应用数学和力学，第2卷第1期（1981年2月）
Applied Mathematics and Mechanics

应用数学和力学编委会编
四川人民出版社出版

用数学弹性力学的方法研究弹性体的稳定问题*

王震鸣

（北京中国科学院力学研究所　1980年2月20日收到）

摘　　要

用数学弹性力学的方法研究弹性体的稳定问题，是一个重要而困难的课题．В．В．Новожилов 在文献［1］中给出了平衡方程和边界条件，由于数学上的困难，没有给出具体问题的解．А．Ю．Ишлинский[2] 用数学弹性力学的方法解决了两边简支的无限宽平板当两边均匀受压时，在平面应变条件下的弹性稳定问题．他说："从 Новожилов 方程的观点看来，我们在平衡方程中忽略了转动分量，同时在边界条件中保存了转动的因素"，以克服数学上的困难．由于引入了一些简化带来了一些误差，他所得的临界载荷略微高于经典理论给出的临界载荷．К．Ф．Войцеховская[3,4] 采用 Ишлинский 的方法，获得了两端简支的圆杆和圆柱壳在轴压作用下的临界载荷，也略微高于经典理论给出的临界载荷．从弹性理论的观点看来，他们的结果是不够严格的．

本文采用 Новожилов 的平衡方程和边界条件，采用胡海昌[6] 的位移函数用以简化微分方程组，克服了数学上的困难，解得了［2—4］中求解过的几个稳定问题，得到的临界载荷略微低于经典理论给出的临界载荷．从数学弹性力学的观点看来，它是严格的．

一、引　　言

高强度材料的发展，使薄壁杆件在现代结构中的应用越来越广，同时也使薄壁结构(杆、拱、环、板和壳等）的稳定问题的研究，显得更为重要了．

从十八世纪 Euler L．研究压杆的屈曲开始至今，在弹性稳定方面，已有大量的研究成果．这些研究活动和 Kirchhoff G．关于线性弹性力学的唯一性定理所引起的影响，曾一度给人们造成这样的印象，即研究弹性体的平衡稳定问题，只有用材料力学和板壳理论的方法，对物体的应力或变形状态作出某种假定后，才能解决．

В．В．Новожилов[1] 用非线性弹性力学研究了弹性体的稳定问题，指出弹性力学问题中解的非唯一性．他指出：同一弹性体在相同的载荷与支承条件下，可以有几个可能的平衡位置．这样的力学问题的解常常不是唯一的．经典弹性力学的唯一性定理，在此情况下之所以不正确，是因为它的公式不够精确，这些公式是略去转动对应变分量和平衡方程的影响而

* 钱伟长推荐．

导得的．其实，在研究弹性体平衡的非单值形式时，考虑转动的影响是完全必要的．上面提到的几个可能的平衡位置，并不都是稳定的，经典弹性力学所给出的平衡位置常常是不稳定的．因此在弹性力学的问题中，不仅要求出弹性平衡的形式，而且要弄清楚它们的稳定性．在文献[1]中阐明了任意形状的、服从Hooke定律的弹性体的平衡稳定理论，提出了确定临界载荷的微分方程、边界条件和确定临界载荷的能量准则．但是，没有提出具体问题的解，这是因为求解这组微分方程有一定的困难．

用数学弹性力学的方法研究弹性体的稳定问题，与用材料力学、板壳理论的方法所得的临界载荷算式作比较，具有重要的理论意义．

А. Ю. Ишлинский[2] 用数学弹性力学的方法解决了一个具体问题，即两边简支的无限宽平板，当两边均匀受压时，在平面应变条件下的弹性稳定问题．他自称："从Новожилов的方程的观点看来，我们在平衡方程中忽略了转动分量，同时在边界条件中保存了转动的因素"．К. Ф. Войцеховская[3,4] 采用[2]中的处理方法，解决了简支的圆杆在两端受均匀轴压时的稳定问题，和简支的圆柱壳在两端受均匀轴压时的轴对称失稳问题．他们的结果都比用细杆理论和板壳理论所得的临界载荷大些．从理论上来说，这是不合理的．

Л. В. Ершов 和 Д. Д. Ивлев[7] 采用[1]中的平衡方程和[2]中的边界条件，解决了无限宽板在两边均匀受压时的稳定问题，与 Л. С. Лейбензон[8] 的结果作了比较，当 $\frac{m\pi h}{l}\leqslant 0.3$ 时，它们的结果是一致的；当 $\frac{m\pi h}{l}$ 很小时，与Euler公式是一致的．

本文严格按照[1]中的平衡方程和边界条件，采用胡海昌[6]的方法将未知函数加以变换，使方程组得到化简，对[2～4]中研究过的三个问题，重新作了计算，得到了比依据细杆理论和板壳理论为小的临界力，改正了[2—4]中不合理的结果．

二、两端简支的圆柱壳在轴压作用下的稳定问题

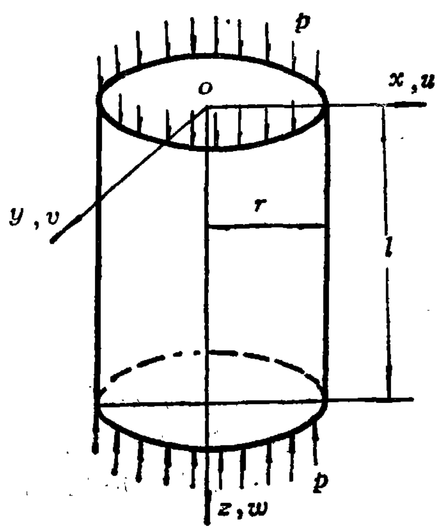

图 1

设圆柱壳的内径为 r_1，外径为 r_2，壳厚 $h=r_2-r_1$，壳长为 l．在均匀轴压 p 的作用下，求开始失稳时的临界压力。

所取坐标如图1．圆柱壳在失稳前的平衡位置，应力为

$$\begin{cases}\sigma_z^0=-p,\\ \sigma_r^0=\sigma_\theta^0=\tau_{r\theta}^0=\tau_{\theta z}^0=0\end{cases}\tag{2.1}$$

位移为

$$u_r^0=\frac{\nu p}{E}r,\quad u_\theta^0=0,\quad u_z^0=-\frac{p}{E}z,\tag{2.2}$$

式中 E 和 ν 分别为弹性模量和泊桑比．

圆柱壳开始失稳，进入与失稳前的平衡位置无限接近的平衡位置时，壳体的柱面已变形，而在内外侧面上依旧没有外力作用．设 u_r, u_θ, u_z 分别为壳体从失稳前的平衡位置到失

稳后的平衡位置的附加位移，是小量；$\varepsilon_r,\varepsilon_\theta,\varepsilon_z,\gamma_{r\theta},\gamma_{rz},\gamma_{\theta z}$ 和 $\sigma_r,\sigma_\theta,\sigma_z,\tau_{r\theta},\tau_{rz},\tau_{\theta z}$ 分别为相应的附加应变和应力分量，亦为小量.

在直角坐标中，应用 [1] 中的平衡方程（V.13），即

$$\frac{\partial}{\partial x}\left\{\sigma_x+\varepsilon_x\sigma_x^0+\left(\frac{1}{2}\gamma_{xy}-\omega_z\right)\tau_{xy}^0+\left(\frac{1}{2}\gamma_{xz}+\omega_y\right)\tau_{xz}^0\right\}$$
$$+\frac{\partial}{\partial y}\left\{\tau_{xy}+\varepsilon_x\tau_{xy}^0+\left(\frac{1}{2}\gamma_{xy}-\omega_z\right)\sigma_y^0+\left(\frac{1}{2}\gamma_{xz}+\omega_y\right)\tau_{yz}^0\right\}$$
$$+\frac{\partial}{\partial z}\left\{\tau_{xz}+\varepsilon_x\tau_{xz}^0+\left(\frac{1}{2}\gamma_{xy}-\omega_z\right)\tau_{yz}^0+\left(\frac{1}{2}\gamma_{xz}+\omega_y\right)\sigma_z^0\right\}=0 \qquad (2.3)$$

其余二式可用循序轮换 x,y,z 得到.

相应的边界条件（侧面上没有外力作用）为

$$\left\{\sigma_x+\varepsilon_x\sigma_x^0+\left(\frac{1}{2}\gamma_{xy}-\omega_z\right)\tau_{xy}^0+\left(\frac{1}{2}\gamma_{xz}+\omega_y\right)\tau_{xz}^0\right\}\cos(n,X)$$
$$+\left\{\tau_{xy}+\varepsilon_x\tau_{xy}^0+\left(\frac{1}{2}\gamma_{xy}-\omega_z\right)\sigma_y^0+\left(\frac{1}{2}\gamma_{xz}+\omega_y\right)\tau_{yz}^0\right\}\cos(n,Y)$$
$$+\left\{\tau_{xz}+\varepsilon_x\tau_{xz}^0+\left(\frac{1}{2}\gamma_{xy}-\omega_z\right)\tau_{yz}^0+\left(\frac{1}{2}\gamma_{xz}+\omega_y\right)\sigma_z^0\right\}\cos(n,Z)=0 \qquad (2.4)$$

其余二式可用循序轮换 x,y,z 得到.

采用小挠度的位移应变关系，

$$\left.\begin{aligned}&\varepsilon_x=\frac{\partial u}{\partial x},\ \varepsilon_y=\frac{\partial v}{\partial y},\ \varepsilon_z=\frac{\partial w}{\partial z}\\&\gamma_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x},\ \gamma_{xz}=\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x},\ \gamma_{yz}=\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\end{aligned}\right\} \qquad (2.5)$$

转动分量为

$$\omega_x=\frac{1}{2}\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right),\ \omega_y=\frac{1}{2}\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right),\ \omega_z=\frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right) \qquad (2.6)$$

而 n 为变形前物体界面的法线方向，X,Y,Z 为空间固定的直角坐标系.

将 (2.1) 代入 (2.3)，稍加简化，可得平衡方程：

$$\left.\begin{aligned}&\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+\frac{\partial\tau_{xz}}{\partial z}-p\frac{\partial^2 u}{\partial z^2}=0\\&\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}+\frac{\partial\tau_{yz}}{\partial z}-p\frac{\partial^2 v}{\partial z^2}=0\\&\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial\sigma_z}{\partial z}-p\frac{\partial^2 w}{\partial z^2}=0\end{aligned}\right\}. \qquad (2.7)$$

其中，σ_x，σ_y，σ_z，τ_{xy}，τ_{yz}，τ_{zx} 为直角坐标系中，从失稳前的平衡位置到失稳后的平衡位置的附加应力分量，u，v，w 为附加的位移分量

在两侧面上，即 $r=r_1$ 及 r_2 处，

$$\cos(n,X)\neq 0,\ \cos(n,Y)\neq 0,\ \cos(n,Z)=0 \qquad (2.8)$$

将 (2.1) 和 (2.8) 代入 (2.4)，可得内外侧面上的边界条件为

$$\sigma_x=\sigma_y=\tau_{xy}=\tau_{xz}=\tau_{yz}=0 \tag{2.9}$$

在 $z=0$ 及 $z=l$ 处，认为是简支条件，即

$$u=v=\frac{\partial w}{\partial z}=0 \tag{2.10}$$

应力应变关系为：

$$\left.\begin{aligned}&\sigma_x=\lambda e+2G\varepsilon_x,\ \sigma_y=\lambda e+2G\varepsilon_y,\ \sigma_z=\lambda e+2G\varepsilon_z\\&\tau_{xy}=G\gamma_{xy},\ \tau_{xz}=G\gamma_{xz},\ \tau_{yz}=G\gamma_{yz}\end{aligned}\right\} \tag{2.11}$$

其中

$$G=\frac{E}{2(1+\nu)},\ \lambda=\frac{\nu E}{(1+\nu)(1-2\nu)} \tag{2.12}$$

$$e=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} \tag{2.13}$$

将 (2.5) 代入 (2.11)，然后代入 (2.7)，可得用位移分量作为未知量的一组方程.

$$\left.\begin{aligned}&(\lambda+G)\frac{\partial e}{\partial x}+G\nabla^2 u-p\frac{\partial^2 u}{\partial z^2}=0\\&(\lambda+G)\frac{\partial e}{\partial y}+G\nabla^2 v-p\frac{\partial^2 v}{\partial z^2}=0\\&(\lambda+G)\frac{\partial e}{\partial z}+G\nabla^2 w-p\frac{\partial^2 w}{\partial z^2}=0\end{aligned}\right\} \tag{2.14}$$

其中

$$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2} \tag{2.15}$$

在 (2.14) 中，每一个方程都包含有未知函数 u, v, w, 难于求解，引用胡海昌[6]的方法，引进位移函数 φ_0, φ_1, φ_2, 使

$$\left.\begin{aligned}&u=\frac{\partial}{\partial x}(\varphi_1+\varphi_2)+\frac{\partial\varphi_0}{\partial y},\ v=\frac{\partial}{\partial y}(\varphi_1+\varphi_2)-\frac{\partial\varphi_0}{\partial x},\\&w=\frac{\partial}{\partial z}(k_1\varphi_1+k_2\varphi_2)\end{aligned}\right\} \tag{2.16}$$

其中当

$$k_1=1-\frac{p}{G},\ k_2=1 \tag{2.17}$$

就可将 (2.14) 这个微分方程组，化为已分离变量的另三个方程.

$$\left.\begin{aligned}&\frac{\partial^2\varphi_0}{\partial x^2}+\frac{\partial^2\varphi_0}{\partial y^2}+\left(1-\frac{p}{G}\right)\frac{\partial^2\varphi_0}{\partial z^2}=0\\&\frac{\partial^2\varphi_1}{\partial x^2}+\frac{\partial^2\varphi_1}{\partial y^2}+\left(1-\frac{p}{G}\right)\frac{\partial^2\varphi_1}{\partial z^2}=0\\&\frac{\partial^2\varphi_2}{\partial x^2}+\frac{\partial^2\varphi_2}{\partial y^2}+\left(1-\frac{p}{2G+\lambda}\right)\frac{\partial^2\varphi_2}{\partial z^2}=0\end{aligned}\right\} \tag{2.18}$$

在采用 (2.16) 后，方程组 (2.14) 和 (2.18) 是完全等价的. 根据 [6]，证明如下：

在方程组(2.14)中，不论 u，v 最后是什么样的函数，总可用另外两个函数 F 和 Ψ 表示如下：

$$u=\frac{\partial^2 F}{\partial x\partial z}+\frac{\partial \Psi}{\partial y},\quad v=\frac{\partial^2 F}{\partial y\partial z}-\frac{\partial \Psi}{\partial x} \tag{2.19}$$

这个表示法不是唯一的，因为与此对应的齐次方程

$$\frac{\partial^2 F_0}{\partial x\partial z}+\frac{\partial \Psi_0}{\partial y}=0,\quad \frac{\partial^2 F_0}{\partial y\partial z}-\frac{\partial \Psi_0}{\partial x}=0 \tag{2.20}$$

有不恒等于零的解存在，事实上，方程(2.20)的解为

$$\Psi_0+i\frac{\partial F_0}{\partial z}=f(x+iy,z) \tag{2.21}$$

其中，$i=\sqrt{-1}$，而 f 为任意的解析函数. 由此可知，若在 F 和 Ψ 中加上由(2.21)规定的 F_0 和 Ψ_0，则并不影响 u 和 v 的大小. 此外，很明显，在 F 加减一个不含 z 的函数，也不影响 u 和 v 的大小.

将 u 和 v 的表示式(2.19)代入(2.14)中的前二式，得到

$$\left.\begin{aligned}&(\lambda+G)\left[\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\frac{\partial^2 F}{\partial x\partial z}+\frac{\partial^2 w}{\partial x\partial z}\right]+G\nabla^2\left(\frac{\partial^2 F}{\partial x\partial z}+\frac{\partial \Psi}{\partial y}\right)\\&\quad-p\frac{\partial^2}{\partial z^2}\left(\frac{\partial^2 F}{\partial x\partial z}+\frac{\partial \Psi}{\partial y}\right)=0\\&(\lambda+G)\left[\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\frac{\partial^2 F}{\partial y\partial z}+\frac{\partial^2 w}{\partial y\partial z}\right]+G\nabla^2\left(\frac{\partial^2 F}{\partial y\partial z}-\frac{\partial \Psi}{\partial x}\right)\\&\quad-p\frac{\partial^2}{\partial z^2}\left(\frac{\partial^2 F}{\partial y\partial z}-\frac{\partial \Psi}{\partial x}\right)=0\end{aligned}\right\} \tag{2.22}$$

化简后得

$$\left.\begin{aligned}&(\lambda+2G)\frac{\partial^2}{\partial x\partial z}\left[\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)F+\frac{G-p}{\lambda+2G}\,\frac{\partial^2 F}{\partial z^2}+\frac{\lambda+G}{\lambda+2G}w\right]\\&\quad+G\frac{\partial}{\partial y}\left[\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\Psi+\frac{G-p}{G}\,\frac{\partial^2 \Psi}{\partial z^2}\right]=0\\&(\lambda+2G)\frac{\partial^2}{\partial y\partial z}\left[\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)F+\frac{G-p}{\lambda+2G}\,\frac{\partial^2 F}{\partial z^2}+\frac{\lambda+G}{\lambda+2G}w\right]\\&\quad-G\frac{\partial}{\partial x}\left[\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\Psi+\frac{G-p}{G}\,\frac{\partial^2 \Psi}{\partial z^2}\right]=0\end{aligned}\right\} \tag{2.23}$$

由此可见，

$$\begin{aligned}&G\left[\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\Psi+\frac{G-p}{G}\frac{\partial^2 \Psi}{\partial z^2}\right]+i(\lambda+2G)\,\frac{\partial}{\partial z}\left[\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)F\right.\\&\quad\left.+\frac{G-p}{\lambda+2G}\,\frac{\partial^2 F}{\partial z^2}+\frac{\lambda+G}{\lambda+2G}w\right]=g(x+iy,z)\end{aligned} \tag{2.24}$$

其中 $g(x+iy,z)$ 亦为一个解析函数. 如果选择(2.21)中的 $f(x+iy,z)$，使适合方程

$$(G-p)\frac{\partial^2 f}{\partial z^2}=g(x+iy,z) \tag{2.25}$$

则在F和Ψ中加上由 (2.21) 式规定的F_0和Ψ_0后可使 (2.24) 式的右端为零.因此可取$g=0$，而无损于解的一般性．于是由 (2.24) 可得

$$\frac{\partial^2\Psi}{\partial x^2}+\frac{\partial^2\Psi}{\partial y^2}+\frac{G-p}{G}\frac{\partial^2\Psi}{\partial z^2}=0 \tag{2.26}$$

$$\frac{\partial}{\partial z}\left[\left(\frac{\partial^2F}{\partial x^2}+\frac{\partial^2F}{\partial y^2}+\frac{G-p}{\lambda+2G}\frac{\partial^2F}{\partial z^2}\right)+\frac{\lambda+G}{\lambda+2G}w\right]=0 \tag{2.27}$$

由 (2.27) 可得

$$\left(\frac{\partial^2F}{\partial x^2}+\frac{\partial^2F}{\partial y^2}+\frac{G-p}{\lambda+2G}\frac{\partial^2F}{\partial z^2}\right)+\frac{\lambda+G}{\lambda+2G}w=h(x,y) \tag{2.28}$$

由于在F中可以加减一个不含z的函数，所以可以取$h(x,y)=0$，而无损于解的一般性，由此可得

$$w=-\frac{\lambda+2G}{\lambda+G}\left(\frac{\partial^2F}{\partial x^2}+\frac{\partial^2F}{\partial y^2}+\frac{G-p}{\lambda+2G}\frac{\partial^2F}{\partial z^2}\right) \tag{2.29}$$

将 (2.19) 和 (2.29) 代入 (2.14) 中第三式，整理后得

$$\nabla_1^2\nabla_1^2F+\frac{4G^2+2G\lambda-3Gp-p\lambda}{G(\lambda+2G)}\nabla_1^2\frac{\partial^2F}{\partial z^2}+\frac{G-p}{G}\cdot\frac{\lambda+2G-p}{\lambda+2G}\frac{\partial^4F}{\partial z^4}=0 \tag{2.30}$$

其中

$$\nabla_1^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2} \tag{2.31}$$

由 (2.30) 经过代数运算后可得

$$\left(\nabla_1^2+\frac{G-p}{G}\frac{\partial^2}{\partial z^2}\right)\left(\nabla_1^2+\frac{\lambda+2G-p}{\lambda+2G}\frac{\partial^2}{\partial z^2}\right)F=0 \tag{2.32}$$

如果

$$\nabla_1^2F_1+\frac{G-p}{G}\frac{\partial^2F_1}{\partial z^2}=0,\ \nabla_1^2F_2+\frac{\lambda+2G-p}{\lambda+2G}\frac{\partial^2F_2}{\partial z^2}=0 \tag{2.33}$$

则

$$F=F_1+F_2 \tag{2.34}$$

便满足 (2.32) 式．将 (2.33) 对z微分，再令

$$\frac{\partial F_1}{\partial z}=\varphi_1,\quad \frac{\partial F_2}{\partial z}=\varphi_2 \tag{2.35}$$

于是有

$$\nabla_1^2\varphi_1+\frac{G-p}{G}\frac{\partial^2\varphi_1}{\partial z^2}=0,\ \nabla_1^2\varphi_2+\frac{\lambda+2G-p}{\lambda+2G}\frac{\partial^2\varphi_2}{\partial z^2}=0 \tag{2.36}$$

由于 (2.33)～(2.35)，可将 (2.29) 化为

$$\begin{aligned}w=&-\frac{\lambda+2G}{\lambda+G}\left\{\left[\nabla_1^2F_1+\frac{G-p}{G}\frac{\partial^2F_1}{\partial z^2}-\frac{(\lambda+G)(G-p)}{G(\lambda+2G)}\frac{\partial^2F_1}{\partial z^2}\right]\right.\\&\left.+\left[\nabla_1^2F_2+\frac{\lambda+2G-p}{\lambda+2G}\frac{\partial^2F_2}{\partial z^2}-\frac{\lambda+G}{\lambda+2G}\frac{\partial^2F_2}{\partial z^2}\right]\right\}=\frac{G-p}{G}\frac{\partial^2F_1}{\partial z^2}\\&+\frac{\partial^2F_2}{\partial z^2}=\frac{G-p}{G}\frac{\partial\varphi_1}{\partial z}+\frac{\partial\varphi_2}{\partial z}\end{aligned} \tag{2.37}$$

取

$$\Psi=\varphi_0 \tag{2.38}$$

于是，由(2.19)，(2.35)，(2.38)，可得

$$\left.\begin{aligned} u&=\frac{\partial^2(F_1+F_2)}{\partial x\partial z}+\frac{\partial\Psi}{\partial y}=\frac{\partial}{\partial x}(\varphi_1+\varphi_2)+\frac{\partial\varphi_0}{\partial y}\\ v&=\frac{\partial^2(F_1+F_2)}{\partial y\partial z}-\frac{\partial\Psi}{\partial x}=\frac{\partial}{\partial y}(\varphi_1+\varphi_2)-\frac{\partial\varphi_0}{\partial x}\end{aligned}\right\} \tag{2.39}$$

将(2.38)代入(2.26)，得

$$\frac{\partial^2\varphi_0}{\partial x^2}+\frac{\partial^2\varphi_0}{\partial y^2}+\frac{G-p}{G}\frac{\partial^2\varphi_0}{\partial z^2}=0 \tag{2.40}$$

由此可见，将 u, v, w 用(2.16)中的位移函数 $\varphi_0, \varphi_1, \varphi_2$ 表示时，方程组(2.14)和(2.18)完全等价.

在圆柱坐标中，方程组(2.18)可化为

$$\left.\begin{aligned} &\frac{\partial^2\varphi_0}{\partial r^2}+\frac{1}{r}\frac{\partial\varphi_0}{\partial r}+\frac{1}{r^2}\frac{\partial^2\varphi_0}{\partial\theta^2}+\frac{G-p}{G}\frac{\partial^2\varphi_0}{\partial z^2}=0\\ &\frac{\partial^2\varphi_1}{\partial r^2}+\frac{1}{r}\frac{\partial\varphi_1}{\partial r}+\frac{1}{r^2}\frac{\partial^2\varphi_1}{\partial\theta^2}+\frac{G-p}{G}\frac{\partial^2\varphi_1}{\partial z^2}=0\\ &\frac{\partial^2\varphi_2}{\partial r^2}+\frac{1}{r}\frac{\partial\varphi_2}{\partial r}+\frac{1}{r^2}\frac{\partial^2\varphi_2}{\partial\theta^2}+\frac{\lambda+2G-p}{\lambda+2G}\frac{\partial^2\varphi_2}{\partial z^2}=0\end{aligned}\right\} \tag{2.41}$$

而(2.16)式化为

$$\left.\begin{aligned} u_r&=\frac{\partial}{\partial r}(\varphi_1+\varphi_2)+\frac{1}{r}\frac{\partial\varphi_0}{\partial\theta},\\ u_\theta&=\frac{1}{r}\frac{\partial}{\partial\theta}(\varphi_1+\varphi_2)-\frac{\partial\varphi_0}{\partial r},\quad e=\frac{p}{\lambda+2G}\frac{\partial^2\varphi_2}{\partial z^2}\\ u_z&=\frac{G-p}{G}\frac{\partial\varphi_1}{\partial z}+\frac{\partial\varphi_2}{\partial z},\end{aligned}\right\} \tag{2.42}$$

位移应变关系为

$$\left.\begin{aligned} &\varepsilon_r=\frac{\partial u_r}{\partial r},\ \varepsilon_\theta=\frac{u_r}{r}+\frac{1}{r}\frac{\partial u_\theta}{\partial\theta},\ \varepsilon_z=\frac{\partial u_z}{\partial z}\\ &\gamma_{r\theta}=\frac{1}{r}\frac{\partial u_r}{\partial\theta}+\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r},\ \gamma_{rz}=\frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r},\ \gamma_{z\theta}=\frac{\partial u_\theta}{\partial z}+\frac{1}{r}\frac{\partial u_z}{\partial\theta}\end{aligned}\right\} \tag{2.43}$$

应力应变关系为

$$\left.\begin{aligned} &\sigma_r=\lambda e+2G\varepsilon_r,\quad \sigma_\theta=\lambda e+2G\varepsilon_\theta,\quad \sigma_z=\lambda e+2G\varepsilon_z\\ &\tau_{r\theta}=G\gamma_{r\theta},\qquad \tau_{rz}=G\gamma_{rz},\qquad \tau_{\theta z}=G\gamma_{\theta z}\end{aligned}\right\} \tag{2.44}$$

边界条件可化为：在 $r=r_1$ 及 $r=r_2$ 处

$$\sigma_r=\tau_{r\theta}=\tau_{rz}=0 \tag{2.45}$$

边界条件(2.10)可化为：在 $z=0$ 及 $z=l$ 处简支，

$$u_r=u_\theta=\frac{\partial u_z}{\partial z}=0 \tag{2.46}$$

考虑到方程 (2.41) 及边界条件 (2.46)，可设

$$\left.\begin{aligned}\varphi_0&=f_0(r)\sin\frac{m\pi z}{l}\sin n\theta\\ \varphi_1&=f_1(r)\sin\frac{m\pi z}{l}\cos n\theta\\ \varphi_2&=f_2(r)\sin\frac{m\pi z}{l}\cos n\theta\end{aligned}\right\}\tag{2.47}$$

其中 m 为失稳时在轴向形成的半波数，n 为周向形成的波数．将 (2.47) 代入 (2.41)，得

$$\left.\begin{aligned}&f_0''(r)+\frac{1}{r}f_0'(r)-\left(\frac{n^2}{r^2}+\frac{G-p}{G}\ \frac{m^2\pi^2}{l^2}\right)f_0(r)=0\\ &f_1''(r)+\frac{1}{r}f_1'(r)-\left(\frac{n^2}{r^2}+\frac{G-p}{G}\ \frac{m^2\pi^2}{l^2}\right)f_1(r)=0\\ &f_2''(r)+\frac{1}{r}f_2'(r)-\left(\frac{n^2}{r^2}+\frac{\lambda+2G-p}{\lambda+2G}\ \frac{m^2\pi^2}{l^2}\right)f_2(r)=0\end{aligned}\right\}\tag{2.48}$$

其解为

$$\left.\begin{aligned}f_0(r)&=A_0I_n(m_0r)+B_0K_n(m_0r)\\ f_1(r)&=A_1I_n(m_1r)+B_1K_n(m_1r)\\ f_2(r)&=A_2I_n(m_2r)+B_2K_n(m_2r)\end{aligned}\right\}\tag{2.49}$$

其中 A_0，B_0，A_1，B_1，A_2，B_2 为积分常数，而

$$m_0=\frac{m\pi}{l}\left(1-\frac{p}{G}\right)^{\frac{1}{2}}=m_1,\qquad m_2=\frac{m\pi}{l}\left(1-\frac{p}{\lambda+2G}\right)^{\frac{1}{2}}\tag{2.50}$$

将 (2.47) 代入 (2.42)，得

$$\left.\begin{aligned}u_r&=\left\{\frac{\partial}{\partial r}[A_1I_n(m_1r)+B_1K_n(m_1r)+A_2I_n(m_2r)+B_2K_n(m_2r)]\right.\\ &\quad\left.+\frac{n}{r}[A_0I_n(m_0r)+B_0K_n(m_0r)]\right\}\sin\frac{m\pi z}{l}\cos n\theta\\ u_\theta&=-\left\{\frac{1}{r}\ \frac{\partial}{\partial\theta}[A_1I_n(m_1r)+B_1K_n(m_1r)+A_2I_n(m_2r)+B_2K_n(m_2r)]\right.\\ &\quad\left.+\frac{\partial}{\partial r}[A_0I_n(m_0r)+B_0K_n(m_0r)]\right\}\sin\frac{m\pi z}{l}\sin n\theta\\ u_z&=\left\{\frac{G-p}{G}[A_1I_n(m_1r)+B_1K_n(m_1r)]+A_2I_n(m_2r)\right.\\ &\quad\left.+B_2K_n(m_2r)]\right\}\frac{m\pi}{l}\cos\frac{m\pi z}{l}\cos n\theta\end{aligned}\right\}\tag{2.51}$$

由 (2.51) 式可见，边界条件 (2.46) 已得到满足．由 (2.42)～(2.44) 可得

$$\left.\begin{aligned}
\sigma_r=&\frac{\lambda p}{\lambda+2G}\frac{\partial^2\varphi_2}{\partial z^2}+2G\Big[-\frac{1}{r}\frac{\partial}{\partial r}(\varphi_1+\varphi_2)-\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}(\varphi_1+\varphi_2)\\
&-\frac{\partial^2}{\partial z^2}\Big(\frac{G-p}{G}\varphi_1+\frac{\lambda+2G-p}{\lambda+2G}\varphi_2\Big)+\frac{1}{r}\frac{\partial^2\varphi_0}{\partial r\partial\theta}-\frac{1}{r^2}\frac{\partial\varphi_0}{\partial\theta}\Big]\\
\tau_{rz}=&2G\frac{\partial^2}{\partial r\partial z}\Big(\frac{2G-p}{2G}\varphi_1+\varphi_2\Big)+\frac{G}{r}\frac{\partial^2\varphi_0}{\partial\theta\partial z}\\
\tau_{r\theta}=&2G\Big[\frac{1}{r}\frac{\partial^2}{\partial r\partial\theta}(\varphi_1+\varphi_2)-\frac{1}{r^2}\frac{\partial}{\partial\theta}(\varphi_1+\varphi_2)+\frac{1}{r^2}\frac{\partial^2\varphi_0}{\partial\theta^2}\\
&+\frac{1}{r}\frac{\partial\varphi_0}{\partial r}+\frac{G-p}{2G}\frac{\partial^2\varphi_0}{\partial z^2}\Big]
\end{aligned}\right\}\tag{2.52}$$

将(2.47)代入(2.52)，然后代入(2.45)，可得在 $r=r_i$ 处($i=1,2$)的边界条件为

$$\left.\begin{aligned}
&\Big(\frac{n^2}{r_i^2}+\frac{G-p}{G}\frac{m^2\pi^2}{l^2}\Big)f_1(r_i)+\Big(\frac{n^2}{r_i^2}+\frac{2G-p}{2G}\frac{m^2\pi^2}{l^2}\Big)f_2(r_i)\\
&\quad-\frac{1}{r_i}[f_1'(r_i)+f_2'(r_i)]-\frac{n}{r_i^2}f_0(r_i)+\frac{n}{r_i}f_0'(r_i)=0\\
&\frac{2G-p}{2G}f_1'(r_i)+f_2'(r_i)+\frac{n}{2r_i}f_0(r_i)=0\\
&\frac{n}{r_i^2}[f_1(r_i)+f_2(r_i)]-\frac{n}{r_i}[f_1'(r_i)+f_2'(r_i)]\\
&\quad-\Big(\frac{n^2}{r_i^2}+\frac{G-p}{2G}\frac{m^2\pi^2}{l^2}\Big)f_0(r_i)+\frac{1}{r_i}f_0'(r_i)=0
\end{aligned}\right\}\tag{2.53}$$

其中，$f'(r)=\frac{d}{dr}f(r)$. 将(2.49)代入(2.53)可得

$$\left.\begin{aligned}
F_{11}A_1+F_{12}B_1+F_{13}A_2+F_{14}B_2+F_{15}A_0+F_{16}B_0=0\\
F_{21}A_1+F_{22}B_1+F_{23}A_2+F_{24}B_2+F_{25}A_0+F_{26}B_0=0\\
F_{31}A_1+F_{32}B_1+F_{33}A_2+F_{34}B_2+F_{35}A_0+F_{36}B_0=0\\
F_{41}A_1+F_{42}B_1+F_{43}A_2+F_{44}B_2+F_{45}A_0+F_{46}B_0=0\\
F_{51}A_1+F_{52}B_1+F_{53}A_2+F_{54}B_2+F_{55}A_0+F_{56}B_0=0\\
F_{61}A_1+F_{62}B_1+F_{63}A_2+F_{64}B_2+F_{65}A_0+F_{66}B_0=0
\end{aligned}\right\}\tag{2.54}$$

其中，

$$\begin{aligned}
F_{11}&=\Big(\frac{n^2}{r_2^2}+\frac{G-p}{G}\frac{m^2\pi^2}{l^2}\Big)I_n(m_1r_2)-\frac{1}{r_2}I_n'(m_1r_2)\\
F_{12}&=\Big(\frac{n^2}{r_2^2}+\frac{G-p}{G}\frac{m^2\pi^2}{l^2}\Big)K_n(m_1r_2)-\frac{1}{r_2}K_n'(m_1r_2)\\
F_{13}&=\Big(\frac{n^2}{r_2^2}+\frac{2G-p}{2G}\frac{m^2\pi^2}{l^2}\Big)I_n(m_2r_2)-\frac{1}{r_2}I_n'(m_2r_2)\\
F_{14}&=\Big(\frac{n^2}{r_2^2}+\frac{2G-p}{2G}\frac{m^2\pi^2}{l^2}\Big)K_n(m_2r_2)-\frac{1}{r_2}I_n'(m_2r_2)
\end{aligned}$$

$$
\left.\begin{aligned}
&F_{15}=-\frac{n}{r_2^2}I_n(m_1r_2)-\frac{n}{r_2}I'_n(m_1r_2)\\
&F_{16}=-\frac{n}{r_2^2}K_n(m_1r_2)-\frac{n}{r_2}K'_n(m_1r_2)\\
&F_{21}=\left(\frac{n^2}{r_1^2}+\frac{G-p}{G}\ \frac{m^2\pi^2}{l^2}\right)I_n(m_1r_1)-\frac{1}{r_1}I'_n(m_1r_1)\\
&F_{22}=\left(\frac{n^2}{r_1^2}+\frac{G-p}{G}\ \frac{m^2\pi^2}{l^2}\right)K_n(m_1r_1)-\frac{1}{r_1}K'_n(m_1r_1)\\
&F_{23}=\left(\frac{n^2}{r_1^2}+\frac{2G-p}{2G}\ \frac{m^2\pi^2}{l^2}\right)I_n(m_2r_1)-\frac{1}{r_1}I'_n(m_2r_1)\\
&F_{24}=\left(\frac{n^2}{r_1^2}+\frac{2G-p}{2G}\ \frac{m^2\pi^2}{l^2}\right)K_n(m_2r_1)-\frac{1}{r_1}K'_n(m_2r_1)\\
&F_{25}=-\frac{n^2}{r_1^2}I_n(m_1r_1)+\frac{n}{r_1}I'_n(m_1r_1)\\
&F_{26}=-\frac{n^2}{r_1^2}K_n(m_1r_1)+\frac{n}{r_1}K'_n(m_1r_1)\\
&F_{31}=\frac{2G-p}{2G}I'_n(m_1r_2)\qquad F_{32}=\frac{2G-p}{2G}K'_n(m_1r_2)\\
&F_{33}=I'_n(m_2r_2)\qquad F_{34}=K'_n(m_2r_2)\\
&F_{35}=\frac{n}{2r_2}I_n(m_1r_2)\qquad F_{36}=\frac{n}{2r_2}K_n(m_1r_2)\\
&F_{41}=\frac{2G-p}{2G}I'_n(m_1r_1)\qquad F_{42}=\frac{2G-p}{2G}K'_n(m_1r_1)\\
&F_{43}=I'_n(m_2r_1)\qquad F_{44}=K'_n(m_2r_1)\\
&F_{45}=\frac{n}{2r_1}I_n(m_1r_1)\qquad F_{46}=\frac{n}{2r_1}K_n(m_1r_1)\\
&F_{51}=\frac{n}{r_2^2}I_n(m_1r_2)-\frac{n}{r_2}I'_n(m_1r_2)\\
&F_{52}=\frac{n}{r_2^2}K_n(m_1r_2)-\frac{n}{r_2}K'_n(m_1r_2)\\
&F_{53}=\frac{n}{r_2}I_n(m_2r_2)-\frac{n}{r_2}I'_n(m_2r_2)\\
&F_{54}=\frac{n}{r_2}K_n(m_2r_2)-\frac{n}{r_2}K'_n(m_2r_2)\\
&F_{55}=-\left(\frac{n^2}{r_2^2}+\frac{G-p}{2G}\ \frac{m^2\pi^2}{l^2}\right)I_n(m_1r_2)+\frac{1}{r_2}I'_n(m_1r_2)\\
&F_{56}=-\left(\frac{n^2}{r_2^2}+\frac{G-p}{2G}\ \frac{m^2\pi^2}{l^2}\right)K_n(m_1r_2)+\frac{1}{r_2}K'_n(m_1r_2)
\end{aligned}\right\}\qquad(2.55)
$$

$$
\left.\begin{aligned}
F_{61}&=\frac{n}{r_1^2}I_n(m_1r_1)-\frac{n}{r_1}I'_n(m_1r_1)\\
F_{62}&=\frac{n}{r_1^2}K_n(m_1r_1)-\frac{n}{r_1}K'_n(m_1r_1)\\
F_{63}&=\frac{n}{r_1^2}I_n(m_2r_1)-\frac{n}{r_1}I'_n(m_2r_1)\\
F_{64}&=\frac{n}{r_1^2}K_n(m_2r_1)-\frac{n}{r_1}K'_n(m_2r_1)\\
F_{65}&=-\left(\frac{n^2}{r_1^2}+\frac{G-p}{2G}\frac{m^2\pi^2}{l^2}\right)I_n(m_1r_1)+\frac{1}{r_1}I'_n(m_1r_1)\\
F_{66}&=-\left(\frac{n^2}{r_1^2}+\frac{G-p}{2G}\frac{m^2\pi^2}{l^2}\right)K_n(m_1r_1)+\frac{1}{r_1}K'_n(m_1r_1)
\end{aligned}\right\}
$$

待定常数 A_1，B_1，A_2，B_2，A_0，B_0，具有非零解的条件为

$$
\begin{vmatrix}
F_{11} & F_{12} & F_{13} & F_{14} & F_{15} & F_{16}\\
F_{21} & F_{22} & F_{23} & F_{24} & F_{25} & F_{26}\\
F_{31} & F_{32} & F_{33} & F_{34} & F_{35} & F_{36}\\
F_{41} & F_{42} & F_{43} & F_{44} & F_{45} & F_{46}\\
F_{51} & F_{52} & F_{53} & F_{54} & F_{55} & F_{56}\\
F_{61} & F_{62} & F_{63} & F_{64} & F_{65} & F_{66}
\end{vmatrix}=0 \qquad (2.56)
$$

由此可以求解两端简支的圆柱壳在均匀轴压下一般失稳时的临界载荷．

由于行列式（2.56）中每一个元素都是超越函数，在超越函数中包涵着待求的临界载荷 p 和失稳时轴向的半波数 m 与周向的波数 n，因此由（2.56）式求解两端简支的圆柱壳在均匀轴压下一般失稳时最小的临界载荷，还相当困难．采用电子计算机，用数值计算的方法求解，是一种途径．现在研究文献［2～4］中研究过的几个问题，用解析法求出临界载荷的算式．

三、两对边简支的无限宽平板均匀受压时的稳定问题

有一无限宽的平板，长为 l，厚为 h，在两简支边上均匀受压，压力为 p，求失稳时的临界压力 p_{cr}．

所取坐标如图 2．无限宽的平板可看作是圆柱壳的半径趋于无限大时的特殊情况．由上一节确定的临界载荷的方程可得：

在失稳前的平衡位置，应力为

$$
\sigma_z^0=-p,\ \sigma_x^0=\sigma_y^0=\tau_{xy}^0=\tau_{yz}^0=\tau_{zx}^0=0 \qquad (3.1)
$$

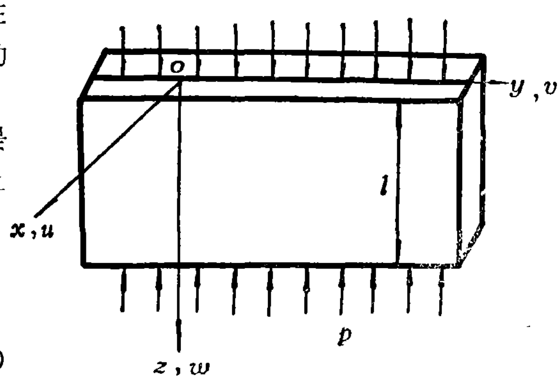

图 2

位移为

$$u^0=\frac{\nu p}{E}x,\quad v^0=0,\quad w^0=-\frac{p}{E}z \tag{3.2}$$

由 (2.7)，平衡方程为

$$\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xz}}{\partial z}-p\frac{\partial^2 u}{\partial z^2}=0,\qquad \frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\sigma_z}{\partial z}-p\frac{\partial^2 w}{\partial z^2}=0 \tag{3.3}$$

由 (2.9)，在 $x=\pm\frac{h}{2}$处，边界条件为

$$\sigma_x=\tau_{xz}=0 \tag{3.4}$$

由 (2.10)，在 $z=0$ 及 $z=l$ 处，简支条件为

$$u=\frac{\partial w}{\partial z}=0 \tag{3.5}$$

位移应变关系，由 (2.5) 得

$$\varepsilon_x=\frac{\partial u}{\partial x},\quad \varepsilon_z=\frac{\partial w}{\partial z},\quad \gamma_{xz}=\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x} \tag{3.6}$$

应力应变关系，由 (2.11) 得

$$\sigma_x=\lambda e+2G\varepsilon_x,\quad \sigma_z=\lambda e+2G\varepsilon_z,\quad \tau_{xz}=G\gamma_{xz} \tag{3.7}$$

其中，$e=\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}$.

用位移分量表示的平衡方程，由 (2.14) 得

$$\left.\begin{aligned}(\lambda+G)\frac{\partial e}{\partial x}+G\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial z^2}\right)-p\frac{\partial^2 u}{\partial z^2}=0\\(\lambda+G)\frac{\partial e}{\partial z}+G\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial z^2}\right)-p\frac{\partial^2 w}{\partial z^2}=0\end{aligned}\right\} \tag{3.8}$$

引入两个位移函数 φ_1 与 φ_2，使得

$$u=\frac{\partial}{\partial x}(\varphi_1+\varphi_2),\ w=\frac{\partial}{\partial z}\left(\frac{G-p}{G}\varphi_1+\varphi_2\right) \tag{3.9}$$

于是 (3.8) 化为已分离变量的方程组. 由 (2.18) 可得

$$\frac{\partial^2\varphi_1}{\partial x^2}+\frac{G-p}{G}\,\frac{\partial^2\varphi_1}{\partial z^2}=0,\qquad \frac{\partial^2\varphi_2}{\partial x^2}+\frac{\lambda+2G-p}{\lambda+2G}\,\frac{\partial^2\varphi_2}{\partial z^2}=0 \tag{3.10}$$

将 (3.9) 代入 (3.7)，注意到 (3.10)，得

$$\left.\begin{aligned}\sigma_x=-2G\left(\frac{G-p}{G}\,\frac{\partial^2\varphi_1}{\partial z^2}+\frac{2G-p}{2G}\,\frac{\partial^2\varphi_2}{\partial z^2}\right)\\\tau_{xz}=2G\left(\frac{2G-p}{2G}\,\frac{\partial^2\varphi_1}{\partial x\partial z}+\frac{\partial^2\varphi_2}{\partial x\partial z}\right)\end{aligned}\right\} \tag{3.11}$$

将 (3.11) 代入边界条件 (3.4) 得

$$s\frac{\partial^2\varphi_1}{\partial z^2}+\frac{\partial^2\varphi_2}{\partial z^2}=0,\ g\frac{\partial^2\varphi_1}{\partial x\partial z}+\frac{\partial^2\varphi_2}{\partial x\partial z}=0 \tag{3.12}$$

其中

$$s=\frac{2(G-p)}{2G-p},\ g=\frac{2G-p}{2G} \tag{3.13}$$

设

$$\varphi_1=f_1(x)\sin\frac{\pi z}{l},\ \varphi_2=f_2(x)\sin\frac{\pi z}{l} \tag{3.14}$$

将(3.14)代入(3.9)，可以看到简支条件(3.5)已满足. 将(3.14)代入(3.10)，可解得

$$f_1(x)=A_1e^{m_1x}+B_1e^{-m_1x},\ f_2(x)=A_2e^{m_2x}+B_2e^{-m_2x} \tag{3.15}$$

其中

$$m_1=\frac{m\pi}{l}\left(1-\frac{p}{G}\right)^{\frac{1}{2}},\ m_2=\left(1-\frac{p}{\lambda+2G}\right)^{\frac{1}{2}} \tag{3.16}$$

由(3.12)～(3.16)，边界条件(3.4)成为

$$\left.\begin{aligned}
&se^{m_1h/2}A_1+se^{-m_1h/2}B_1+e^{m_2h/2}A_2+e^{-m_2h/2}B_2=0\\
&se^{-m_1h/2}A_1+se^{m_1h/2}B_1+e^{-m_2h/2}A_2+e^{m_2h/2}B_2=0\\
&gm_1e^{m_1h/2}A_1-gm_1e^{-m_1h/2}B_1+m_2e^{m_2h/2}A_2-m_2e^{-m_2h/2}B_2=0\\
&gm_1e^{-m_1h/2}A_1-gm_1e^{m_1h/2}B_1+m_2e^{-m_2h/2}A_2-m_2e^{m_2h/2}B_2=0
\end{aligned}\right\} \tag{3.17}$$

(3.17)也可由(2.54)直接得到，只要将贝塞尔函数作渐近展开，约去公因子就行了.

附加位移不等于零，即积分常数 A_1，B_1，A_2，B_2，具有非零解的条件为

$$\begin{vmatrix}
se^{m_1h/2} & se^{-m_1h/2} & e^{m_2h/2} & e^{-m_2h/2}\\
se^{-m_1h/2} & se^{m_1h/2} & e^{-m_2h/2} & e^{m_2h/2}\\
gm_1e^{m_1h/2} & -gm_1e^{-m_1h/2} & m_2e^{m_2h/2} & -m_2e^{-m_2h/2}\\
gm_1e^{-m_1h/2} & -gm_1e^{m_1h/2} & m_2e^{-m_2h/2} & -m_2e^{m_2h/2}
\end{vmatrix}=0 \tag{3.18}$$

由此可求解临界载荷. 由(3.18)化简可得

$$\left[\left(\frac{sm_2}{gm_1}+1\right)sh\,\frac{m_2-m_1}{2}h-\left(\frac{sm_2}{gm_1}-1\right)sh\,\frac{m_1+m_2}{2}h\right]$$
$$\cdot\left[\left(\frac{sm_2}{gm_1}+1\right)sh\,\frac{m_2-m_1}{2}h+\left(\frac{sm_2}{gm_1}-1\right)sh\,\frac{m_1+m_2}{2}h\right]=0 \tag{3.19}$$

在平板中，反对称失稳时，

$$u(x,z)=u(-x,z),\ w(x,z)=-w(-x,z) \tag{3.20}$$

由(3.9)，(3.14)，(3.15)及(3.20)得

$$A_1=-B_1,\ A_2=-B_2 \tag{3.21}$$

于是(3.17)成为

$$\left.\begin{aligned}
&sA_1(e^{m_1h/2}-e^{-m_1h/2})+A_2(e^{m_2h/2}-e^{-m_2h/2})=0\\
&gm_1A_1(e^{m_1h/2}+e^{-m_1h/2})+m_2A_2(e^{m_2h/2}+e^{-m_2h/2})=0
\end{aligned}\right\} \tag{3.22}$$

产生反对称失稳的条件为 A_1，A_2 的系数的行列式等于零，化简后得

$$\left(\frac{sm_2}{gm_1}+1\right)sh\,\frac{m_2-m_1}{2}h-\left(\frac{sm_2}{gm_1}-1\right)sh\,\frac{m_1+m_2}{2}h=0 \tag{3.23}$$

与(3.19)中第一个因子等于零的条件相同.

在平板中，对称失稳时，

$$u(x,z)=-u(-x,z),\ w(x,z)=w(-x,z) \tag{3.24}$$

用相同的方法可得

$$A_1=B_1,\ A_2=B_2 \tag{3.25}$$

$$\left.\begin{aligned}
&sA_1(e^{m_1h/2}+e^{-m_1h/2})+A_2(e^{m_2h/2}+e^{-m_2h/2})=0\\
&gm_1A_1(e^{m_1h/2}-e^{-m_1h/2})+m_2A_2(e^{m_2h/2}-e^{-m_2h/2})=0
\end{aligned}\right\} \tag{3.26}$$

产生对称失稳的条件为

$$\left(\frac{sm_2}{gm_1}+1\right)sh\,\frac{m_2-m_1}{2}h+\left(\frac{sm_2}{gm_1}-1\right)sh\,\frac{m_1+m_2}{2}h=0 \tag{3.27}$$

与(3.19)中第二个因子等于零的条件相同.

由此可见，(3.19)式中包含了反对称失稳与对称失稳两种可能性.

将(3.23)中各项展成级数，注意到$\frac{\pi h}{l}$为一阶小量，$\frac{p}{G}$为二阶小量，计算后得到反对称失稳时的临界载荷为

$$\frac{p_{cr}}{G}=\frac{\pi^2h^2}{3l^2}\cdot\frac{\lambda+G}{\lambda+2G}\left(1-\beta_1\frac{\pi^2h^2}{l^2}+\cdots\right) \tag{3.28}$$

其中

$$\beta_1=\frac{17-7\nu}{60(1-\nu)} \tag{3.29}$$

当 $\nu=0.3$ 时，

$$\beta_1=0.355 \tag{3.30}$$

(3.28)中，取一级近似，得

$$\frac{p'_{cr}}{G}=\frac{\pi^2h^2}{3l^2}\,\frac{\lambda+G}{\lambda+2G} \tag{3.31}$$

这就是 Euler 公式.

将(3.27)展成级数，化简后得

$$\frac{p_{cr}}{G}=\frac{4\left(1+\frac{\pi^2h^2}{3l^2}\right)}{\frac{G}{\lambda+G}\left(1+\frac{2\pi^2h^2}{3l^2}\right)-1} \tag{3.32}$$

当$\frac{\pi^2h^2}{l^2}$为任何正值时，由(3.32)算得之临界应力都超过弹性极限，因此不可能产生弹性的对称失稳.

当采用[1]中(V 16)式时，平衡方程为

$$\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xz}}{\partial z}-\frac{p}{2}\left(\frac{\partial^2u}{\partial z^2}-\frac{\partial^2w}{\partial x\partial z}\right)=0,\quad \frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\sigma_z}{\partial z}=0 \tag{3.33}$$

采用边界条件(V 23)式时，所得之边界条件与(3.4)式相同.

用相同的方法，可得在反对称失稳时的临界应力

$$\frac{p_{cr}}{G}=\frac{\pi^2h^2}{3l^2}\,\frac{\lambda+G}{\lambda+2G}\left(1-\beta'_1\frac{\pi^2h^2}{l^2}+\cdots\right) \tag{3.34}$$

其中

$$\beta'_1=\frac{7-2\nu}{60(1-\nu)} \tag{3.35}$$

当 $\nu=0.3$ 时，

$$\beta'_1=0.152$$

并且同样不可能产生弹性的对称失稳形式.

由于在弹性阶段产生反对称失稳时，$\frac{\pi^2h^2}{l^2}$与1相比为很小的值．以钢为例，若 $\sigma_T=4200\ \mathrm{kg/cm^2}$，$E=2.1\times10^6\ \mathrm{kg/cm^2}$，$p_{cr}=4200\ \mathrm{kg/cm^2}$，此时

$$\frac{\pi^2h^2}{l^2}=0.022,\qquad \beta_1\frac{\pi^2h^2}{l^2}:1=0.78\% \tag{3.36}$$

所以$\left(\beta_1\frac{\pi^2h^2}{l^2}:1\right)$一般小于1%，是一个很小的量．因此由[1]中(Ⅴ13)和(Ⅴ16)所算得的系数 β_1 和 β_1'，如(3.29)和(3.35)中所表示的那样，差别虽然很大，但算得的临界应力都很接近，而且都与 Euler 公式接近，都比用 Euler 公式算得的应力要小一些．这说明Euler公式是相当精确的，用数学弹性力学的方法算得的临界应力，要比用板壳理论算得的结果小一些．

四、两端简支的圆杆在均匀轴压作用下的稳定问题

有一两端简支的细长圆杆，长为 l，半径为 r_2，在两端受均匀轴压时，求失稳时的临界应力 p_{cr}．

圆杆可以看作是圆柱壳的内径等于零的特殊情况，因此第2节中的公式可以应用．由于在中心轴上即 $r=0$ 处，位移、应力和应变都应为有限值，而 $K_n(0)=\infty$，$(n=0,1,2\cdots)$，因此(2.49)中之 B_0，B_1，B_2 应等于零．在圆杆失稳时，由物理现象可知 $n=1$．于是由(2.56)，求临界载荷的方程为

$$\begin{vmatrix} F_{11} & F_{13} & F_{15} \\ F_{31} & F_{33} & F_{35} \\ F_{51} & F_{53} & F_{55} \end{vmatrix}=0 \tag{4.1}$$

其中，由(2.55)可得

$$\left.\begin{aligned}
&F_{11}=\left(\frac{1}{r_2^2}+\frac{G-p}{G}\frac{m^2\pi^2}{l^2}\right)I_1(m_1r_2)-\frac{1}{r_2}I_1'(m_1r_2)\\
&F_{13}=\left(\frac{1}{r_2^2}+\frac{2G-p}{2G}\frac{m^2\pi^2}{l^2}\right)I_1(m_2r_2)-\frac{1}{r_2}I_1'(m_2r_2)\\
&F_{15}=-\frac{1}{r_2^2}I_1(m_1r_2)+\frac{1}{r_2}I_1'(m_1r_2)\\
&F_{31}=\frac{2G-p}{2G}I_1'(m_1r_2)\\
&F_{33}=I_1'(m_2r_2)\qquad F_{35}=\frac{1}{2r_2}I_1(m_1r_2)\\
&F_{51}=\frac{1}{r_2^2}I_1(m_1r_2)-\frac{1}{r_2}I_1'(m_1r_2)\\
&F_{53}=\frac{1}{r_2^2}I_1(m_2r_2)-\frac{1}{r_2}I_1'(m_2r_2)
\end{aligned}\right\} \tag{4.2}$$

$$F_{55}=-\left(\frac{1}{r_2^2}+\frac{G-p}{2G}\cdot\frac{m^2\pi^2}{l^2}\right)I_1(m_1r_2)+\frac{1}{r_2}I_1'(m_1r_2)$$

对 (4.1) 作如下运算

$$\begin{vmatrix} (F_{11}-F_{51})\frac{G}{G-p}\cdot\frac{l^2}{m^2\pi^2} & (F_{13}-F_{53})\frac{G}{G-p}\cdot\frac{l^2}{m^2\pi^2} & (F_{15}-F_{55})\frac{G}{G-p}\cdot\frac{l^2}{m^2\pi^2} \\ r_2F_{31} & r_2F_{33} & r_2F_{35} \\ r_2^2F_{51} & r_2^2F_{53} & r_2^2F_{55} \end{vmatrix}$$

$$=\begin{vmatrix} I_1(m_1r_2) & \frac{2G-p}{2(G-p)}I_1(m_2r_2) & \frac{1}{2}I_1(m_1r_2) \\ r_2\cdot\frac{2G-p}{2G}I_1'(m_1r_2) & r_2I_1'(m_2r_2) & \frac{1}{2}I_1(m_1r_2) \\ I_1(m_1r_2)-r_2I_1'(m_1r_2) & I_1(m_2r_2)-r_2I_1'(m_2r_2) & \left[-\left(1+\frac{G-p}{2G}\cdot\frac{m^2\pi^2r_2^2}{l^2}\right)I_1(m_1r_2)+r_2I_1'(m_1r_2)\right] \end{vmatrix}=0 \tag{4.3}$$

记行列式 (4.3) 为

$$\begin{vmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{vmatrix}=0 \tag{4.4}$$

然后作如下运算

$$\begin{vmatrix} H_{11} & H_{12} & H_{13} \\ H_{21}-H_{11} & H_{22}-H_{12} & H_{23}-H_{13} \\ H_{31}-\frac{2(G-p)}{2G-p}H_{11}+H_{21} & H_{32}-\frac{2(G-p)}{2G-p}H_{12}+H_{22} & H_{33}-\frac{2(G-p)}{2G-p}H_{13}+H_{23} \end{vmatrix}=0 \tag{4.5}$$

由于

$$H_{23}-H_{13}=0,\ H_{32}-\frac{2(G-p)}{2G-p}H_{12}+H_{22}=0 \tag{4.6}$$

(4.5) 化为

$$(H_{11}H_{22}-H_{12}H_{21})\left[H_{33}-\frac{2(G-p)}{2G-p}H_{13}+H_{23}\right]$$

$$-H_{13}(H_{22}-H_{12})\left[H_{31}-\frac{2(G-p)}{2G-p}H_{11}+H_{21}\right]=0 \tag{4.7}$$

当 n 为整数时，

$$I_n(x)=\sum_{k=0}^{\infty}\frac{1}{k!(n+k)!}\left(\frac{x}{2}\right)^{n+2k} \tag{4.8}$$

于是，

$$\left.\begin{aligned}&I_0(x)=1+\frac{x^2}{4}+\frac{x^4}{64}+\frac{x^6}{2304}+\frac{x^8}{147456}+\cdots\\&I_1(x)=I_0'(x)=\frac{x}{2}\left(1+\frac{x^2}{8}+\frac{x^4}{192}+\frac{x^6}{9216}+\cdots\right)\\&I_1'(x)=\frac{1}{2}\left(1+\frac{3x^2}{8}+\frac{5x^4}{192}+\frac{7x^6}{9216}+\cdots\right)\end{aligned}\right\}\tag{4.9}$$

在 (4.7) 中

$$\left.\begin{aligned}&H_{11}H_{22}-H_{12}H_{21}=r_2I_1(m_1r_2)I_1'(m_2r_2)-r_2\frac{(2G-p)^2}{4G(G-p)}I_1(m_2r_2)I_1'(m_1r_2)\\&H_{33}-\frac{2(G-p)}{2G-p}H_{13}+H_{23}=-\left[1-\frac{p}{2(2G-p)}+\frac{m^2\pi^2r_2^2}{l^2}\cdot\frac{G-p}{2G}\right]I_1(m_1r_2)+r_2I_1'(m_1r_2)\\&H_{13}=\frac{1}{2}I_1(m_1r_2)\\&H_{22}-H_{12}=r_2I_1'(m_2r_2)-\frac{2G-p}{2(G-p)}I_1(m_2r_2)\\&H_{31}-\frac{2(G-p)}{2G-p}H_{11}+H_{21}=\frac{p}{2G-p}I_1(m_1r_2)-r_2\frac{p}{2G}I_1'(m_1r_2)\end{aligned}\right\}\tag{4.10}$$

注意到

$$m_1=\frac{m\pi}{l}\left(1-\frac{p}{G}\right)^{\frac{1}{2}},\quad m_2=\frac{m\pi}{l}\left(1-\frac{p}{\lambda+2G}\right)^{\frac{1}{2}}\tag{4.11}$$

利用 (4.9) 式，将 (4.10) 式展成幂级数，认为 $\frac{p}{G}$ 与 $\left(\frac{m\pi r_2}{l}\right)^2$ 是二阶小量．代入 (4.7)，化简后得

$$\begin{aligned}&\left[\left(\frac{m^2\pi^2r_2^2}{l^2}\cdot\frac{\lambda+G}{\lambda+2G}-\frac{p}{G}\right)+\left(\frac{m^4\pi^4r_2^4}{6l^4}\frac{\lambda+G}{\lambda+2G}-\frac{p}{2G}\frac{m^2\pi^2r_2^2}{l^2}-\frac{p^2}{G^2}\right)\right]\\&\cdot\left[\left(1-\frac{p}{G}+\frac{m^2\pi^2r_2^2}{l^2}\right)+\left(-\frac{p^2}{2G^2}+\frac{m^4\pi^4r_2^4}{6l^4}-\frac{9}{8}\frac{m^2\pi^2r_2^2}{l^2}\cdot\frac{p}{G}\right)\right]\\&+\left(1+\frac{m^2\pi^2r_2^2}{8l^2}\right)\left[\left(\frac{m^2\pi^2r_2^2}{2l^2}-\frac{p}{G}\right)+\left(\frac{1}{24}\frac{m^4\pi^4r_2^4}{l^4}-\frac{p^2}{G^2}-\frac{1}{24}\frac{m^4\pi^4r_2^4}{l^4}\right.\right.\\&\left.\left.+\frac{5}{8}\frac{p}{G}\frac{m^2\pi^2r_2^2}{l^2}\right)\right]=0\end{aligned}\tag{4.12}$$

即

$$\begin{aligned}&\left[\left(\frac{m^2\pi^2r_2^2}{l^2}\frac{\lambda+G}{\lambda+2G}-\frac{p}{G}\right)\left(-\frac{p}{G}+\frac{m^2\pi^2r_2^2}{l^2}\right)-\left(\frac{m^2\pi^2r_2^2}{2l^2}-\frac{p}{G}\right)^2\right]\\&+\left\{\left[\left(\frac{p}{G}-\frac{m^2\pi^2r_2^2}{l^2}\cdot\frac{\lambda+G}{\lambda+2G}\right)\left(\frac{p^2}{2G^2}+\frac{9}{8}\frac{m^2\pi^2r_2^2}{l^2}\frac{p}{G}-\frac{m^4\pi^4r_2^4}{6l^4}\right)\right.\right.\\&\left.+\left(\frac{p}{G}-\frac{m^2\pi^2r_2^2}{l^2}\right)\left(\frac{p^2}{G^2}-\frac{p}{2G}\frac{m^2\pi^2r_2^2}{l^2}-\frac{m^4\pi^4r_2^4}{6l^4}\frac{\lambda+G}{\lambda+2G}\right)\right]\end{aligned}$$

$$+\left[-\frac{m^2\pi^2r_2^2}{8l^2}\cdot\left(\frac{m^2\pi^2r_2^2}{2l^2}-\frac{p}{G}\right)^2+\left(\frac{m^2\pi^2r_2^2}{2l^2}-\frac{p}{G}\right)\left(\frac{p^2}{2G}-\frac{1}{24}\frac{m^4\pi^4r_2^4}{l^4}\right.\right.$$

$$+\frac{5}{8}\frac{p}{G}\frac{m^2\pi^2r_2^2}{l^2}\right)+\left(\frac{p}{G}-\frac{1}{2}\frac{m^2\pi^2r_2^2}{l^2}\right)\cdot\left(\frac{1}{24}\frac{m^4\pi^4r_2^4}{l^4}-\frac{p^2}{G^2}\right.$$

$$\left.\left.-\frac{1}{2}\frac{m^2\pi^2r_2^2}{l^2}\frac{p}{\lambda+2G}-\frac{p}{8G}\frac{m^2\pi^2r_2^2}{l^2}\right)\right]\Bigg\}=0 \tag{4.13}$$

由四阶小量的部分等于零，可得一级近似，

$$\frac{p_{cr}}{G}=\frac{m^2\pi^2r_2^2}{4l^2}\cdot\frac{3\lambda+2G}{\lambda+G} \tag{4.14}$$

这就是 Euler 公式.由 (4.3) 可得

$$\frac{p_{cr}}{G}=\frac{m^2\pi^2r_2^2}{4l^2}\cdot\frac{3\lambda+2G}{\lambda+G}\left(1-\beta_2\frac{m^2\pi^2r_2^2}{l^2}+\cdots\right) \tag{4.15}$$

其中

$$\beta_2=\frac{37\lambda^2+55\lambda G+20G^2}{12(\lambda+G)(3\lambda+2G)} \tag{4.16}$$

当 $\nu=0.3$ 时，$\beta_2=0.955$.

如以钢为例，若 $\sigma_T=4200\ \mathrm{kg/cm^2}$，$E=2.1\times10^6\ \mathrm{kg/cm^2}$，设 $p_{cr}=4200\ \mathrm{kg/cm^2}$，此时

$$\frac{m^2\pi^2r_2^2}{l^2}=\frac{p_{cr}}{G}\frac{4\lambda+4G}{3\lambda+2G}=\frac{8.0}{1000},\quad \frac{m^2\pi^2r_2^2}{l^2}\beta_2:1=0.76\% \tag{4.17}$$

因此，Euler 公式是相当精确的.

五、两端简支的圆柱壳在均匀轴压作用下的轴对称失稳问题

有一两端简支的圆柱壳，长为 l，厚为 $h=r_2-r_1$，在均匀轴压 p 的作用下，求轴对称失稳时的临界载荷 p_{cr}.

在轴对称问题中，u_r 与 θ 无关. 而且第 2 节中的

$$u_\theta=\gamma_{r\theta}=\gamma_{\theta z}=n=\varphi_0=f_0(r)=0 \tag{5.1}$$

确定临界载荷的方程，由 (2.56) 式可得

$$\begin{vmatrix} F_{11} & F_{12} & F_{13} & F_{14} \\ F_{21} & F_{22} & F_{23} & F_{24} \\ F_{31} & F_{32} & F_{33} & F_{34} \\ F_{41} & F_{42} & F_{43} & F_{44} \end{vmatrix}=0 \tag{5.2}$$

其中，由 (2.55) 及虚变量圆柱函数的微分公式有

$$F_{11}=\frac{G-p}{G}\frac{m^2\pi^2}{l^2}I_0(m_1r_2)-\frac{m_1}{r_2}I_1(m_1r_2)$$

$$F_{12}=\frac{G-p}{G}\frac{m^2\pi^2}{l^2}K_0(m_1r_2)+\frac{m_1}{r_2}K_1(m_1r_2)$$

$$\left.\begin{aligned}
&F_{13}=\frac{2G-p}{2G}\frac{m^2\pi^2}{l^2}I_0(m_2r_2)-\frac{m_2}{r_2}I_1(m_2r_2)\\
&F_{14}=\frac{2G-p}{2G}\frac{m^2\pi^2}{l^2}K_0(m_2r_2)+\frac{m_2}{r_2}K_1(m_2r_2)\\
&F_{21}=\frac{G-p}{G}\cdot\frac{m^2\pi^2}{l^2}I_0(m_1r_1)-\frac{m_1}{r_1}I_1(m_1r_1)\\
&F_{22}=\frac{G-p}{G}\cdot\frac{m^2\pi^2}{l^2}K_0(m_1r_1)+\frac{m_1}{r_1}K_1(m_1r_1)\\
&F_{23}=\frac{2G-p}{2G}\frac{m^2\pi^2}{l^2}I_0(m_2r_1)-\frac{m_2}{r_1}I_1(m_2r_1)\\
&F_{24}=\frac{2G-p}{2G}\frac{m^2\pi^2}{l^2}K_0(m_2r_1)+\frac{m_2}{r_1}K_1(m_2r_1)\\
&F_{33}=m_2I_1(m_2r_2),\qquad F_{34}=-m_2K_1(m_2r_2)\\
&F_{41}=\frac{2G-p}{2G}m_1I_1(m_1r_1),\qquad F_{42}=-\frac{2G-p}{2G}\cdot m_1K_1(m_1r_1)\\
&F_{43}=m_2I_1(m_2r_1),\qquad F_{44}=-m_2K_1(m_2r_1).
\end{aligned}\right\}\tag{5.3}$$

作如下运算.

$$\begin{vmatrix}
F_{11}+\frac{1}{r_2}F_{31} & F_{12}+\frac{1}{r_2}F_{32} & F_{13}+\frac{1}{r_2}F_{33} & F_{14}+\frac{1}{r_2}F_{34}\\
F_{21}+\frac{1}{r_1}F_{41} & F_{22}+\frac{1}{r_1}F_{42} & F_{23}+\frac{1}{r_1}F_{43} & F_{24}+\frac{1}{r_1}F_{44}\\
F_{31} & F_{32} & F_{33} & F_{34}\\
F_{41} & F_{42} & F_{43} & F_{44}
\end{vmatrix}=0\tag{5.4}$$

约去公因子$\frac{2G-p}{2G}\cdot\frac{m^2\pi^2}{l^2}$，得

$$\begin{vmatrix}
sI_0(m_1r_2)-\frac{t}{r_2}m_1I_1(m_1r_2) & sK_0(m_1r_2)+\frac{t}{r_2}m_1K_1(m_1r_2) & I_0(m_2r_2) & K_0(m_2r_2)\\
sI_0(m_1r_1)-\frac{t}{r_1}m_1I_1(m_1r_1) & sK_0(m_1r_1)+\frac{t}{r_1}m_1K_1(m_1r_1) & I_0(m_2r_1) & K_0(m_2r_1)\\
gm_1I_1(m_1r_2) & -gm_1K_1(m_1r_2) & m_2I_1(m_2r_2) & -m_2K_1(m_2r_2)\\
gm_1I_1(m_1r_1) & -gm_1K_1(m_1r_1) & m_2I_1(m_2r_1) & -m_2K_1(m_2r_1)
\end{vmatrix}=0\tag{5.5}$$

其中

$$s=\frac{2(G-p)}{2G-p},\ t=\frac{l^2}{m^2\pi^2}\cdot\frac{p}{2G-p},\ g=\frac{2G-p}{2G}\tag{5.6}$$

在行列式(5.5)中，第一行和第二行，第三行和第四行，其差别仅在于r_2和r_1的不同所引起，由于$r_2-r_1=h$相当小，在作相减运算时，可将第一行和第三行用泰勒级数对r_1展

开、注意到，若$\frac{m\pi h}{l}$和$\frac{l}{m\pi r_1}$为一阶小量，则$\frac{p}{G}$和$\frac{h}{r_1}$为二阶小量，利用虚变量圆柱函数的微分与递推公式，可得

$$\left.\begin{aligned}
&\frac{I_0(m_1r_2)-I_0(m_1r_1)}{m_1(r_2-r_1)}=c_1I_0(m_1r_1)+d_1I_1(m_1r_1)\\
&\frac{K_0(m_1r_2)-K_0(m_1r_1)}{m_1(r_2-r_1)}=c_1K_0(m_1r_1)-d_1K_1(m_1r_1)\\
&\frac{I_0(m_2r_2)-I_0(m_2r_1)}{m_2(r_2-r_1)}=c_2I_0(m_2r_1)+d_2I_1(m_2r_1)\\
&\frac{K_0(m_2r_2)-K_0(m_2r_1)}{m_2(r_2-r_1)}=c_2K_0(m_2r_1)-d_2K_1(m_2r_1)\\
&\frac{I_1(m_1r_2)-I_1(m_1r_1)}{m_1(r_2-r_1)}=\alpha_1I_0(m_1r_1)+\beta_1I_1(m_1r_1)\\
&\frac{K_1(m_1r_2)-K_1(m_1r_1)}{m_1(r_2-r_1)}=-\alpha_1K_0(m_1r_1)+\beta_1K_1(m_1r_1)\\
&\frac{I_1(m_2r_2)-I_1(m_2r_1)}{m_2(r_2-r_1)}=\alpha_2I_0(m_2r_1)+\beta_2I_1(m_2r_1)\\
&\frac{K_1(m_2r_2)-K_1(m_2r_1)}{m_2(r_2-r_1)}=-\alpha_2K_0(m_2r_1)+\beta_2K_1(m_2r_1)
\end{aligned}\right\}\tag{5.7}$$

其中系数

$$\left.\begin{aligned}
c_1=&\frac{m_1h}{2}\left[1+\left(\frac{m_1^2h^2}{12}-\frac{h}{3r_1}\right)+\left(\frac{m_1^4h^4}{360}+\frac{h^2}{4r_1^2}-\frac{m_1^2h^3}{30r_1}\right)+\cdots\right]\\
d_1=&1+\left(\frac{m_1^2h^2}{6}-\frac{h}{2r_1}\right)+\left(\frac{m_1^4h^4}{120}+\frac{h^2}{3r_1^2}-\frac{m_1^2h^3}{12r_1}\right)+\left(\frac{m_1^6h^6}{5040}-\frac{m_1^4h^5}{240r_1}\right.\\
&\left.+\frac{7}{120}\frac{m_1^2h^4}{r_1^2}-\frac{h^3}{4r_1^3}\right)+\cdots\\
c_2=&\frac{m_2h}{2}\left[1+\left(\frac{m_2^2h^2}{12}-\frac{h}{3r_1}\right)+\left(\frac{m_2^4h^4}{360}+\frac{h^2}{4r_1^2}-\frac{m_2^2h^3}{30r_1}\right)+\cdots\right]\\
d_2=&1+\left(\frac{m_2^2h^2}{6}-\frac{h}{2r_1}\right)+\left(\frac{m_2^4h^4}{120}+\frac{h^2}{3r_1^2}-\frac{m_1^2h^3}{12r_1}\right)+\left(\frac{m_2^6h^6}{5040}-\frac{m_2^4h^5}{240r_1}\right.\\
&\left.+\frac{7}{120}\frac{m_2^2h^4}{r_1^2}-\frac{h^3}{4r_1^3}\right)+\cdots\\
\alpha_1=&1+\left(\frac{m_1^2h^2}{6}-\frac{h}{2r_1}\right)+\left(\frac{m_1^4h^4}{120}+\frac{h^2}{2r_1^2}-\frac{m_1^2h^3}{12r_1}\right)+\cdots\\
\beta_1=&\left(-\frac{1}{m_1r_1}+\frac{m_1h}{2}\right)+\left(\frac{h}{m_1r_1^2}-\frac{m_1h^2}{3r_1}+\frac{m_1^3h^3}{24}\right)+\left(\frac{m_1^5h^5}{720}-\frac{h^2}{m_1r_1^3}\right.\\
&\left.+\frac{7}{24}\frac{m_1h^3}{r_1^2}-\frac{m_1^3h^4}{40r_1}\right)+\cdots
\end{aligned}\right\}\tag{5.8}$$

$$\left.\begin{aligned}\alpha_2&=1+\left(\frac{m_2^2h^2}{6}-\frac{h}{2r_1}\right)+\left(\frac{m_2^4h^4}{120}+\frac{h^2}{2r_1^2}-\frac{m_2^2h^3}{12r_1}\right)+\cdots,\\ \beta_2&=\left(-\frac{1}{m_2r_1}+\frac{m_2h}{2}\right)+\left(\frac{h}{m_2r_1^2}-\frac{m_2h^2}{3r_1}+\frac{m_2^3h^3}{24}\right)+\left(\frac{m_2^5h^5}{720}-\frac{h^2}{m_2r_1^3}\right.\\ &\quad\left.+\frac{7}{24}\cdot\frac{m_2h^3}{r_1}-\frac{m_2^3h^4}{40r_1}\right)+\cdots\end{aligned}\right\}$$

将 (5.5) 中第一行减去第二行，第三行减去第四行，分别除以 h，作为新行列式的第一行和第三行，考虑到 (5.7) 和 (5.8) 式，有

$$\left|\begin{matrix}\left(sm_1c_1-\frac{tm_1^2}{r_2}\alpha_1\right)I_0(m_1r_1) & \left(sm_1c_1-\frac{tm_1^2}{r_2}\alpha_1\right)K_0(m_1r_1)\\ +\left[sm_1\alpha_1-\frac{tm_1^2}{r_2}\left(\beta_1-\frac{1}{m_1r_1}\right)\right]I_1(m_1r_1) & -\left[sm_1\alpha_1-\frac{tm_1^2}{r_2}\left(\beta_1-\frac{1}{m_1r_1}\right)\right]K_1(m_1r_1)\\ sI_0(m_1r_1)-\frac{tm_1}{r_1}I_1(m_1r_1) & sK_0(m_1r_1)+\frac{tm_1}{r_1}K_1(m_1r_1)\\ gm_1^2[\alpha_1I_0(m_1r_1)+\beta_1I_1(m_1r_1)] & gm_1^2[\alpha_1K_0(m_1r_1)-\beta_1K_1(m_1r_1)]\\ gm_1I_1(m_1r_1) & -gm_1K_1(m_1r_1)\end{matrix}\right.$$

$$\left.\begin{matrix}m_2[c_2I_0(m_2r_1)+d_2I_1(m_2r_1)] & m_2[c_2K_0(m_2r_1)-d_2K_1(m_2r_1)]\\ I_0(m_2r_1) & K_0(m_2r_1)\\ m_2^2\alpha_2I_0(m_2r_1)+m_2\beta_2I_1(m_2r_1) & m_2^2\alpha_2K_0(m_2r_1)-m_2\beta_2K_1(m_2r_1)\\ m_2I_1(m_2r_1) & -m_2K_1(m_2r_1)\end{matrix}\right|=0 \qquad (5.9)$$

将 (5.9) 中第一行减去 (m_2c_2 乘第三行加 d_2 乘第四行)，作为新行列式的第一行，第三行减去 ($m_2^2\alpha_2$ 乘第二行加 β_2 乘第四行)，作为新行列式的第三行，原第二和第四行保留在新行列式中，经化简后可得

$$\begin{aligned}&[I_0(m_2r_1)K_1(m_2r_1)+K_0(m_2r_1)I_1(m_2r_1)]\\ &\cdot\left\{\left[s(m_2c_2-m_1c_1)+\frac{tm_1^2}{r_2}\alpha_1\right]\cdot\left[g(m_2\beta_2-m_1\beta_1)-\frac{tm_2^2}{r_2}\alpha_2\right]\right.\\ &\left.-(sm_2^2\alpha_2-gm_1^2\alpha_1)\left[gd_2-sd_1+\frac{tm_1}{r_2}\left(\beta_1-\frac{1}{m_1r_1}\right)-\frac{m_2c_2t}{r_1}\right]\right\}=0\end{aligned} \qquad (5.10)$$

由于 $I_0(m_2r_1)K_1(m_2r_1)+K_0(m_2r_1)I_1(m_2r_1)\neq 0$，所以

$$\begin{aligned}&\left[s(m_2c_2-m_1c_1)+\frac{tm_1^2}{r_2}\alpha_1\right]\cdot\left[g(m_2\beta_2-m_1\beta_1)-\frac{tm_2^2}{r_2}\alpha_2\right]\\ &-(sm_2^2\alpha_2-gm_1^2\alpha_1)\left[gd_2-sd_1+\frac{tm_1}{r_2}\left(\beta_1-\frac{1}{m_1r_1}\right)-\frac{m_2c_2t}{r_1}\right]=0\end{aligned} \qquad (5.11)$$

在化简过程中，注意到

$$\left.\begin{aligned}
&s=\frac{2G-2p}{2G-p},\quad t=\frac{l^2p}{m^2\pi^2(2G-p)},\quad g=\frac{2G-p}{2G},\quad sg=1-\frac{p}{G},\\
&gt=\frac{l^2}{m^2\pi^2}\cdot\frac{p}{2G},\quad g^2=1-\frac{p}{G}+\frac{p^2}{4G^2},\quad \frac{s}{g}=1-\frac{p^2}{4G^2}-\frac{p^3}{4G^3}+\cdots,\\
&\frac{g}{s}=1+\frac{p^2}{4G^2}+\frac{p^3}{4G^3}+\cdots,\quad \frac{t}{g}=\frac{l^2}{m^2\pi^2}\cdot\frac{p}{2G}\left(1+\frac{p}{G}+\frac{3}{4}\frac{p^2}{G^2}+\frac{p^3}{2G^3}+\cdots\right),\\
&m_1^2=\frac{m^2\pi^2}{l^2}\left(1-\frac{p}{G}\right),\quad m_2^2=\frac{m^2\pi^2}{l^2}\left(1-\frac{p}{\lambda+2G}\right),\quad m_2^2-m_1^2=\frac{m^2\pi^2}{l^2}\cdot\frac{p}{G}\cdot\frac{\lambda+G}{\lambda+2G}\\
&m_1^2+m_2^2=\frac{m^2\pi^2}{l^2}\left(2-\frac{\lambda+3G}{\lambda+2G}\cdot\frac{p}{G}\right),\quad a=\frac{r_1+r_2}{2},\quad r_1=a\left(1-\frac{h}{2a}\right),\quad r_2=a\left(1+\frac{h}{2a}\right)
\end{aligned}\right\}\quad(5.12)$$

由 (5.11) 可得

$$\left[sg(m_2c_2-m_1c_1)+\frac{gtm_1^2\alpha_1}{r_2}\right]\left[(m_2\beta_2-m_1\beta_1)-\frac{t}{g}\cdot\frac{m_2^2\alpha_2}{r_1}\right]$$

$$-(sgm_2^2\alpha_2-g^2m_1^2\alpha_1)\left[d_2-\frac{s}{g}d_1+\frac{tm_1}{gr_2}\left(\beta_1-\frac{1}{m_1r_1}\right)-\frac{m_2c_2t}{r_1g}\right]=0 \qquad (5.13)$$

由于 (5.12)，可将 (5.13) 化为

$$\left[\frac{l^2}{m^2\pi^2h^2}(m_2c_2-m_1c_1)+\frac{gt\alpha_1}{hr_2}\right]\left[h(m_2\beta_2-m_1\beta_1)-\frac{t}{g}\frac{m_2^2\alpha_2}{r_1}h\right]$$

$$-\left[\frac{l^2}{m^2\pi^2}m_2^2\alpha_2-g^2\alpha_1\right]\left[d_2-\frac{s}{g}d_1+\frac{tm_1}{gr_2}\left(\beta_1-\frac{1}{m_1r_1}\right)-\frac{m_2c_2t}{gr_1}\right]=0 \qquad (5.14)$$

其中，

$$\left.\begin{aligned}
&\frac{l^2}{m^2\pi^2h^2}(m_2c_2-m_1c_1)+\frac{gt\alpha_1}{hr_2}=\frac{p}{2G}\Big[\Big(\frac{\lambda+G}{\lambda+2G}+\frac{l^2}{m^2\pi^2ah}\\
&\quad+\Big(\frac{\lambda+G}{\lambda+2G}\cdot\frac{m^2\pi^2h^2}{6l^2}-\frac{h}{6a}\cdot\frac{\lambda}{\lambda+2G}-\frac{l^2}{m^2\pi^2a^2}\Big)\Big]+\cdots\\
&h(m_2\beta_2-m_1\beta_1)-\frac{t}{g}\frac{m_2^2\alpha_2}{r_1}h=\frac{p}{2G}\Big[\Big(\frac{m^2\pi^2h^2}{l^2}\frac{\lambda+G}{\lambda+2G}-\frac{h}{a}\Big)\\
&\quad+\Big(\frac{m^4\pi^4h^4}{6l^4}\cdot\frac{\lambda+G}{\lambda+2G}-\frac{p}{G}\cdot\frac{\lambda+G}{\lambda+2G}\cdot\frac{h}{a}-\frac{5\lambda+6G}{\lambda+2G}\cdot\frac{m^2\pi^2h^3}{6al^2}\Big)\Big]+\cdots\\
&\frac{l^2}{m^2\pi^2}m_2^2\alpha_2-g^2\alpha_1=\frac{p}{2G}\Big[\frac{2(\lambda+G)}{\lambda+2G}+\Big(\frac{2}{3}\frac{m^2\pi^2h^2}{l^2}-\frac{h}{a}\cdot\frac{\lambda+G}{\lambda+2G}-\frac{p}{2G}\Big)\Big]+\cdots\\
&d_2-\frac{s}{g}d_1+\frac{tm_1}{gr_2}\left(\beta_1-\frac{1}{m_1r_1}\right)-\frac{m_2c_2t}{gr_1}=\frac{p}{2G}\Big[\Big(\frac{m^2\pi^2h^2}{3l^2}\cdot\frac{\lambda+G}{\lambda+2G}+\frac{p}{2G}\\
&\quad-\frac{2l^2}{m^2\pi^2a^2}\Big)+\Big(\frac{m^4\pi^4h^4}{30l^4}\cdot\frac{\lambda+G}{\lambda+2G}-\frac{m^2\pi^2h^3}{6al^2}\frac{\lambda+G}{\lambda+2G}+\frac{p^2}{G^2}\\
&\quad+\frac{p}{G}\frac{m^2\pi^2h^2}{12l^2}+\frac{hl^2}{m^2\pi^2a^3}-\frac{2}{3}\frac{h^2}{a^2}-\frac{ph}{4Ga}\cdot\frac{3\lambda+4G}{\lambda+2G}-\frac{2l^2}{m^2\pi^2a^2}\cdot\frac{p}{G}\Big)\Big]
\end{aligned}\right\}\quad(5.15)$$

将 (5.15) 代入 (5.14)，可得

$$
\begin{aligned}
&\Big[\Big(\frac{\lambda+G}{\lambda+2G}+\frac{l^2}{m^2\pi^2ah}\Big)\Big(\frac{m^2\pi^2h^2}{l^2}\frac{\lambda+G}{\lambda+2G}-\frac{h}{a}\Big)-\frac{2(\lambda+G)}{\lambda+2G}\Big(\frac{m^2\pi^2h^3}{3l^2}\cdot\frac{\lambda+G}{\lambda+2G}\\
&+\frac{p}{2G}-\frac{2l^2}{m^2\pi^2a^2}\Big)\Big]+\Big[\Big(\frac{\lambda+G}{\lambda+2G}+\frac{l^2}{m^2\pi^2ah}\Big)\Big(\frac{m^4\pi^4h^4}{6l^4}\cdot\frac{\lambda+G}{\lambda+2G}-\frac{p}{G}\cdot\frac{\lambda+G}{\lambda+2G}\frac{h}{a}\\
&-\frac{5\lambda+6G}{\lambda+2G}\frac{m^2\pi^2h^3}{6al^2}+\Big(\frac{m^2\pi^2h^2}{l^2}\cdot\frac{\lambda+G}{\lambda+2G}-\frac{h}{a}\Big)\Big(\frac{\lambda+G}{\lambda+2G}\frac{m^2\pi^2h^2}{6l^2}-\frac{h}{6a}\cdot\frac{\lambda}{\lambda+2G}-\frac{l^2}{m^2\pi^2a^2}\Big)\\
&-\frac{2(\lambda+G)}{\lambda+2G}\Big(\frac{m^4\pi^4h^4}{30l^4}\frac{\lambda+G}{\lambda+2G}-\frac{m^2\pi^2h^3}{6al^2}\cdot\frac{\lambda+G}{\lambda+2G}+\frac{p^2}{2G^2}+\frac{p}{G}\frac{m^2\pi^2h^2}{12l^2}\\
&+\frac{hl^2}{m^2\pi^2a^3}-\frac{2}{3}\frac{h^2}{a^2}-\frac{ph}{4Ga}\cdot\frac{3\lambda+4G}{\lambda+2G}-\frac{2l^2}{m^2\pi^2a^2}\cdot\frac{p}{G}\Big)\\
&-\Big(\frac{m^2\pi^2h^2}{3l^2}\cdot\frac{\lambda+G}{\lambda+2G}+\frac{p}{2G}-\frac{2l^2}{m^2\pi^2a^2}\Big)\cdot\Big(\frac{2}{3}\frac{m^2\pi^2h^2}{l^2}\frac{\lambda+G}{\lambda+2G}\\
&-\frac{h}{a}\frac{\lambda+G}{\lambda+2G}-\frac{p}{2G}\Big)\Big]+\cdots=0
\end{aligned}
\tag{5 16}
$$

由其中第一个方括号中的式子等于零，可得一级近似

$$
\frac{p_{cr}}{G}=\frac{\lambda+G}{\lambda+2G}\cdot\frac{m^2\pi^2h^2}{3l^2}+\frac{3\lambda+2G}{\lambda+G}\cdot\frac{l^2}{m^2\pi^2a^2} \tag{5.17}
$$

即

$$
\frac{p'_{cr}}{E}=\frac{m^2\pi^2h^2}{12(1-\nu^2)l^2}+\frac{l^2}{m^2\pi^2a^2} \tag{5.18}
$$

与 Timoshenko[5] 书上的公式一致.

设在屈曲时在轴向形成了好多个波，使得$\frac{p'_{cr}}{E}$可近似地看作是$\frac{m\pi}{l}$的连续函数，于是$\frac{p'_{cr}}{E}$为最小值的条件为

$$
\partial\Big(\frac{p'}{E}\Big)\Big/\partial\Big(m\frac{\pi}{l}\Big)=0 \tag{5.19}
$$

由此可得

$$
\frac{m^2\pi^2h^2}{12(1-\nu^2)l^2}=\frac{l^2}{m^2\pi^2a^2} \tag{5.20}
$$

将(5.20)代入(5.18)，可得

$$
\frac{p}{E}=\frac{m^2\pi^2h^2}{6(1-\nu^2)l^2} \tag{5.21}
$$

于是

$$
\frac{p'}{G}=\frac{m^2\pi^2h^2}{3(1-\nu)l^2}=\frac{2m^2\pi^2h^2}{3l^2}\cdot\frac{\lambda+G}{\lambda+2G} \tag{5.22}
$$

设

$$
\frac{p}{G}=\frac{2}{3}\frac{m^2\pi^2h^2}{l^2}\cdot\frac{\lambda+G}{\lambda+2G}\Big(1-\beta_3\frac{m^2\pi^2h^2}{l^2}\Big) \tag{5.23}
$$

代入(5.16)得

$$\beta_3=\left[\frac{4}{15}-\frac{1+\nu-\nu^2}{2\sqrt{3(1-\nu^2)}\,(1-\nu^2)}+\frac{2\nu-1}{3(1-\nu^2)}\right]+\frac{1+2\nu}{2(1+\nu)}\left[\frac{1}{3}-\frac{1}{2\sqrt{3(1-\nu^2)}}\right]$$

$$-\frac{1}{4}\left[1+\sqrt{\frac{1-\nu}{3(1+\nu)}}\right]\left[1-\frac{(2-\nu)^2}{\sqrt{3(1-\nu^2)}\,(1-\nu)}-\frac{\nu}{4}\left[1-\sqrt{\frac{1-\nu}{3(1+\nu)}}\right]\right]\left[\frac{1}{1+\nu}\right.$$

$$\left.-\frac{1}{\sqrt{3(1-\nu^2)}}\right] \tag{5.24}$$

当 $\nu=0.3$ 时，$\beta_3=0.261$

仍以钢为例，若 $\sigma_T=4200\text{kg/cm}^2$，$E=2.1\times10^6\text{kg/cm}^2$，设 $p_{cr}=4200\text{kg/cm}^2$，由(5.21)得

$$\frac{m^2\pi^2h^2}{l^2}=0.109 \tag{5.25}$$

$$\beta_3\frac{m^2\pi^2h^2}{l^2}:1=0.285\% \tag{5.26}$$

因此，[5]中所引用的公式是相当精确的.

在(5.16)中，当 $a\to\infty$，就化为两边简支的无限宽的平板在两边均匀受压时的稳定问题.

$$\frac{p_{cr}}{G}=\frac{m^2\pi^2h^2}{3l^2}\ \frac{\lambda+G}{\lambda+2G}\left(1-\beta_1\frac{m^2\pi^2h^2}{l^2}\right) \tag{5.27}$$

其中

$$\beta_1=\frac{17-7\nu}{60(1-\nu)}$$

与(3.29)式相一致.

六、讨　　论

(1) 用数学弹性力学的方法，研究弹性体的稳定问题，如文献[1]中所提出的那样，不但在原则上是可行的，而且本文的工作有力地表明，在事实上亦是可行的.

(2) 用细杆理论和板壳理论所得的临界载荷公式，与本文中用数学弹性力学所得的结果相比，误差相当小(小于1%)，相当精确.

(3) 用数学弹性力学方法算得的临界载荷应该比用细杆理论和板壳理论算得的结果小，这才合理. 因为在细杆理论和板壳理论中，对应力状态或变形的分布作了一些简化假定，这相当于将原来是各向同性的(或正交各向异性的)物体，假定为一个弹性常数增大了的各向异性体. 如以圆柱壳为例，采用直法线假定，相当于将E_r，G_{rz}，$G_{r\theta}$从有限值提高到无限大，因此临界载荷就提高了.

(4) 我们认为，Ишлинский[2]和Войцеховская[3,4]所得的临界载荷比用细杆理论或板壳理论算得的结果还要大些，在理论上是不合理的. 探讨其原因，注意到[2～4]中所用的直角坐标(x_1,y_1,z_1)是对于已变形物体而言的，而[1]中所用的直角坐标(x,y,z)是对于变形前的物体而言的. 所以[2～4]中的平衡方程和边界条件，会和[1]中的平衡方程和边界条件在形式上不同. 由于$x_1=x+u(x,y,z)$，$y_1=y+v(x,y,z)$，$z_1=z+w(x,y,z)$，所以已

变形物体上的直角座标中，包含了未知的位移$u(x,y,z)$，$v(x,y,z)$，$w(x,y,z)$，这使边界条件变得复杂和困难了．[2]中采用简单的$\frac{dy_1}{dx_1}\approx\frac{dy}{dx}$这种近似方法来克服这个困难，取得了一定的成果，即算得的临界载荷与板壳理论的结果相接近．但这种做法却带来了这样的后果，即用数学弹性力学的方法算得的临界载荷比用板壳理论算得的临界载荷还要大些的不合现象．

本文是研究生论文中的一篇，是1962年初在导师胡海昌指导下完成的．本文与文献[9]都是采用位移函数求解具体问题的．我采用的是胡海昌在1953年发表的位移函数[6]，[9]中采用的是K. Heki和M. Habara在1965年发表的位移函数(引文见[9]中第四篇)．[9]中算得的临界载荷亦比经典理论值要小(只做了杆的问题)．这再次说明Ишлинский和Войцеховская的结果不够合理．

参 考 文 献

1. В. В.诺沃日洛夫，«非线性弹性力学基础»，科学出版社，(1958)，131—142.
2. Ишлинский, А. Ю., Рассмотрение вопросов об устойчивости равновесия упругих тел с точки зрения математической теории Упругости, Укр. матм. жур. т.6, №.2, (1954), 140—146.
3. Войцеховская, К. Ф., Устойчивость равновесия стержней с точки зрения математической теории упругости, *ДАНСССР*, т.119, №5, (1958), 903—906.
4. Войцеховская, К. Ф., Устойчивость цилиндрических оболочек с точки зрения математической теории упругости, *ДАНСССР*, т.123, №4. (1958), 623—626.
5. S. 铁摩辛柯，«弹性稳定理论»，科学出版社，(1958)，429—431.
6. 胡海昌，横观各向同性弹性体的空间问题，物理学报，第九卷(1953)，130—144.
7. Ершов, Л. В. и Ивлев, Д. Д., Об устойчивость полосы при сжатии, ДАДСССР, т.138, №5, (1961), 1047—1049.
8. Лейбензон, Л. С., О применении гармонических функций к вопросу оь устойчивости сферической и цилиндрической оболочек, сбор тр. 1. (1951), 81—85.
9. Renton, J. D., Note on generalized displacement functions in the presence of initial stress. *J. Structural Mechanics*, 7, 4, (1979), 365—373.

A Study of Elastic Stability Problems Based on Mathematical Theory of Elasticity

Wang Zhen-ming

(Institute of Mechanics, Academia Sinica, Beijing)

Abstract

It is important but difficult to study the stability problems of elastic bodies based on mathematical theory of elasticity. In [1], B. B. Новожилов had obtained the equilibrium equations and boundary conditions, but he did not give any practical solution.

A. Ю. Ишлинский had given a solution for the stability problem of elastic plate of infinite width with two edges simply supported and under uniform compression, on the plane strain condition, using mathematical theory of elasticity. He said, the mathematical difficulties were overcome "by neglecting the rotationary terms in the equilibrium equations, and retaining the rotationary terms in the boundary conditions, as viewed from the Новожилов's equations." But in his equations he introduced some simplifications and brought some errors, which made the critical load slightly higher than the result from the classical theory.

К. Ф. Войцеховская used the same method as Ишлинский and solved the problem of a circular bar and a circular cylindrical shell with two edges simply supported under axial compression, but the critical loads were also slightly higher than the results from the classical theory. From the viewpoint of theoretical elasticity, their results were not rigorous enough.

In this paper, we used Новожилов's equilibrium equations and boundary conditions, and introduced Hu Hai-chang's displacement functions to simiplify the differential equations and overcame the mathematical difficulties. We solved the same problems as [2 ~ 4]. The results of critical load were slightly lower than the results from classical theory. From the viewpoint of mathematical theory of elasticity, it was rigorous.