sigma}{2},\quad \xi\frac{da}{d\xi}=x^*\frac{da}{dx^*}=\frac{\sigma}{2}\frac{da}{d\sigma}
\end{aligned}$$

$$\left.\begin{aligned}
\frac{(-v_0)\delta_1}{\nu} &= (2x^*)^{\frac{1}{2}}\sum_{i=0}^{\infty} d_i^{(\delta_1)}\,\Gamma\left(\frac{i+1}{3}\right)=\sigma\sum_{i=0}^{\infty} d_i^{(\delta_1)}\,\Gamma\left(\frac{i+1}{3}\right)\\
\frac{(-v_0)\delta_2}{\nu} &= (2x^*)^{\frac{1}{2}}\sum_{=0}^{\infty}\left[(2d_i^{(\delta_2)}-d_i^{(\delta_1)})\Gamma\left(\frac{i+1}{3}\right)\right]\\
&=\sigma\sum_{i=0}^{\infty}\left[(2d_i^{(\delta_2)}-d_i^{(\delta_1)})\Gamma\left(\frac{i+1}{3}\right)\right]\\
\frac{\tau_0\delta_1}{\mu U_\infty} &= \frac{(-v_0)\delta_1}{\nu}\,\frac{a}{(2x^*)^{\frac{1}{2}}}=\frac{(-v_0)\delta_1}{\nu}\,\frac{a}{\sigma}\\
c_f &= \frac{-v_0}{U_\infty}\,\frac{2a}{(2x^*)^{\frac{1}{2}}}=\frac{-v_0}{U_\infty}\,\frac{2a}{\sigma}
\end{aligned}\right\}\qquad(5.4)$$

我们明确取σ为自变数．既经由(2.15)就一组σ值（步长$\Delta\sigma$如表1所示）求出a和$\frac{da}{d\sigma}$的数值．便能由(5.4)计算边界层特性参数$(-v_0)\delta_1/\nu$、$(-v_0)\delta_2/\nu$和$\tau_0\delta_1/(\mu U_\infty)$的对应值，由(2.16)计算边界层内的速度分布．

鉴于Iglisch[5]已经圆满地解决了本例题，他的精确解与实验结果很好地符合[6]．我们就没有必要再处理整个问题了，因此，为了检验我们的解，仅就几个边界层特性参数作些计算．我们不拟在此给出计算过程，表1列出我们的最终结果并附Iglisch的精确计算值以资比较．由该表可以看出，彼此的结果是一致的．图1画出a对σ的曲线．边界层参数$(-v_0)\delta_1/\nu$、$(-v_0)\delta_2/\nu$和$\tau_0\delta_1/(\mu U_\infty)$随$\sigma$的变化如图2所示．

表 1 有吸除的平板绕流情形的一些数值结果 *

σ	a	$\frac{da}{d\sigma}$	$\frac{(-v_0)\delta_1}{\nu}$	$\frac{(-v_0)\delta_2}{\nu}$	$\frac{\tau_0\delta_1}{\mu U_\infty}$
0	0.4696	—	0 (0)	0 (0)	0.571(0.571)
0.1	0.532	0.674	0.114(0.114)	0.046(0.045)	0.606(0.607)
0.2	0.599	0.657	0.211(0.211)	0.088(0.086)	0.631(0.631)
0.3	0.665	0.671	0.303(0.303)	0.126(0.125)	0.672(0.671)
0.4	0.732	0.732	0.383(0.381)	0.160(0.160)	0.701(0.699)
0.5	0.805	0.768	0.451(0.450)	0.194(0.192)	0.727(0.726)
0.6	0.882	0.744	0.511(0.511)	0.220(0.221)	0.751(0.750)
0.7	0.956	0.759	0.566(0.566)	0.249(0.248)	0.773(0.773)
0.8	1.032	0.786	0.613(0.614)	0.270(0.273)	0.791(0.794)
0.9	1.111	0.853	0.659(0.658)	0.298(0.295)	0.813(0.813)
1.0	1.196	0.830	0.695(0.695)	0.315(0.315)	0.832(0.830)
1.2	1.362	0.846	0.759(0.761)	0.350(0.351)	0.862(0.864)
1.4	1.532	0.901	0.814(0.812)	0.382(0.380)	0.891(0.889)
1.6	1.712	0.906	0.850(0.853)	0.404(0.404)	0.910(0.911)
2.0	2.074	0.930	0.909(0.911)	0.446(0.440)	0.943(0.944)

* 括号内的数值系Iglisch的精确计算值[5]．

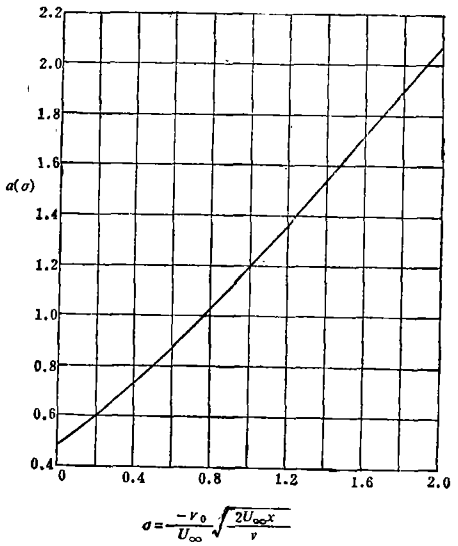

图 1 有吸除的平板绕流情形，a 对 σ 的曲线.

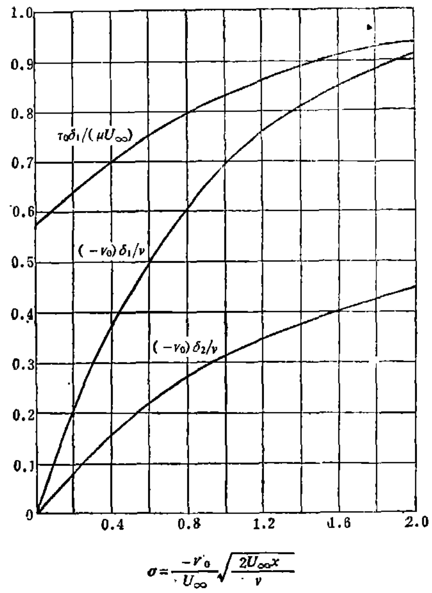

图 2 有吸除的平板绕流情形，几个边界层特性参数随 σ 的变化.

参 考 文 献

1. Meksyn, D., Integration of the boundary layer equations, *Proc. Roy. Soc.*, **A237** (1956), 543—559.
2. Meksyn, D., *New Methods in Laminar Boundary-Layer Theory*, Pergamon Press, Oxford (1961).
3. Meksyn, D., The boundary-layer equation for axially symmetric flow past a body of revolution-motion of a sphere, *J. Aero/Space Sci.*, 25(1958), 631—634, 664.
4. Meksyn, D., The boundary layer equation of compressible flow. Separation, *Z angew. Math. Mech.*, 38(1958), 372—379.
5. Iglisch, R., Exact calculation of laminar boundary layer in longitudinal flow over a flat plate with homogeneous suction, *NACA Tech. Memor.*, №1205, (1949).
6. Schlichting, H., *Grenzschicht-Theorie*, 5. Aufl., G. Braun, Karlsruhe (1965).

On Laminar Boundary Layers with Suction

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Abstract

In this paper, we obtain the asymptotic solution of the general equation for laminar boundary-layer flows with suction. Formulae for calculating the displacement thickness, momentum thickness, and skin friction are then derived. And further, the problem of determining the separation point is dealt with. Finally, as a numerical example, we compute certain characteristic boundary-layer parameters for the case of uniform flow over a flat plate with constant suction. Our numerical results obtained are in good agreement with those of Iglisch's.