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非均匀变截面弹性圆环在任意载荷下的弯曲问题

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摘　　要

本文在等刚度弹性圆环的初参数公式的基础上，利用〔2〕提出的阶梯折算法，进一步研究非均匀变截面弹性圆环的弯曲，得到了这类问题的通解．应当指出，这组通解对非均匀变截面圆柱拱的相应问题也是适用的．为验证所得的公式并说明这种方法的应用，文末给出了示例并进行了求解．

圆环、圆拱是工程上经常采用的结构，它们的弯曲，Timoshenko, S.[5]，Barber, J. R.[3]，Roark, R. J.[4]，津村利光[6]等曾作过很多研究．然而，迄今只求得了均匀材料、等截面圆环的通解。对变截面问题，仅仅求得了抗弯刚度是坐标的线性函数这一特殊情况的解．由于非均匀变截面问题常常导出变系数微分方程，它们的求解遇到很大的数学困难．本文通过阶梯折算法把非均匀变截面弹性圆环弯曲问题的变系数微分方程转化成一等效的等刚度圆环弯曲的常系数微分方程．为保证内力连续，引入虚拟内力，并以[1]导出的初参数公式为影响函数，通过积分构造出了非齐次解，从而求得了非均匀变截面弹性圆环弯曲问题的通解．

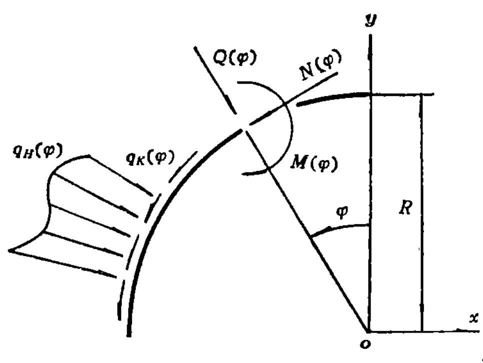
一、等刚度弹性圆环弯曲的基本结果

为了以下的需要，根据[1]简要给出均匀等截面弹性圆环弯曲的基本公式．

（1）圆环的内力及其微分关系

在图(1)所示的极坐标系中，圆环任一截面A上作用的内力有弯矩M(φ)、剪力Q(φ)、轴向力N(φ)，它们与环上作用的法向和切向分布载荷$q_H(\varphi)$、$q_K(\varphi)$间的关系，由微元体的平衡得到为：

$$\left.\begin{aligned}\frac{dM(\varphi)}{d\varphi}&=RQ(\varphi)\\ \frac{dQ(\varphi)}{d\varphi}&=-N(\varphi)+Rq_H(\varphi)\\ \frac{dN(\varphi)}{d\varphi}&=Q(\varphi)+Rq_K(\varphi)\end{aligned}\right\}\quad(1.1)$$



图(1)

式中R为圆环中性轴的半径.

（2）圆环的位移分量及其微分关系

圆环中性轴上任一点及与该点相联系的微元，变形后它们的位置由以下三个位移分量确定：

$W(\varphi)$——中性轴上任一点A在变形过程中沿径向的位移分量，以指向圆心为正，此位移称为圆环的挠度.

$u(\varphi)$——中性轴上任一点A在变形过程中沿周向的位移，以指向φ减小的一方为正，此位移称为圆环的轴向位移.

$\theta(\varphi)$——中性轴上任一点A的切线在变形过程中沿逆时针方向转过的角度，此角称为圆环的倾角.

当圆环的弯曲采用直梁弯曲的直法线假定时，以上三种位移存在以下微分关系：

$$\left.\begin{aligned}&\frac{du(\varphi)}{d\varphi}=-W(\varphi)\\&\theta(\varphi)=\frac{1}{R}\left[\frac{dW(\varphi)}{d\varphi}-u(\varphi)\right]=-\frac{1}{R}\left[\frac{d^2u(\varphi)}{d\varphi^2}+u(\varphi)\right]\end{aligned}\right\} \tag{1.2}$$

（3）挠曲微分方程及初参数解

采用直梁的弯曲假定，即认为：

$$M(\varphi)=D(\varphi)\chi \tag{1.3}$$

式中 $D(\varphi)$ 为圆环的抗弯刚度，对非均匀变截面圆环，它是φ的函数 $D(\varphi)=E(\varphi)\cdot J(\varphi)$. χ为圆环任一点在变形过程中曲率的改变值，它与挠度有以下关系：

$$\chi=\frac{1}{R^2}\left[\frac{d^2W(\varphi)}{d\varphi^2}+W(\varphi)\right] \tag{1.4}$$

利用上述各式，得到非均匀变截面弹性圆环的挠曲微分方程为：

$$\left[\frac{d^3}{d\varphi^3}+\frac{d}{d\varphi}\right]\left(\frac{D(\varphi)}{R^4}\right)\left(\frac{d^2}{d\varphi^2}+1\right)W(\varphi)=\frac{dq_H(\varphi)}{d\varphi}-q_K(\varphi) \tag{1.5}$$

当圆环的抗弯刚度$D(\varphi)=EJ=D_0$为常数时，上述方程便成为常系数微分方程：

$$\frac{d^5W(\varphi)}{d\varphi^5}+2\frac{d^3W(\varphi)}{d\varphi^3}+W(\varphi)=\frac{R^4}{D_0}\left[\frac{dq_H(\varphi)}{d\varphi}-q_K(\varphi)\right] \tag{1.6}$$

这个方程的解是大家熟知的，它由齐次方程的通解和非齐次方程的特解迭加得到．若通解用初参数来表示，则特解便可用通解作为影响函数按积分构造出来，故在此处给出齐次方程的初参数解.

设圆环起始截面$\varphi=0$处的内力和位移初参数为弯矩M_0、剪力Q_0、轴向力N_0及挠度W_0、轴向位移u_0、倾角θ_0，则（1.6）齐次方程的初参数解为：

$$W_*(\varphi)=W_0\cos\varphi+R\theta_0\sin\varphi+u_0\sin\varphi+\frac{R^2M_0}{D_0}\left(1-\cos\varphi\right)+\frac{R^3Q_0}{D_0}\left(\frac{1}{2}\sin\varphi-\frac{1}{2}\varphi\cos\varphi\right)+\frac{R^3N_0}{D_0}\left(-1+\cos\varphi+\frac{1}{2}\varphi\sin\varphi\right) \tag{1.7}$$

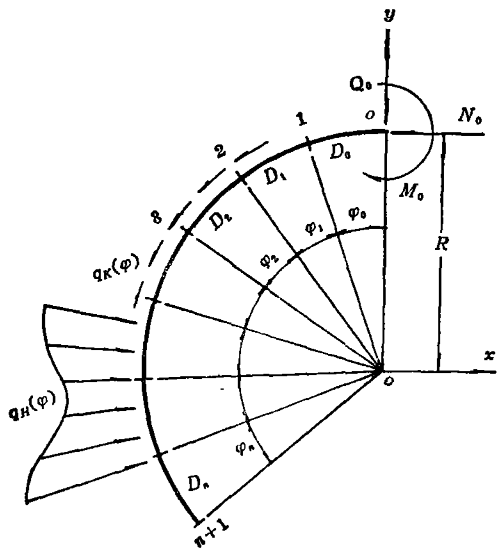
把求得的 $W_*(\varphi)$ 代入微分关系（1.1）和（1.2），则得任一截面用初参数表示的其他位移和内力为：

$$\left.\begin{aligned}u_*(\varphi)=&-W_0\sin\varphi-R\theta_0(1-\cos\varphi)+u_0\cos\varphi+\frac{R^2M_0}{D_0}\left(\sin\varphi-\varphi\right)\\&+\frac{R^3Q_0}{D_0}\left(-1+\cos\varphi+\frac{1}{2}\varphi\sin\varphi\right)\\&+\frac{R^3N_0}{D_0}\left(\varphi-\frac{3}{2}\sin\varphi+\frac{1}{2}\varphi\cos\varphi\right)\\R\theta_*(\varphi)=&-u_*(\varphi)-W_0\sin\varphi+R\theta_0\cos\varphi+u_0\cos\varphi+\frac{R^2M_0}{D_0}\sin\varphi\\&+\frac{R^3Q_0}{2D_0}\varphi\sin\varphi+\frac{R^3N_0}{2D_0}(\varphi\cos\varphi-\sin\varphi)\end{aligned}\right\}\tag{1.8}$$

$$\left.\begin{aligned}M_*(\varphi)&=M_0+RQ_0\sin\varphi+RN_0(-1+\cos\varphi)\\Q_*(\varphi)&=Q_0\cos\varphi-N_0\sin\varphi\\N_*(\varphi)&=Q_0\sin\varphi+N_0\cos\varphi\end{aligned}\right\}\tag{1.9}$$

二、非均匀变截面圆环弯曲的一般解

现在转而考虑图（2）所示的非均匀变截面圆环的弯曲，设环的抗弯刚度 $D(\varphi)$ 是极坐标φ的函数，圆环起始截面的位移和内力初参数为 W_0、u_0、θ_0 及 M_0、Q_0、N_0，除此而外，环上还作用法向和切向分布载荷 $q_H(\varphi)$、$q_K(\varphi)$．此时圆环的挠曲微分方程由（1.5）式知为：



图（2）

$$\left[\frac{d^3}{d\varphi^3}+\frac{d}{d\varphi}\right]\left(\frac{D(\varphi)}{R^4}\right)\left(\frac{d^2}{d\varphi^2}+1\right)W(\varphi)=\frac{dq_H(\varphi)}{d\varphi}-q_K(\varphi)\tag{2.1}$$

这是一个变系数微分方程，它的求解一般是困难的．这里使用文献[2]提出的阶梯折算法对它求解．为此，将抗弯刚度曲线 $D(\varphi)$ 用一根近似的阶梯折线代替，若在圆环上取分点φ_0，φ_1，φ_2，……φ_{n+1}，则$D(\varphi)$ 在各分段上均为常数，即：

$$D(\varphi)=D(\varphi_i)=D_i,\qquad(\varphi_i\leqslant\varphi<\varphi_{i+1})\tag{2.2}$$
$$(i=0,\ 1,\ 2\cdots\cdots n)$$

利用 Heviside*函数$H(\varphi\ \varphi_i)$ 有：

$$\frac{1}{D(\varphi)}=\frac{1}{D_0}\left[\delta_0+\sum_{i=1}^{n} H(\varphi-\varphi_i)(\delta_i-\delta_{i-1})\right]$$

式中：$\varphi_i \leqslant \varphi < \varphi_{i+1}$；而 δ_i 为圆环的抗弯刚度比：

$$\delta_i=\frac{D_0}{D_i},\ (i=0,\ 1,\ 2\cdots n) \tag{2.3}$$

将 $\frac{1}{D(\varphi)}$ 代入（2.1）式，并注意在各分段上 $\frac{1}{D(\varphi)}$ 都是常数，于是变系数常微分方程（2.1）化为下述常系数常微分方程：

$$\frac{d^5W(\varphi)}{d\varphi^5}+2\frac{d^3W(\varphi)}{d\varphi^3}+W(\varphi)=\frac{R^4}{D_0}\left[\frac{dq_H(\varphi)}{d\varphi}-q_K(\varphi)\right]$$
$$\cdot\left[\delta_0+\sum_{i=1}^{n} H(\varphi-\varphi_i)(\delta_i-\delta_{i-1})\right] \tag{2.4}$$

这是刚度为D_0的均匀圆环在折算载荷：

$$\left[\delta_0+\sum_{i=1}^{n} H(\varphi-\varphi_i)(\delta_i-\delta_{i-1})\right]q_H(\varphi) \tag{2.5}$$

及

$$\left[\delta_0+\sum_{i=1}^{n} H(\varphi-\varphi_i)(\delta_i-\delta_{i-1})\right]q_K(\varphi) \tag{2.6}$$

作用下的挠度 $W(\varphi)$ 所满足的微分方程．因此，非均匀变截面弹性圆环的弯曲问题就转化成折算载荷作用下等刚度弹性圆环弯曲问题．在此等刚度圆环上求得的挠度 $W(\varphi)$ 及其四阶以下的导数都是连续的，但由于上述阶梯刚度在各分点上的间断性，不能满足各分点上的弯矩，剪力及轴向力的连续条件．为此，在此等刚度圆环的各分点上，应加上适当的虚拟弯矩G_i，剪力V_i及轴向力S_i，以证保原非均匀变载面圆环各分点上的内力连续性．

方程（2.4）右端的折算载荷函数是间断可积的，因而此方程的求解可按通常的方法进行．它的解可表示为：

$$W(\varphi)=W_1(\varphi)+W_2(\varphi)+W_3(\varphi) \tag{2.7}$$

式中 $W_1(\varphi)$为由初参数W_0，u_0，θ_0及M_0，Q_0，N_0在等刚度圆环上引起的挠度，$W_2(\varphi)$为折算载荷在等刚度圆环上引起的挠度，$W_3(\varphi)$为虚拟载荷G_i，V_i，S_i在等刚度圆环上引起的挠度．

由（1.7）式：

$$W_1(\varphi)=W_0\cos\varphi+R\theta_0\sin\varphi+u_0\sin\varphi+\frac{R^2M_0}{D_0}(1-\cos\varphi)$$
$$+\frac{R^3Q_0}{D_0}\left(\frac{1}{2}\sin\varphi-\frac{1}{2}\varphi\cos\varphi\right)+\frac{R^3N_0}{D_0}\left(-1+\cos\varphi+\frac{1}{2}\varphi\sin\varphi\right) \tag{2.8}$$

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* Heviside函数 $H(\varphi-\varphi_i)=\begin{cases}0, & 当\varphi<\varphi_i\\ 1, & 当\varphi\geqslant\varphi_i\end{cases}$

将（1 7）式作为影响函数，用积分构造出：

$$W_2(\varphi)=\frac{R^4}{D_0}\int_0^{\varphi}\left[\delta_0+\sum_{i=1}^{n}H(\xi-\varphi_i)(\delta_i-\delta_{i-1})\right]\left[\frac{1}{2}\sin(\varphi-\xi)\right.$$

$$\left.-\frac{1}{2}(\varphi-\xi)\cos(\varphi-\xi)\right]\cdot q_H(\xi)d\xi+\frac{R^4}{D_0}\int_0^{\varphi}\left[\delta_0+\sum_{i=1}^{n}H(\xi-\varphi_i)(\delta_i-\delta_{i-1})\right]$$

$$\cdot\left[-1+\cos(\varphi-\xi)+\frac{1}{2}(\varphi-\xi)\cdot\sin(\varphi-\xi)\right]q_K(\xi)d\xi \tag{2.9}$$

$$W_3(\varphi)=\sum_{i=1}^{n}H(\varphi-\varphi_i)\frac{R^2G_i}{D_0}[1-\cos(\varphi-\varphi_i)]+\sum_{i=1}^{n}H(\varphi-\varphi_i)\frac{R^3V_i}{D_0}$$

$$\cdot\left[\frac{1}{2}\sin(\varphi-\varphi_i)-\frac{1}{2}(\varphi-\varphi_i)\cos(\varphi-\varphi_i)\right]+\sum_{i=1}^{n}H(\varphi-\varphi_i)$$

$$\cdot\frac{R^3S_i}{D_0}\left[-1+\cos(\varphi-\varphi_i)+\frac{1}{2}(\varphi-\varphi_i)\sin(\varphi-\varphi_i)\right] \tag{2.10}$$

非均匀圆环上分点φ_i的弯矩连续条件为：

$$M(\varphi_i)=\frac{D_{i-1}}{R^2}\left[\frac{d^2W(\varphi_i-0)}{d\varphi^2}+W(\varphi-0)\right]=\frac{D_i}{R^2}\left[\frac{d^2W(\varphi_i+0)}{d\varphi^2}+W(\varphi_i+0)\right] \tag{2.11}$$

在等刚度圆环上作用于分点φ_i的虚拟弯矩G_i引起曲率跳跃：

$$\frac{R^2M(\varphi_i)}{D_i}-\frac{R^2M(\varphi_i)}{D_{i-1}}=M(\varphi_i)R^2\left(\frac{1}{D_i}-\frac{1}{D_{i-1}}\right) \tag{2.12}$$

将(2.11)代入上式并利用弯矩与曲率的关系立即可以得到：

$$\left.\begin{aligned}
G_i&=\frac{D_{i-1}}{R^2}\left[\frac{d^2W(\varphi_i-0)}{d\varphi^2}+W(\varphi-0)\right](\delta_i-\delta_{i-1})\\
&=\frac{D_i}{R^2}\left[\frac{d^2W(\varphi_i+0)}{d\varphi^2}+W(\varphi_i+0)\right](\delta_i-\delta_{i-1})\\
&\text{类似地可以得到：}\\
V_i&=\frac{D_{i-1}}{R^3}\left[\frac{d^3W(\varphi_i-0)}{d\varphi^3}+\frac{dW(\varphi_i-0)}{d\varphi}\right](\delta_i-\delta_{i-1})\\
&=\frac{D_i}{R^3}\left[\frac{d^3W(\varphi_i+0)}{d\varphi^3}+\frac{dW(\varphi_i+0)}{d\varphi}\right](\delta_i-\delta_{i-1})\\
S_i&=\left\{Rq_H(\varphi_i)-\frac{D_{i-1}}{R^3}\left[\frac{d^4W(\varphi_i-0)}{d\varphi^4}+\frac{d^2W(\varphi_i-0)}{d\varphi^2}\right]\right\}(\delta_i-\delta_{i-1})\\
&=\left\{Rq_H(\varphi_i)-\frac{D_i}{R^3}\left[\frac{d^4W(\varphi_i+0)}{d\varphi^4}+\frac{d^2W(\varphi_i+0)}{d\varphi^2}\right]\right\}(\delta_i-\delta_{i-1})
\end{aligned}\right\} \tag{2.13}$$

上述公式右端除去因子$(\delta_i-\delta_{i-1})$外的部分正是圆环上分点φ_i处的弯矩，剪力及轴向力，因此，我们无须直接利用（2.13）式去求G_i、V_i、S_i，根据圆环段$0\leqslant\varphi\leqslant\varphi_i$上各力的平衡关系可通过积分给出：

$$\left.\begin{aligned}
G_i=&\Big\{M_0+Q_0R\sin\varphi_i-N_0R(1-\cos\varphi_i)+\int_0^{\varphi_i}R^2\sin(\varphi_i-\xi)q_H(\xi)d\xi\\
&-\int_0^{\varphi_i}R^2[1-\cos(\varphi_i-\xi)]q_K(\xi)d\xi\Big\}(\delta_i-\delta_{i-1})\\
V_i=&\Big[Q_0\cos\varphi_i-N_0\sin\varphi_i+\int_0^{\varphi_i}R\cos(\varphi_i-\xi)q_H(\xi)d\xi\\
&-\int_0^{\varphi_i}R\sin(\varphi_i-\xi)q_K(\xi)d\xi\Big](\delta_i-\delta_{i-1})\\
S_i=&\Big[Q_0\sin\varphi_i+N_0\cos\varphi_i+\int_0^{\varphi_i}R\sin(\varphi_i-\xi)q_H(\xi)d\xi\\
&+\int_0^{\varphi_i}R\cos(\varphi_i-\xi)q_K(\xi)d\xi\Big](\delta_i-\delta_{i-1})
\end{aligned}\right\}\tag{2.14}$$

现在只需将解（2.7）按次序进行整理，并利用微分关系（1.1）和（1.2），则得非均匀变截面弹性圆环变曲的一般解为：

$$\begin{aligned}
W(\varphi)=&W_0\cos\varphi+R\theta_0\sin\varphi+u_0\sin\varphi+\frac{R^2M_0}{D_0}\Big\{(1-\cos\varphi)\\
&+\sum_{i=1}^{n}H(\varphi-\varphi_i)(\delta_i-\delta_{i-1})[1-\cos(\varphi-\varphi_i)]\Big\}+\frac{R^3Q_0}{D_0}\Big\{\Big(\frac{1}{2}\sin\varphi\\
&-\frac{1}{2}\varphi\cos\varphi\Big)+\sum_{i=1}^{n}H(\varphi-\varphi_i)(\delta_i-\delta_{i-1})\sin\varphi_i[-\cos(\varphi-\varphi_i)]\\
&+\sum_{i=1}^{n}H(\varphi-\varphi_i)(\delta_i-\delta_{i-1})\cos\varphi_i\Big[\frac{1}{2}\sin(\varphi-\varphi_i)-\frac{1}{2}(\varphi-\varphi_i)\\
&\cdot\cos(\varphi-\varphi_i)\Big]+\sum_{i=1}^{n}H(\varphi-\varphi_i)(\delta_i-\delta_{i-1})\sin\varphi_i\Big[-1+\cos(\varphi-\varphi_i)\\
&+\frac{1}{2}(\varphi-\varphi_i)\sin(\varphi-\varphi_i)\Big]\Big\}+\frac{R^3N_0}{D_0}\Big\{(-1+\cos\varphi+\frac{1}{2}\varphi\sin\varphi)\\
&-\sum_{i=1}^{n}H(\varphi-\varphi_i)(\delta_i-\delta_{i-1})(1-\cos\varphi_i)[1-\cos(\varphi-\varphi_i)]\\
&-\sum_{i=1}^{n}H(\varphi-\varphi_i)(\delta_i-\delta_{i-1})\sin\varphi_i\Big[\frac{1}{2}\sin(\varphi-\varphi_i)-\frac{1}{2}(\varphi-\varphi_i)\\
&\cdot\cos(\varphi-\varphi_i)\Big]+\sum_{i=1}^{n}H(\varphi-\varphi_i)(\delta_i-\delta_{i-1})\cos\varphi_i\Big[-1+\cos(\varphi-\varphi_i)\\
&+\frac{1}{2}(\varphi-\varphi_i)\sin(\varphi-\varphi_i)\Big]\Big\}+W_q(\varphi)
\end{aligned}$$

$$u(\varphi)=-W_0\sin\varphi-R\theta_0(1-\cos\varphi)+u_0\cos\varphi-\frac{R^2M_0}{D_0}\Big\{(\varphi-\sin\varphi)$$

$$
\begin{aligned}
&+\sum_{i=1}^{n}H(\varphi-\varphi_i)(\delta_i-\delta_{i-1})[(\varphi-\varphi_i)-\sin(\varphi-\varphi_i)]+\frac{R^3Q_0}{D_0}\\
&\cdot\left\{\left(-1+\cos\varphi+\frac{1}{2}\varphi\sin\varphi\right)-\sum_{i=1}^{n}H(\varphi-\varphi_i)(\delta_i-\delta_{i-1})\sin\varphi_i[(\varphi-\varphi_i)\right.\\
&-\sin(\varphi-\varphi_i)]+\sum_{i=1}^{n}H(\varphi-\varphi_i)\cdot(\delta_i-\delta_{i-1})\cos\varphi_i\Big[-1+\cos(\varphi-\varphi_i)\\
&+\frac{1}{2}(\varphi-\varphi_i)\sin(\varphi-\varphi_i)\Big]-\sum_{i=1}^{n}H(\varphi-\varphi_i)(\delta_i-\delta_{i-1})\sin\varphi_i\Big[-(\varphi-\varphi_i)\\
&\left.+\frac{3}{2}\sin(\varphi-\varphi_i)-\frac{1}{2}(\varphi-\varphi_i)\cos(\varphi-\varphi_i)\Big]\right\}+\frac{R^3N_0}{D_0}\\
&\cdot\left\{\left(\varphi-\frac{3}{2}\sin\varphi+\frac{1}{2}\varphi\cos\right)+\sum_{i=1}^{n}H(\varphi-\varphi_i)(\delta_i-\delta_{i-1})(1-\cos\varphi_i)\right.\\
&[(\varphi-\varphi_i)-\sin(\varphi-\varphi_i)]-\sum_{i=1}^{n}H(\varphi-\varphi_i)(\delta_i-\delta_{i-1})\sin\varphi\\
&\cdot\Big[-1+\cos(\varphi-\varphi_i)+\frac{1}{2}(\varphi-\varphi_i)\cdot\sin(\varphi-\varphi_i)\Big]-\sum_{i=1}^{n}H(\varphi-\varphi_i)\\
&\cdot(\delta_i-\delta_{i-1})\cos\varphi_i\Big[-(\varphi-\varphi_i)+\frac{3}{2}\sin(\varphi-\varphi_i)\\
&\left.-\frac{1}{2}(\varphi-\varphi_i)\cos(\varphi-\varphi_i)\Big]\right\}+u_q(\varphi)\\
&R\theta(\varphi)+u(\varphi)=-W_0\sin\varphi+R\theta_0\cos\varphi+u_0\cos\varphi+\frac{R^2M_0}{D_0}\Big[\sin\varphi+\sum_{i=1}^{n}H(\varphi-\varphi_i)\\
&\cdot(\delta-\delta_{i-1})\sin(\varphi-\varphi_i)\Big]+\frac{R^3Q_0}{D_0}\left\{\frac{\varphi}{2}\sin\varphi+\sum_{i=1}^{n}H(\varphi-\varphi_i)(\delta_i-\delta_{i-1})\sin\varphi_i\right.\\
&\cdot\sin(\varphi-\varphi_i)+\sum_{i=1}^{n}H(\varphi-\varphi_i)(\delta_i-\delta_{i-1})\cos\varphi_i\Big[\frac{1}{2}(\varphi-\varphi_i)\sin(\varphi-\varphi_i)\Big]\\
&\left.+\sum_{i=1}^{n}H(\varphi-\varphi_i)(\delta_i-\delta_{i-1})\sin\varphi_i\Big[-\frac{1}{2}\sin(\varphi-\varphi_i)+\frac{1}{2}(\varphi-\varphi_i)\cos(\varphi-\varphi_i)\Big]\right\}\\
&+\frac{R^3N_0}{D_0}\left\{\left(-\frac{1}{2}\sin\varphi+\frac{1}{2}\varphi\cos\varphi\right)+\sum_{i=1}^{n}H(\varphi-\varphi_i)(\delta_i-\delta_{i-1})(\cos\varphi_i-1)\right.
\end{aligned}
\quad (2.15)
$$

$$\cdot \sin(\varphi-\varphi_i)-\sum_{i=1}^{n} H(\varphi-\varphi_i)(\delta_i-\delta_{i-1})\sin\varphi_i \frac{1}{2}\Big[(\varphi-\varphi_i)$$

$$\cdot \sin(\varphi-\varphi_i)\Big]+\sum_{i=1}^{n} H(\varphi-\varphi_i)(\delta_i-\delta_{i-1})\cos\varphi_i\Big[-\frac{1}{2}\sin(\varphi-\varphi_i)$$

$$+\frac{1}{2}(\varphi-\varphi_i)\cos(\varphi-\varphi_i)\Big]\Big\}+\theta_q(\varphi)$$

$$\left.\begin{aligned} M(\varphi)&=M_0+RQ_0\sin\varphi+RN_0(\cos\varphi-1)+M_q(\varphi)\\ Q(\varphi)&=Q_0\cos\varphi-N_0\sin\varphi+Q_q(\varphi)\\ N(\varphi)&=Q_0\sin\varphi+N_0\cos\varphi+N_q(\varphi)\end{aligned}\right\}\quad(2.16)$$

以上各式中，$W_q(\varphi)$，$u_q(\varphi)$ $\theta_q(\varphi)$为对应于法向和切向分布载荷产生的挠度，轴向位移和倾角，而$M_q(\varphi)$，$Q_q(\varphi)$，$N_q(\varphi)$为对应于法向和切向分布载荷在截面φ处产生的弯矩，剪力和轴向力．一般解（2.15）和（2.16）还可写为以下更简洁的形式：

$$W(\varphi)=W_0\cos\varphi+R\theta_0\sin\varphi+u_0\sin\varphi+M_0\eta_{WM}(\varphi)+Q_0\eta_{WQ}(\varphi)+N_0\eta_{WN}(\varphi)+W_q(\varphi)\quad(2.17)$$

式中：

$$\eta_{WM}(\varphi)=\frac{R^2}{D_0}\Big\{(1-\cos\varphi)+\sum_{i=1}^{n} H(\varphi-\varphi_i)(\delta_i-\delta_{i-1})[1-\cos(\varphi-\varphi_i)]\Big\}$$

$$\eta_{WQ}(\varphi)=\frac{R^3}{D_0}\Big\{\Big(\frac{1}{2}\sin\varphi-\frac{1}{2}\varphi\cos\varphi\Big)+\sum_{i=1}^{n} H(\varphi-\varphi_i)(\delta_i-\delta_{i-1})\sin\varphi_i$$

$$\cdot[1-\cos(\varphi-\varphi_i)]+\sum_{i=1}^{n} H(\varphi-\varphi_i)(\delta_i-\delta_{i-1})\cos\varphi_i\Big[\frac{1}{2}\sin(\varphi-\varphi_i)$$

$$-\frac{1}{2}(\varphi-\varphi_i)\cos(\varphi-\varphi_i)\Big]+\sum_{i=1}^{n} H(\varphi-\varphi_i)(\delta_i-\delta_{i-1})\sin\varphi_i$$

$$\cdot\Big[-1+\cos(\varphi-\varphi_i)+\frac{1}{2}(\varphi-\varphi_i)\sin(\varphi-\varphi_i)\Big]\Big\}$$

$$\eta_{WN}(\varphi)=\frac{R^3}{D_0}\Big\{\Big(-1+\cos\varphi+\frac{1}{2}\varphi\sin\varphi\Big)-\sum_{i=1}^{n} H(\varphi-\varphi_i)(\delta_i-\delta_{i-1})$$

$$\cdot(1-\cos\varphi_i)[1-\cos(\varphi-\varphi_i)]-\sum_{i=1}^{n} H(\varphi-\varphi_i)(\delta_i-\delta_{i-1})$$

$$\cdot\sin\varphi_i\Big[\frac{1}{2}\sin(\varphi-\varphi_i)-\frac{1}{2}(\varphi-\varphi_i)\cos(\varphi-\varphi_i)\Big]$$

$$+\sum_{i=1}^{n} H(\varphi-\varphi_i)(\delta_i-\delta_{i-1})\cos\varphi_i[-1+\cos(\varphi-\varphi_i)$$

$$+\frac{1}{2}(\varphi-\varphi_i)\sin(\varphi-\varphi_i)]\Big\}$$

$$W_q(\varphi)=\int_0^{\varphi}[q_H(\xi)\eta_{WQ}(\varphi-\xi)+q_K(\xi)\eta_{WN}(\varphi-\xi)]Rd\xi\quad(2.17.1)$$

$$u(\varphi)=-W_0\sin\varphi-R\theta_0(1-\cos\varphi)+u_0\cos\varphi+M_0\eta_{uM}(\varphi)+Q_0\eta_{uQ}(\varphi)$$

$$+N_0\eta_{uN}(\varphi)+u_q(\varphi)\quad(2.18)$$

式中：

$$\left.\begin{aligned}
\eta_{uM}(\varphi) &= -\frac{R^2}{D_0}\Big\{(\varphi-\sin\varphi)+\sum_{i=1}^{n}H(\varphi-\varphi_i)(\delta_i-\delta_{i-1})[(\varphi-\varphi_i)-\sin(\varphi-\varphi_i)]\Big\}\\
\eta_{uQ}(\varphi) &= \frac{R^3}{D_0}\Big\{(-1+\cos\varphi+\frac{1}{2}\varphi\sin\varphi)+\sum_{i=1}^{n}H(\varphi-\varphi_i)(\delta_i-\delta_{i-1})\\
&\quad\cdot\Big[-\sin\varphi_i((\varphi-\varphi_i)-\sin(\varphi-\varphi_i))+\cos\varphi_i(-1+\cos(\varphi-\varphi_i)\\
&\quad+\frac{1}{2}(\varphi-\varphi_i)\sin(\varphi-\varphi_i))-\sin\varphi_i\Big(-(\varphi-\varphi_i)+\frac{3}{2}\sin(\varphi-\varphi_i)\\
&\quad-\frac{1}{2}(\varphi-\varphi_i)\cos(\varphi-\varphi_i)\Big)\Big]\Big\}\\
\eta_{uN}(\varphi) &= \frac{R^3}{D_0}\Big\{\Big(\varphi-\frac{3}{2}\sin\varphi+\frac{1}{2}\varphi\cos\varphi\Big)+\sum_{i=1}^{n}H(\varphi-\varphi_i)(\delta_i-\delta_{i-1})\\
&\quad\cdot\Big[(1-\cos\varphi_i)((\varphi-\varphi_i)-\sin(\varphi-\varphi_i))-\sin\varphi_i\Big(-1+\cos(\varphi-\varphi_i)\\
&\quad+\frac{1}{2}(\varphi-\varphi_i)\sin(\varphi-\varphi_i)\Big)-\cos\varphi_i\Big(-(\varphi-\varphi_i)+\frac{3}{2}\sin(\varphi-\varphi_i)\\
&\quad-\frac{1}{2}(\varphi-\varphi_i)\cos(\varphi-\varphi_i)\Big)\Big]\Big\}\\
u_q(\varphi) &= \int_0^{\varphi}\Big[q_H(\xi)\eta_{uQ}(\varphi-\xi)+q_K(\xi)\eta_{uN}(\varphi-\xi)\Big]Rd\xi
\end{aligned}\right\}\quad(2.18.1)$$

$$\begin{aligned}R\theta(\varphi)+u(\varphi) &= -W_0\sin\varphi+R\theta_0\cos\varphi+u_0\cos\varphi+M_0\eta_{\theta M}(\varphi)\\
&\quad+Q_0\eta_{\theta Q}(\varphi)+N_0\eta_{\theta N}(\varphi)+\theta_q(\varphi)\end{aligned}\qquad(2.19)$$

式中：

$$\left.\begin{aligned}
\eta_{\theta M}(\varphi) &= \frac{R^2}{D_0}\Big[\sin\varphi+\sum_{i=1}^{n}H(\varphi-\varphi_i)(\delta_i-\delta_{i-1})\sin(\varphi-\varphi_i)\Big]\\
\eta_{\theta Q}(\varphi) &= \frac{R^3}{D_0}\Big\{\frac{1}{2}\varphi\sin\varphi+\sum_{i=1}^{n}H(\varphi-\varphi_i)(\delta_i-\delta_{i-1})\Big[\sin\varphi_i\sin(\varphi-\varphi_i)\\
&\quad+\cos\varphi_i\Big(\frac{1}{2}(\varphi-\varphi_i)\sin(\varphi-\varphi_i)\Big)+\sin\varphi_i\Big(-\frac{1}{2}\sin(\varphi-\varphi_i)\\
&\quad+\frac{1}{2}(\varphi-\varphi_i)\cos(\varphi-\varphi_i)\Big)\Big]\Big\}\\
\eta_{\theta N}(\varphi) &= \frac{R^3}{D_0}\Big\{\Big(-\frac{1}{2}\sin\varphi+\frac{1}{2}\varphi\cos\varphi\Big)+\sum_{i=1}^{n}H(\varphi-\varphi_i)(\delta_i-\delta_{i-1})\\
&\quad\cdot\Big[(\cos\varphi_i-1)\sin(\varphi-\varphi_i)-\sin\varphi_i\Big(\frac{1}{2}(\varphi-\varphi_i)\sin(\varphi-\varphi_i)\Big)\\
&\quad+\cos\varphi_i\Big(-\frac{1}{2}\sin(\varphi-\varphi_i)+\frac{1}{2}(\varphi-\varphi_i)\cos(\varphi-\varphi_i)\Big)\Big]\Big\}\\
\theta_q(\varphi) &= \int_0^{\varphi}[q_H(\xi)\eta_{\theta Q}(\varphi-\xi)+q_K(\xi)\eta_{\theta N}(\varphi-\xi)]Rd\xi
\end{aligned}\right\}\quad(2.19.1)$$

$$\left.\begin{aligned}M(\varphi)&=M_0+RQ_0\sin\varphi+RN_0(\cos\varphi-1)+M_q(\varphi)\\Q(\varphi)&=Q_0\cos\varphi-N_0\sin\varphi+Q_q(\varphi)\\N(\varphi)&=Q_0\sin\varphi+N_0\cos\varphi+N_q(\varphi)\end{aligned}\right\}\quad(2.20)$$

式中
$$\left.\begin{aligned}M_q(\varphi)&=\int_0^\varphi[q_H(\xi)R\sin(\varphi-\xi)+q_K(\xi)R[\cos(\varphi-\xi)-1]Rd\xi\\Q_q(\varphi)&=\int_0^\varphi[q_H(\xi)\cos(\varphi-\xi)-q_K(\xi)\sin(\varphi-\xi)]Rd\xi\\N_q(\varphi)&=\int_0^\varphi[q_H(\xi)\sin(\varphi-\xi)+q_K(\xi)\cos(\varphi-\xi)]Rd\xi\end{aligned}\right\}\quad(2.20.1)$$

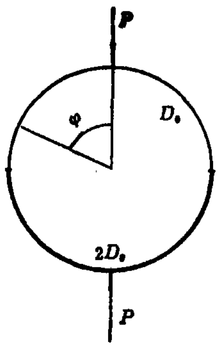
以上我们得到的公式(2.15)——(2.20)，即为非均匀变截面弹性圆环弯曲问题的通解，利用这组公式即可解决任意载荷作用下静定或静不定圆环的各种计算.

［例］ 图(3)所示的圆环，上半圆抗弯刚度为D_0，下半部为$2D_0$，半径为R，受到对称的集中力P. 试确定环中的内力.

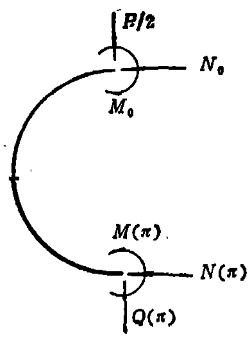
［解］ 由结构及载荷的对称性，将力P作图(4)的替代后，显然有：

$$Q(0)=\frac{P}{2}$$

$$u(0)=\theta(0)=0$$
$$W(\pi)=u(\pi)=\theta(\pi)=0$$



图(3)



图(4)

将其代入(2.17)、(2.18)、(2.19)，可解得：

$$M(0)=M_0=\frac{14-12\pi}{9\pi^2-8}RP$$

$$N(0)=N_0=\frac{12-3\pi}{9\pi^2-8}P$$

$$W(0)=W_0=\frac{1}{16(9\pi^2-8)}(96-252\pi+27\pi^3)\frac{R^3P}{D_0}$$

将M_0、N_0、W_0再代入(2.17)、(2.18)、(2.19)、(2.20)得到

$$W(\varphi)=\frac{R^3P}{D_0}\frac{1}{16(9\pi^2-8)}\Big\{(32-144\pi)+(36\pi^2-32)\sin\varphi+(27\pi^3-108\pi+64)\cos\varphi$$

$$+(96-24\pi)\varphi\sin\varphi-(36\pi^2-32)\varphi\cos\varphi+H\left(\varphi-\frac{\pi}{2}\right)$$

$$\left[(72\pi-16)+(48-12\pi)\sin\left(\varphi-\frac{\pi}{2}\right)+(16-72\pi)\cos\left(\varphi-\frac{\pi}{2}\right)\right.$$

$$-(18\pi^2-16)\left(\varphi-\frac{\pi}{2}\right)\sin\left(\varphi-\frac{\pi}{2}\right)-(48-12\pi)$$

$$\left.\cdot\left(\varphi-\frac{\pi}{2}\right)\cos\left(\varphi-\frac{\pi}{2}\right)\right]\Bigg\}$$

$$u(\varphi)=\frac{R^3P}{D_0}\frac{1}{16(9\pi^2-8)}\Bigg\{(64-72\pi^2)+(144\pi-32)\varphi$$

$$+(-160+132\pi-27\pi^3)\sin\varphi+(72\pi^2-64)\cos\varphi+(36\pi^2-32)\varphi\sin\varphi$$

$$+(96-24\pi)\varphi\cos\varphi+H\left(\varphi-\frac{\pi}{2}\right)\left[(-96+24\pi)+(16-72\pi)\right.$$

$$\cdot\left(\varphi-\frac{\pi}{2}\right)+(-32+72\pi+18\pi^2)\sin\left(\varphi-\frac{\pi}{2}\right)+(96-24\pi)$$

$$\cdot\cos\left(\varphi-\frac{\pi}{2}\right)+(48-12\pi)\left(\varphi-\frac{\pi}{2}\right)\sin\left(\varphi-\frac{\pi}{2}\right)$$

$$\left.+(16-18\pi^2)\left(\varphi-\frac{\pi}{2}\right)\cos\left(\varphi-\frac{\pi}{2}\right)\right]\Bigg\}$$

$$R\theta(\varphi)+u(\varphi)=\frac{R^3P}{D_0}\frac{1}{16(9\pi^2-8)}\Bigg\{(32+84\pi-27\pi^3)\sin\varphi$$

$$+(-32+36\pi^2)\varphi\sin\varphi+(96-24\pi)\varphi\cos\varphi$$

$$+H\left(\varphi-\frac{\pi}{2}\right)\left[(72\pi-18\pi^2)\sin\left(\varphi-\frac{\pi}{2}\right)\right.$$

$$+(48-12\pi)\left(\varphi-\frac{\pi}{2}\right)\sin\left(\varphi-\frac{\pi}{2}\right)$$

$$\left.+(16-18\pi^2)\left(\varphi-\frac{\pi}{2}\right)\cos\left(\varphi-\frac{\pi}{2}\right)\right]\Bigg\}$$

$$M(\varphi)=RP\left(\frac{2-9\pi}{9\pi^2-8}+\frac{1}{2}\sin\varphi+\frac{12-3\pi}{9\pi^2-8}\cos\varphi\right)$$

$$Q(\varphi)=P\left(\frac{1}{2}\cos\varphi+\frac{3\pi-12}{9\pi^2-8}\sin\varphi\right)$$

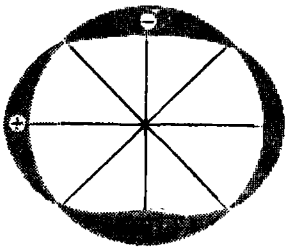
$$N(\varphi)=P\left(-\frac{1}{2}\sin\varphi+\frac{12-3\pi}{9\pi^2-8}\cos\varphi\right)$$

$$(0\leqslant\varphi\leqslant\pi)$$

此结果与卡氏定理求得的结果完全一致.

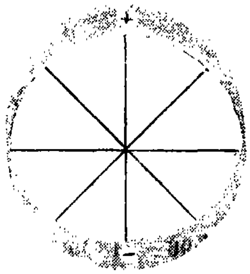
在半圆上等距的五个分点上的内力如下表，并绘出内力图于后.

点 内力	$\varphi=0$	$\varphi=\frac{\pi}{4}$	$\varphi=\frac{\pi}{2}$	$\varphi=\frac{3}{4}\pi$	$\varphi=\pi$
M	−0.294RP	−0.064	+0.174	+0.007	−0.357
Q	+0.500P	+0.330	−0.032	−0.378	−0.500
N	+0.032P	+0.378	+0.500	+0.330	−0.032



$M(\varphi)$

图(5)



$Q(\varphi)$

图(6)

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Bending of Non-homogeneous Variable Thickness Elastic Circular Ring under Arbitrarily Distributed Loads

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Abstract

On the foundation of the initial parameter formulae of elastic circular ring with constant flexural rigidity, using the stepped reduction method, suggested in [2], we investigate the bending problem of non-homogeneous variable thickness elastic circular ring under arbitrarily distributed loads and obtain the general solution of this problem which is also suitable for the corresponding problem of non-homogeneous variable thickness cylindrical arch. In the end, an example is carried out to examine the gained formulae and assure the exactness of this method.