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半圆弧波纹管的计算——环壳一般解的应用

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摘　　要

本文利用前文[1]所得圆环壳的一般解，计算了半圆弧波纹管在轴向力作用下的变形和应力分布.

一、引　　言

波纹管是圆柱形的薄壁壳，沿着侧面在轴向制成有波纹的折皱，最简单的折皱单元是由正负两个半圆弧连接而成（图1），称为半圆弧波纹管．它的平均半径为R，半圆弧的半径为a，壁厚为h，波纹总长为$4na$，n为波纹总数，如果取一个单元AFGEB如图2，则在计算

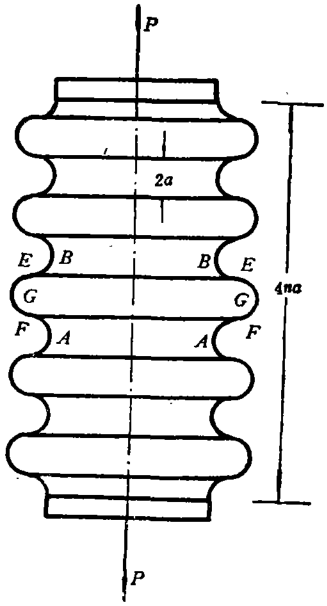


图1　半圆弧波纹管尺寸

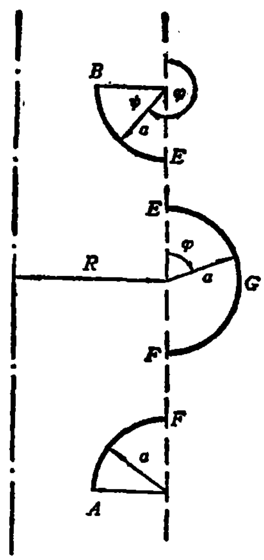


图2　半圆弧波纹管单元坐标

时，应该分段取不同的坐标．在BE段中，用角ψ表示角坐标，$\psi=\varphi-\pi$；在EGF中，可用φ为角坐标．N_φ，N_θ，M_φ和M_θ都对称于G点($\varphi=\pi/2$)和B点($\psi=\pi/2$)．

波纹管的计算，长期以来除了用近似法[如C. E. Turner-H. Ford(1957)[2]]外，由于求解方程的困难，一直未得妥善解决，文[3]根据细环壳的一般解，把波纹管单元分为正负两个环壳处理．本文将不限于细环壳，对任何粗细的圆环壳，利用其一般解进行计算．

二、半圆弧波纹管单元受轴力作用时在不同区域内解的形式

这个问题可以分为两个区域求解，即区域EGF和BE（图2），首先让我们写出EGF区域的解，在这个区域内，文[3]中的(1)式可以写成

$$(1+\alpha\sin\varphi)\frac{d^2V_{\rm I}}{d\varphi^2}-\alpha\cos\varphi\frac{dV_{\rm I}}{d\varphi}+2\mu i\sin\varphi V_{\rm I}=2\mu P_0\cos\varphi$$

$$P_0=\frac{Q_0}{\alpha}2\mu \tag{2.1}$$

它的解可以分为非齐次解$V_{\rm I}^*$和齐次解$V_{\rm I}^{(1)}+V_{\rm I}^{(2)}$两部分

$$\begin{aligned}V_{\rm I}^*=-4\mu\frac{Q_0}{\alpha}\{&A_1\cos\varphi+A_2\sin 2\varphi-A_3\cos 3\varphi-A_4\sin 4\varphi+\cdots\\&+A_{4n+1}\cos(4n+1)\varphi+A_{4n+2}\sin(4n+2)\varphi-A_{4n+3}\cos(4n+3)\varphi\\&+A_{4n+4}\sin(4n+4)\varphi+\cdots\}\end{aligned} \tag{2.2}$$

$$\left.\begin{aligned}V_{\rm I}^{(1)}&=(C_0'+i\bar{C}_0')e^{-\beta(\pi/2-\varphi)}(\cos\gamma\varphi+i\sin\gamma\varphi)[f_1(\varphi)+if_2(\varphi)]\\V_{\rm I}^{(2)}&=(B_0'+i\bar{B}_0')e^{-\beta\varphi}(\cos\gamma\varphi-i\sin\gamma\varphi)[g_1(\varphi)+ig_2(\varphi)]\end{aligned}\right\} \tag{2.3A, B}$$

而

$$V_{\rm I}=V_{\rm I}^*+V_{\rm I}^{(1)}+V_{\rm I}^{(2)}$$

其中A_n、β、γ见文[1]．而$Q_0=P/2\pi R$，P是轴向合力，拉力为正．$f_1(\varphi)$、$f_2(\varphi)$、$g_1(\varphi)$和$g_2(\varphi)$可分别从文[3]的(18A、B、C、D)式证明为

$$\left.\begin{aligned}f_1(\varphi)=G_1(\varphi)+F_1(\varphi),\quad f_2(\varphi)=G_2(\varphi)+F_2(\varphi)\\g_1(\varphi)=G_1(\varphi)-F_1(\varphi),\quad g_2(\varphi)=G_2(\varphi)-F_2(\varphi)\end{aligned}\right\} \tag{2.4}$$

式中$G_1(\varphi)$、$G_2(\varphi)$是对称于G点($\varphi=\pi/2$)的函数，而$F_1(\varphi)$、$F_2(\varphi)$是反对称于G的函数，它们是

$$\left.\begin{aligned}F_1(\varphi)&=\sum_{n=1,3,5\cdots}^{\infty}p_n\cos n\varphi-\sum_{n=2,4,6\cdots}^{\infty}q_n'\sin n\varphi\\F_2(\varphi)&=\sum_{n=1,3,5\cdots}^{\infty}q_n\cos n\varphi+\sum_{n=2,4,6\cdots}^{\infty}p_n'\sin n\varphi\\G_1(\varphi)&=1+\sum_{n=2,4,6\cdots}^{\infty}p_n\cos n\varphi-\sum_{n=1,3,5\cdots}^{\infty}q_n'\sin n\varphi\end{aligned}\right\} \tag{2.5A, B, C, D}$$

$$G_2(\varphi)=\sum_{n=2,4,6\cdots}^{\infty} q_n\cos n\varphi+\sum_{n=1,3,5\cdots}^{\infty} p_n\sin n\varphi$$

其中，$p_n=\frac{1}{2}(a_n+a_{-n})$，$q_n=\frac{1}{2}(b_n+b_{-n})$，$p_n=\frac{1}{2}(a_n-a_{-n})$，$q_n=\frac{1}{2}(b_n-b_{-n})$，而$a_n$、$a_{-n}$，$b_n$及$b_{-n}$见文[1].

利用

$$\cos\gamma\varphi+i\sin\gamma\varphi=e^{i\gamma\frac{\pi}{2}}\left[\cos\gamma\left(\frac{\pi}{2}-\varphi\right)-i\sin\gamma\left(\frac{\pi}{2}-\varphi\right)\right]$$

$$\cos\gamma\varphi-i\sin\gamma\varphi=e^{-i\gamma\frac{\pi}{2}}\left[\cos\gamma\left(\frac{\pi}{2}-\varphi\right)+i\sin\gamma\left(\frac{\pi}{2}-\varphi\right)\right]$$

把$V_{\rm I}^{(1)}$、$V_{\rm I}^{(2)}$分成对称和反对称两部分

$$V_{\rm I}^{(1)}=(C_0'+i\bar{C}_0')e^{\gamma\frac{\pi}{2}}e^{-\beta\left(\frac{\pi}{2}-\varphi\right)}\Big\{\Big[(G_1(\varphi)+iG_2(\varphi))\cos\gamma\left(\frac{\pi}{2}-\varphi\right)-i(F_1(\varphi)+iF_2(\varphi))\sin\gamma\left(\frac{\pi}{2}-\varphi\right)\Big]+\Big[(F_1(\varphi)+iF_2(\varphi))\cos\gamma\left(\frac{\pi}{2}-\varphi\right)-i(G_1(\varphi)+iG_2(\varphi))\sin\gamma\left(\frac{\pi}{2}-\varphi\right)\Big]\Big\}$$

$$V_{\rm I}^{(2)}=(B_0'+i\bar{B}_0')e^{-\gamma\frac{\pi}{2}}e^{-\beta\left(\frac{\pi}{2}-\varphi\right)}\Big\{\Big[(G_1(\varphi)+iG_2(\varphi))\cos\gamma\left(\frac{\pi}{2}-\varphi\right)-i(F_1(\varphi)+iF_2(\varphi))\sin\gamma\left(\frac{\pi}{2}-\varphi\right)\Big]+\Big[-(F_1(\varphi)+iF_2(\varphi))\cos\gamma\left(\frac{\pi}{2}-\varphi\right)+i(G_1(\varphi)+iG_2(\varphi))\sin\gamma\left(\frac{\pi}{2}-\varphi\right)\Big]\Big\}$$

如果把$V_{\rm I}=V_{\rm I}^*+V_{\rm I}^{(1)}+V_{\rm I}^{(2)}$写成实部和虚部，则内力素和位移可表为

$$\left.\begin{aligned}
N_\varphi&=\frac{-a\cos\varphi}{2\mu(1+a\sin\varphi)^2}I_mV_{\rm I}+\frac{P}{2\pi R}\frac{a+\sin\varphi}{(1+a\sin\varphi)^2}\\
N_\theta&=-\frac{1}{2\mu}\frac{d}{d\varphi}\left[\frac{I_mV_{\rm I}}{1+a\sin\varphi}\right]-\frac{P}{2\pi R}\frac{a+\sin\varphi}{(1+a\sin\varphi)^2}\\
M_\varphi&=\frac{a\alpha}{4\mu^2}\left\{\frac{d}{d\varphi}\left[\frac{R_eV_{\rm I}}{1+a\sin\varphi}\right]+\nu\frac{a\cos\varphi}{(1+a\sin\varphi)^2}R_eV_{\rm I}\right\}\\
M_\theta&=\frac{a\alpha}{4\mu^2}\left\{\nu\frac{d}{d\varphi}\left[\frac{ReV_{\rm I}}{1+a\sin\varphi}\right]+\frac{a\cos\varphi}{(1+a\sin\varphi)^2}R_eV_{\rm I}\right\}\\
Q&=\frac{a}{2\mu}\frac{\sin\varphi}{(1+a\sin\varphi)^2}I_mV_{\rm I}+\frac{P}{2\pi R}\frac{\cos\varphi}{(1+a\sin\varphi)^2}
\end{aligned}\right\}\qquad(2.6)$$

$$X=\frac{-1}{Eha(1+a\sin\varphi)}R_eV_{\rm I}$$

$$\left.\begin{aligned} Y&=\frac{R}{Eh}(1+\alpha\sin\varphi)(N_\theta-\nu N_\varphi)\\ Z&=\int_0^{\pi/2}\frac{R}{Eh}\,\frac{\cos\varphi}{(1+\alpha\sin\varphi)}R_eV_{\rm I}d\varphi \end{aligned}\right\} \tag{2.7}$$

以上各式适用于EGF 区域 $0\leqslant\varphi\leqslant\pi$. 这里业已利用了 $Z_0=Z_{\varphi_0=\varphi/2}=0$ 的条件，即取G点的轴向位移为零. 这是根据轴向位移的计算起点从G点开始决定的. 必须指出$Z>0$ 相当于压缩的轴向位移.

在这个问题里，N_φ、N_θ、M_φ和M_θ都是对称于G点，因当$\varphi=\pi/2$，非齐次解$V_{\rm I}^*=0$，齐次解一定是反对称的.

如果我们把$V_{\rm I}^{(1)}+V_{\rm I}^{(2)}$写成对称和反对称两部分：

$$\begin{aligned} V_{\rm I}^{(1)}+V_{\rm I}^{(2)}=&\Big[(C_0'+i\bar{C}_0')e^{i\gamma\frac{\pi}{2}}+(B_0'+i\bar{B}_0')e^{-(\beta+i\gamma)\frac{\pi}{2}}\Big]\\ &\cdot\Big\{\Big[(G_1(\varphi)+iG_2(\varphi))\cos\gamma\Big(\frac{\pi}{2}-\varphi\Big)\\ &+(F_2(\varphi)-iF_1(\varphi))\sin\gamma\Big(\frac{\pi}{2}-\varphi\Big)\Big]\operatorname{coh}\beta\Big(\frac{\pi}{2}-\varphi\Big)\\ &-\Big[(F_1(\varphi)+iF_2(\varphi))\cos\gamma\Big(\frac{\pi}{2}-\varphi\Big)\\ &+(G_1(\varphi)-iG_2(\varphi))\sin\gamma\Big(\frac{\pi}{2}-\varphi\Big)\Big]\operatorname{sh}\beta\Big(\frac{\pi}{2}-\varphi\Big)\Big\}\\ &+\Big[(C_0'+i\bar{C}_0')e^{i\gamma\frac{\pi}{2}}-(B_0'+i\bar{B}_0')e^{-(\beta+i\gamma)\frac{\pi}{2}}\Big]\Big\{\Big[(F_1(\varphi)\\ &+iF_2(\varphi))\cos\gamma\Big(\frac{\pi}{2}-\varphi\Big)\\ &+(G_1(\varphi)-iG_2(\varphi))\sin\gamma\Big(\frac{\pi}{2}-\varphi\Big)\Big]\operatorname{coh}\beta\Big(\frac{\pi}{2}-\varphi\Big)\\ &-\Big[(G_1(\varphi)+iG_2(\varphi))\cos\gamma\Big(\frac{\pi}{2}-\varphi\Big)\\ &+(F_2(\varphi)-iF_1(\varphi))\sin\gamma\Big(\frac{\pi}{2}-\varphi\Big)\Big]\operatorname{sh}\beta\Big(\frac{\pi}{2}-\varphi\Big)\Big\} \end{aligned} \tag{2.8}$$

由于$V_{\rm I}^{(1)}+V_{\rm I}^{(2)}$有反对称要求，第一项必须恒等于零，这就要求

$$(C_0'+i\bar{C}_0')e^{\gamma\frac{\pi}{2}}=-(B_0'+i\bar{B}_0')e^{-(\beta+i\gamma)\frac{\pi}{2}}$$

为了便于计算，引进新的待定系数$C_{\rm I}$、$C_{\rm I}'$，令

$$B_0'+i\bar{B}_0'=-\frac{1}{2}(C_{\rm I}+iC_{\rm I}')\frac{2\mu P}{\pi a}$$

于是$V_{\rm I}$可写成虚部$I_mV_{\rm I}$和实部$R_eV_{\rm I}$

$$I_mV_1=\frac{-2\mu P}{\pi a}[\kappa(\varphi)-C_1\Omega_2(\varphi)-C_1'\Omega_1(\varphi)]$$

$$R_eV_1=\frac{-2\mu P}{\pi a}[J(\varphi)-C_1\Omega_1(\varphi)+C_1'\Omega_2(\varphi)]$$

其中 $A_n=J_n+i\kappa_n$ 可由环壳计算表中得到

$$J(\varphi)=J_1\cos\varphi+J_2\sin 2\varphi-J_3\cos 3\varphi-J_4\sin 4\varphi+\cdots$$
$$+J_{4n+1}\cos(4n+1)\varphi+J_{4n+2}\sin(4n+2)\varphi-J_{4n+3}\cos(4n+3)\varphi$$
$$-J_{4n+4}\sin(4n+4)\varphi+\cdots$$

$$\kappa(\varphi)=\kappa_1\cos\varphi+\kappa_2\sin 2\varphi-\kappa_3\cos 3\varphi-\kappa_4\sin 4\varphi+\cdots$$
$$+\kappa_{4n+1}\cos(4n+1)\varphi+\kappa_{4n+2}\sin(4n+2)\varphi-\kappa_{4n+3}\cos(4n+3)\varphi$$
$$-\kappa_{4n+4}\sin(4n+4)\varphi+\cdots$$

$$\Omega_1(\varphi)=e^{-\beta\frac{\pi}{2}}\left[\omega_1(\varphi)\cos\gamma\frac{\pi}{2}+\omega_2(\varphi)\sin\gamma\frac{\pi}{2}\right]$$

$$\Omega_2(\varphi)=e^{-\beta\frac{\pi}{2}}\left[\omega_2(\varphi)\cos\gamma\frac{\pi}{2}-\omega_1(\varphi)\sin\gamma\frac{\pi}{2}\right]$$

$$\omega_1(\varphi)=\left\{\left[F_1(\varphi)\cos\gamma\left(\frac{\pi}{2}-\varphi\right)+G_2(\varphi)\sin\gamma\left(\frac{\pi}{2}-\varphi\right)\right]\mathrm{coh}\,\beta\left(\frac{\pi}{2}-\varphi\right)\right.$$
$$\left.-\left[G_1(\varphi)\cos\gamma\left(\frac{\pi}{2}-\varphi\right)+F_2(\varphi)\sin\gamma\left(\frac{\pi}{2}-\varphi\right)\right]\mathrm{sh}\,\beta\left(\frac{\pi}{2}-\varphi\right)\right\}$$

$$\omega_2(\varphi)=\left\{\left[F_2(\varphi)\cos\gamma\left(\frac{\pi}{2}-\varphi\right)-G_1(\varphi)\sin\gamma\left(\frac{\pi}{2}-\varphi\right)\right]\mathrm{coh}\,\beta\left(\frac{\pi}{2}-\varphi\right)\right.$$
$$\left.-\left[G_2(\varphi)\cos\gamma\left(\frac{\pi}{2}-\varphi\right)-F_1(\varphi)\sin\gamma\left(\frac{\pi}{2}-\varphi\right)\right]\mathrm{sh}\,\beta\left(\frac{\pi}{2}-\varphi\right)\right\}$$

把 I_mV_1 和 R_eV_1 代入(2.6)，(2.7)式，即得 EGF 区域中诸内力素及位移表达式：

$$N_\varphi\Big/\frac{P}{2\pi R}=\frac{1}{(1+\alpha\sin\varphi)^2}\{2\cos\varphi[\kappa(\varphi)-C_1\Omega(\varphi)-C_1'\Omega_1(\varphi)]$$
$$+\alpha+\sin\varphi\}$$

$$N_\theta\Big/\frac{P}{2\pi R}=\frac{1}{(1+\alpha\sin\varphi)^2}\left\{\frac{2(1+\alpha\sin\varphi)}{\alpha}\left[\frac{d\kappa(\varphi)}{d\varphi}-C_1\frac{d\Omega_2(\varphi)}{d\varphi}\right.\right.$$
$$\left.-C_1'\frac{d\Omega_1(\varphi)}{d\varphi}\right]-2\cos\varphi[\kappa(\varphi)-C_1\Omega_2(\varphi)-C_1'\Omega_1(\varphi)]$$
$$\left.-(\alpha+\sin\varphi)\right\}$$

$$M_\varphi\Big/\frac{\alpha P}{2\pi\mu}=\frac{-1}{(1+\alpha\sin\varphi)^2}\left\{(1+\alpha\sin\varphi)\left[\frac{dJ(\varphi)}{d\varphi}-C_1\frac{d\Omega_1(\varphi)}{d\varphi}\right.\right.$$
$$\left.+C_1'\frac{d\Omega_2(\varphi)}{d\varphi}\right]-(1-\nu)\alpha\cos\varphi[J(\varphi)-C_1\Omega_1(\varphi)$$
$$\left.+C_1'\Omega_2(\varphi)]\right\}$$

$$\left.\begin{aligned}
&M_\varphi\Big/\frac{aP}{2\pi\mu}=\frac{-1}{(1+\alpha\sin\varphi)^2}\Big\{\nu(1+\alpha\sin\varphi)\Big[\frac{dJ(\varphi)}{d\varphi}-C_1\frac{d\Omega_1(\varphi)}{d\varphi}\\
&\qquad+C_1'\frac{d\Omega_2(\varphi)}{d\varphi}\Big]+(1-\nu)\alpha\cos\varphi[J(\varphi)-C_1\Omega_1(\varphi)\\
&\qquad+C_1'\Omega_2(\varphi)]\Big\}\\
&Q\Big/\frac{P}{2\pi R}=\frac{1}{(1+\alpha\sin\varphi)^2}\Big\{-2\sin\varphi[\kappa(\varphi)-C_1\Omega_2(\varphi)\\
&\qquad-C_1\Omega_1(\varphi)]+\cos\varphi\Big\}\\
&X=\frac{1}{Eh\,\alpha}\,\frac{2\mu P}{\pi a(1+\alpha\sin\varphi)}[J(\varphi)-C_1\Omega_1(\varphi)+C_1'\Omega_2(\varphi)]\\
&Y=\frac{R}{Eh}(1+\alpha\sin\varphi)(N_\theta-\nu N_\varphi)\\
&Z=\int_0^{\frac{\pi}{2}}\frac{-R}{Eh}\,\frac{\cos\varphi}{1+\alpha\sin\varphi}\,\frac{2\mu P}{\pi a}[J(\varphi)-C_1\Omega_1(\varphi)\\
&\qquad+C_1'\Omega_2(\varphi)]d\varphi
\end{aligned}\right\}\quad(2.9)$$

在 BE 区域内，$\overline{Q}_0=-P/2\pi R$ 且 $\psi=\varphi-\pi$

其解可写成：

非齐次解 $V_{\rm II}^*$

$$V_{\rm II}^*=\frac{-2\mu P}{\pi a}\{A_1\cos\psi-A_2\sin2\psi-A_3\cos3\psi+A_4\sin4\psi+\cdots$$
$$+A_{4n+1}\cos(4n+1)\psi-A_{4n+2}\sin(4n+2)\psi-A_{4n+3}\cos(4n+3)\psi$$
$$+A_{4n+4}\sin(4n+4)\psi+\cdots\}$$

齐次解

$$V_{\rm II}^{(1)}=\frac{\mu P}{\pi a}(C_{\rm II}+iC_{\rm II}')e^{-\beta\left(\frac{\pi}{2}-\psi\right)}(\cos\gamma\psi+i\sin\gamma\psi)[\bar{f}_1(\psi)+i\bar{f}_2(\psi)]$$

$$V_{\rm II}^{(2)}=\frac{\mu P}{\pi a}(B_{\rm II}+iB_{\rm II}')e^{-\beta\psi}(\cos\gamma\psi-i\sin\gamma\psi)[\bar{g}_1(\psi)+i\bar{g}_2(\psi)]$$

其中

$$\bar{f}_1(\psi)=\bar{G}_1(\psi)+\overline{F}_1(\psi),\quad \bar{f}_2(\psi)=\bar{G}_2(\psi)+\overline{F}_2(\psi)$$

$$\bar{g}_1(\psi)=\bar{G}_1(\psi)-\overline{F}_1(\psi),\quad \bar{g}_2(\psi)=\bar{G}_2(\psi)-\overline{F}_2(\psi)$$

而

$$\overline{F}_1(\psi)=-\sum_{n=1,3,5\cdots}^{\infty}p_n\cos n\psi-\sum_{n=2,4,6\cdots}^{\infty}q_n'\sin n\psi$$

$$\overline{F}_2(\psi)=-\sum_{n=1,3,5\cdots}^{\infty}q_n\cos n\psi+\sum_{n=2,4,6\cdots}^{\infty}p_n\sin n\psi$$

$$\bar{G}_1(\psi)=1+\sum_{n=2,4,6\cdots}^{\infty} p_n\cos n\psi+\sum_{n=1,3,5\cdots}^{\infty} q_n'\sin n\psi$$

$$\bar{G}_2(\psi)=\sum_{n=2,4,6\cdots}^{\infty} q_n\cos n\psi-\sum_{n=1,3,5\cdots}^{\infty} p_n\sin n\psi$$

利用

$$\cos\gamma\psi+i\sin\gamma\psi=e^{i\gamma\frac{\pi}{2}}\left[\cos\gamma\left(\frac{\pi}{2}-\psi\right)-i\sin\gamma\left(\frac{\pi}{2}-\psi\right)\right]$$

$$\cos\gamma\psi-i\sin\gamma\psi=e^{-i\gamma\frac{\pi}{2}}\left[\cos\gamma\left(\frac{\pi}{2}-\psi\right)+i\sin\gamma\left(\frac{\pi}{2}-\psi\right)\right]$$

把$V_{\text{II}}^{(1)}+V_{\text{II}}^{(2)}$写成对称和反对称两部分

$$V_{\text{II}}^{(1)}+V_{\text{II}}^{(2)}=\frac{\mu P}{\pi a}\left[(C_{\text{II}}+iC_{\text{II}}')e^{i\gamma\frac{\pi}{2}}+(B_{\text{II}}+iB_{\text{II}}')e^{-(\beta+i\gamma)\frac{\pi}{2}}\right]\left\{\left[\left(\bar{G}_1(\psi)+i\bar{G}_2(\psi)\right]\cos\gamma\left(\frac{\pi}{2}-\psi\right)+\left(\bar{F}_2(\psi)-i\bar{F}_1(\psi)\right)\sin\gamma\left(\frac{\pi}{2}-\psi\right)\right]\operatorname{coh}\beta\left(\frac{\pi}{2}-\psi\right)-\left[\left(\bar{F}_1(\psi)+i\bar{F}_2(\psi)\right)\cos\gamma\left(\frac{\pi}{2}-\psi\right)+\left(\bar{G}_2(\psi)-i\bar{G}_1(\psi)\right)\sin\gamma\left(\frac{\pi}{2}-\psi\right)\right]\operatorname{sh}\beta\left(\frac{\pi}{2}-\psi\right)\right\}$$

$$+\frac{\mu P}{\pi a}\left[(C_{\text{II}}+iC_{\text{II}}')e^{-i\gamma\frac{\pi}{2}}-(B_{\text{II}}+iB_{\text{II}}')^{-(\beta+i\gamma)\frac{\pi}{2}}\right]\left\{\left[\left(\bar{F}_1(\psi)+i\bar{F}_2(\psi)\right)\cos\gamma\left(\frac{\pi}{2}-\psi\right)+\left(\bar{G}_2(\psi)-i\bar{G}_1(\psi)\right)\sin\gamma\left(\frac{\pi}{2}-\psi\right)\right]\operatorname{coh}\beta\left(\frac{\pi}{2}-\psi\right)-\left[\left(\bar{G}_1(\psi)+i\bar{G}_2(\psi)\right)\cos\gamma\left(\frac{\pi}{2}-\psi\right)+\left(\bar{F}_2(\psi)-i\bar{F}_1(\psi)\right)\sin\gamma\left(\frac{\pi}{2}-\psi\right)\right]\operatorname{sh}\beta\left(\frac{\pi}{2}-\psi\right)\right\}$$

N_φ，N_θ，M_φ和M_θ都对称于B点$\left(\psi=\frac{\pi}{2}\right)$所以$V_{\text{II}}^{(1)}+V_{\text{II}}^{(2)}$所决定的齐次解对$B$点而言，一定是反对称的，于是$V_{\text{II}}^{(1)}+V_{\text{II}}^{(2)}$式的对称项必须为零，即

$$(C_{\text{II}}+iC_{\text{II}}')e^{i\gamma\frac{\pi}{2}}=-(B_{\text{II}}+iB_{\text{II}}')\,e^{-(\beta+i\gamma)\frac{\pi}{2}}$$

于是可化简

$$V_{\text{II}}^{(2)}+V_{\text{II}}^{(1)}=\frac{-2\mu P}{\pi a}(B_{\text{II}}+iB_{\text{II}}')\,e^{-(\beta+i\gamma)\frac{\pi}{2}}\left\{\left[\left(\bar{F}_1(\psi)+i\bar{F}_2(\psi)\right)\cos\gamma\left(\frac{\pi}{2}-\psi\right)+\left(\bar{G}_2(\psi)\right.\right.\right.$$

$$-i\bar{G}_1(\psi))\sin\gamma\left(\frac{\pi}{2}-\psi\right)\Big]\mathrm{coh}\,\beta\left(\frac{\pi}{2}-\psi\right)$$

$$-\Big[(\bar{G}_1(\psi)+i\bar{G}_2(\psi))\cos\gamma\left(\frac{\pi}{2}-\psi\right)$$

$$+(\bar{F}_2(\psi)-i\bar{F}_1(\psi))\sin\gamma\left(\frac{\pi}{2}-\psi\right)\Big]\mathrm{sh}\,\beta\left(\frac{\pi}{2}-\psi\right)\Big\};$$

这样V_{II}的实部R_eV_{II}和虚部I_mV_{II}分别为

$$R_eV_{\text{II}}=\frac{-2\mu P}{\pi a}\{\bar{J}(\psi)+B_{\text{II}}\bar{\Omega}_1(\psi)-B_{\text{II}}'\bar{\Omega}_2(\psi)\}$$

$$I_mV_{\text{II}}=\frac{-2\mu P}{\pi a}\{\kappa(\psi)+B_{\text{II}}\bar{\Omega}_2(\psi)+B_{\text{II}}'\bar{\Omega}_1(\psi)\}$$

其中

$$\begin{aligned}\bar{J}(\psi)=&J_1\cos\psi-J_2\sin2\psi-J_3\cos3\psi+J_4\sin4\psi+\cdots\\&+J_{4n+1}\cos(4n+1)\psi-J_{4n+2}\sin(4n+2)\psi-J_{4n+3}\cos(4n+3)\psi\\&+J_{4n+4}\sin(4n+4)\psi+\cdots\end{aligned}$$

$$\begin{aligned}\bar{\kappa}(\psi)=&\kappa_1\cos\psi-\kappa_2\sin2\psi-\kappa_3\cos3\psi+\kappa_4\sin4\psi+\cdots\\&+\kappa_{4n+1}\cos(4n+1)\psi-\kappa_{4n+2}\sin(4n+2)\psi-\kappa_{4n+3}\cos(4n+3)\psi\\&+\kappa_{4n+4}\sin(4n+4)\psi+\cdots\end{aligned}$$

$$\bar{\Omega}_1(\psi)=e^{-\beta\frac{\pi}{2}}\Big[\bar{\omega}_1(\psi)\cos\gamma\frac{\pi}{2}+\bar{\omega}_2(\psi))\sin\gamma\frac{\pi}{2}\Big]$$

$$\bar{\Omega}_2(\psi)=e^{-\beta\frac{\pi}{2}}\Big[\bar{\omega}_2(\psi)\cos\gamma\frac{\pi}{2}-\bar{\omega}_1(\psi)\sin\gamma\frac{\pi}{2}\Big]$$

$$\begin{aligned}\bar{\omega}_1(\psi)=&\Big\{\Big[\bar{F}_1(\psi)\cos\gamma\left(\frac{\pi}{2}-\psi\right)+\bar{G}_2(\psi)\sin\gamma\left(\frac{\pi}{2}-\psi\right)\Big]\mathrm{coh}\,\beta\left(\frac{\pi}{2}-\psi\right)\\&-\Big[\bar{G}_1(\psi)\cos\gamma\left(\frac{\pi}{2}-\psi\right)+\bar{F}_2(\psi)\sin\gamma\left(\frac{\pi}{2}-\psi\right)\Big]\mathrm{sh}\,\beta\left(\frac{\pi}{2}-\psi\right)\Big\}\end{aligned}$$

$$\begin{aligned}\bar{\omega}_2(\psi)=&\Big\{\Big[\bar{F}_2(\psi)\cos\gamma\left(\frac{\pi}{2}-\psi\right)-\bar{G}_1(\psi)\sin\gamma\left(\frac{\pi}{2}-\psi\right)\Big]\mathrm{coh}\,\beta\left(\frac{\pi}{2}-\psi\right)\\&-\Big[\bar{G}_2(\psi)\cos\gamma\left(\frac{\pi}{2}-\psi\right)-\bar{F}_1(\psi)\sin\gamma\left(\frac{\pi}{2}-\psi\right)\Big]\mathrm{sh}\,\beta\left(\frac{\pi}{2}-\psi\right)\Big\}\end{aligned}$$

这样在BE段内的内力素和位移可表为：

$$\begin{aligned}N_\varphi\Big/\frac{P}{2\pi R}=&\frac{-1}{(1-a\sin\psi)^2}\{2\cos\psi[\bar{\kappa}(\psi)+B_{\text{II}}\bar{\Omega}_2(\psi)+B_{\text{II}}'\bar{\Omega}_1(\psi)]\\&+a-\sin\psi\}\end{aligned}$$

$$\begin{aligned}N_r\Big/\frac{P}{2\pi R}=&\frac{1}{(1-a\sin\psi)^2}\Big\{\frac{2(1-a\sin\psi)}{a}\Big[\frac{d\bar{\kappa}(\psi)}{d\psi}+B_{\text{II}}\frac{d\bar{\Omega}_2(\psi)}{d\psi}\\&+B_{\text{II}}'\frac{d\bar{\Omega}_1(\psi)}{d\psi}\Big]+2\cos\psi[\bar{\kappa}(\psi)+B_{\text{II}}\bar{\Omega}_2(\psi)+B_{\text{II}}'\bar{\Omega}_1(\psi)]\end{aligned}$$

$$
\left.\begin{aligned}
&\qquad +\alpha-\sin\psi\Big\}\\
&\overline{M}_\varphi\Big/\frac{aP}{2\pi\mu}=\frac{-1}{(1-\alpha\sin\psi)^2}\Big\{(1-\alpha\sin\psi)\Big[\frac{d\overline{J}(\psi)}{d\psi}+B_{\text{II}}\frac{d\overline{\Omega}_1(\psi)}{d\psi}\\
&\qquad -B'_{\text{II}}\frac{d\overline{\Omega}_2(\psi)}{d\psi}\Big]+(1-\nu)\alpha\cos\psi[\overline{J}(\psi)+B_{\text{II}}\overline{\Omega}_1(\psi)\\
&\qquad -B'_{\text{II}}\overline{\Omega}_2(\psi)]\Big\}\\
&\overline{M}_\theta\Big/\frac{aP}{2\pi\mu}=\frac{-1}{(1-\alpha\sin\psi)^2}\Big\{\nu(1-\alpha\sin\psi)\Big[\frac{d\overline{J}(\psi)}{d\psi}+B_{\text{II}}\frac{d\overline{\Omega}_1(\psi)}{d\psi}\\
&\qquad -B'_{\text{II}}\frac{d\overline{\Omega}_2(\psi)}{d\psi}\Big]-(1-\nu)\alpha\cos\psi[\overline{J}(\psi)+B_{\text{II}}\overline{\Omega}_1(\psi)\\
&\qquad -B'_{\text{II}}\overline{\Omega}_2(\psi)]\Big\}\\
&\overline{Q}\Big/\frac{P}{2\pi R}=\frac{1}{(1-\alpha\sin\psi)^2}\{2\sin\psi[\overline{\kappa}(\psi)+B_{\text{II}}\overline{\Omega}_2(\psi)+B_{\text{II}}'\overline{\Omega}_1(\psi)]\\
&\qquad +\cos\psi\}\\
&\overline{X}=\frac{1}{Eh\,a(1-\alpha\sin\psi)}\frac{2\mu P}{\pi a}\{\overline{J}(\psi)+B_{\text{II}}\overline{\Omega}_1(\psi)-B_{\text{II}}'\overline{\Omega}_2(\psi)\}\\
&\overline{Y}=\frac{R}{Eh}(1-\alpha\sin\psi)(\overline{N}_\theta-\nu\overline{N}_\varphi)\\
&\overline{Z}=Z_E+\int_0^\psi\frac{R}{Eh}\frac{\cos\psi}{(1-\alpha\sin\psi)}\left(\frac{-2\mu P}{\pi a}\right)\{\overline{J}(\psi)+B_{\text{II}}\overline{\Omega}_1(\psi)\\
&\qquad -B'_{\text{II}}\Omega_2(\psi)\}d\psi
\end{aligned}\right\}\quad(2.10)
$$

其中Z_E为E点的轴向位移.

(2.9)，(2.10)中共有4个待定常数C_{I}、C_{I}'、B_{II}和B_{II}'，它们由E点的连继条件决定.

现在让我们求波纹管单元的轴向位移，在轴向拉力作用下的总伸长等于B点相对于G点的轴向位移的二倍，即$2\overline{Z}_B$，从(2.9)，(2.10)

$$
\begin{aligned}
\delta_1=-2\overline{Z}_B&=-2\Big\{Z_E+\int_0^{\pi/2}\frac{R\cos\psi}{Eh(1-\alpha\sin\psi)}R_eV_{\text{II}}d\psi\Big\}\\
&=\frac{4\mu RP}{Eh\pi a}\Big\{\int_0^{\pi/2}\frac{\cos\varphi}{1+\alpha\sin\psi}[J(\varphi)-C_{\text{I}}\Omega_1(\varphi)+C_{\text{I}}'\Omega_2(\varphi)]d\varphi\\
&\quad+\int_0^{\pi/2}\frac{\cos\psi}{1-\alpha\sin\psi}[\overline{J}(\psi)+B_{\text{II}}\overline{\Omega}_1(\psi)-B_{\text{II}}'\overline{\Omega}_2(\psi)]d\psi\Big\}
\end{aligned}\quad(2.11)
$$

当解得了待定系数C_{I}、C_{I}'、B_{II}和B_{II}'后，上式积分可以用辛普生公式通过数值积分求得.

三、波纹管单元在轴力作用下的应力和变形

决定待定常数C_{I}、C_{I}'、B_{II}和B_{II}'的4个E点连续条件为：

$$\left.\begin{aligned}X_E&=\bar{X}_E\\Y_E&=\bar{Y}_E\\N_{\varphi E}&=\bar{N}_{\varphi E}\\M_{\varphi E}&=-\bar{M}_{\varphi E}\end{aligned}\right\}\tag{3.1A, B, C, D}$$

由(3.1A)得

$$\Omega_1(0)C_{\rm I}-\Omega_2(0)C_{\rm I}'+\bar{\Omega}_1(0)B_{\rm II}-\bar{\Omega}_2(0)B_{\rm II}'=0\tag{3.2A}$$

由(3.1B)得

$$\begin{aligned}&\left[\left(\frac{d\Omega_2(\varphi)}{d\varphi}\right)_{\varphi=0}-\alpha(1+\nu)\Omega_2(0)\right]C_{\rm I}+\left[\left(\frac{d\Omega_1(\varphi)}{d\varphi}\right)_{\varphi=0}-\alpha(1+\nu)\Omega_1(0)\right]C_{\rm I}'\\&\qquad+\left[\left(\frac{d\bar{\Omega}_2(\psi)}{d\psi}\right)_{\psi=0}+\alpha(1+\nu)\bar{\Omega}_2(0)\right]B_{\rm II}\\&\qquad+\left[\left(\frac{d\bar{\Omega}_1(\psi)}{d\psi}\right)_{\psi=0}+\alpha(1+\nu)\bar{\Omega}_1(0)\right]B_{\rm II}'\\&\qquad-2\left[\left(\frac{d\kappa(\varphi)}{d\varphi}\right)_{\varphi=0}-\alpha(1+\nu)\kappa(0)\right]+\alpha^2(1+\nu)=0\end{aligned}\tag{3.2B}$$

由(3.1C)得:

$$\Omega_2(0)C_{\rm I}+\Omega_1(0)C_{\rm I}'-\bar{\Omega}_2(0)B_{\rm II}-\bar{\Omega}_1(0)B_{\rm II}'-\nu\kappa(0)-\alpha=0\tag{3.2C}$$

由(3.1*D*)得:

$$\begin{aligned}&\left[\left(\frac{d\Omega_1(\varphi)}{d\varphi}\right)_{\varphi=0}-\alpha(1-\nu)\Omega_1(0)\right]C_{\rm I}-\left[\left(\frac{d\Omega_2(\varphi)}{d\varphi}\right)_{\varphi=0}-\alpha(1-\nu)\Omega_2(0)\right]C_{\rm I}'\\&-\left[\left(\frac{d\bar{\Omega}_1(\psi)}{d\psi}\right)_{\psi=0}+\alpha(1-\nu)\bar{\Omega}_1(0)\right]B_{\rm II}+\left[\left(\frac{d\bar{\Omega}_2(\psi)}{d\psi}\right)_{\psi=0}+\alpha(1-\nu)\bar{\Omega}_2(0)\right]B_{\rm II}'=0\end{aligned}\tag{3.2D}$$

由(3.2A, B, C, D)解得

$$C_{\rm I}=\frac{\Delta_1}{\Delta},\quad C_{\rm I}'=\frac{\Delta_2}{\Delta},\quad B_{\rm II}=\frac{\Delta_3}{\Delta},\quad B_{\rm II}'=\frac{\Delta_4}{\Delta}\tag{3.3}$$

其中

$$\begin{aligned}\Delta=&(\Omega_1^2(0)+\Omega_2^2(0))(T_3T_8-T_4T_7)+(\bar{\Omega}_1^2(0)+\bar{\Omega}_2^2(0))(T_2T_5-T_1T_6)\\&+(\Omega_1(0)\bar{\Omega}_1(0)-\Omega_2(0)\bar{\Omega}_2(0))(T_3T_6+T_4T_5-T_1T_8-T_2T_7)\\&+(\Omega_1(0)\bar{\Omega}_2(0)+\Omega_2(0)\bar{\Omega}_1(0))(T_2T_8+T_3T_5-T_1T_7-T_4T_6)\\\Delta_1=&S_1[\Omega_2(0)(T_3T_8-T_4T_7)+\bar{\Omega}_1(0)(T_2T_8-T_4T_6)\\&+\bar{\Omega}_2(0)(T_2T_7-T_3T_6)]-(\bar{\Omega}_1^2(0)+\bar{\Omega}_2^2(0))(S_2T_6)-(\Omega_1(0)\bar{\Omega}_1(0)\\&-\Omega_2(0)\bar{\Omega}_2(0))(S_2T_8)-(\Omega_1(0)\bar{\Omega}_2(0)+\Omega_2(0)\bar{\Omega}_1(0))(S_2T_7)\\\Delta_2=&S_1[\Omega_1(0)(T_3T_8-T_4T_7)+\bar{\Omega}_1(0)(T_4T_5-T_1T_8)+\bar{\Omega}_2(0)(T_3T_5-T_1T_7)]\\&+(\bar{\Omega}_1^2(0)+\bar{\Omega}_2^2(0))(S_2T_5)-(\Omega_1(0)\bar{\Omega}_1(0)-\Omega_2(0)\bar{\Omega}_2(0))(S_2T_7)\\&+(\Omega_1(0)\bar{\Omega}_2(0)+\Omega_2(0)\bar{\Omega}_1(0))(S_2T_8)\\\Delta_3=&S_1[\Omega_1(0)(T_4T_6-T_2T_8)+\Omega_2(0)(T_4T_5-T_1T_8)+\bar{\Omega}_2(0)(T_1T_6-T_2T_5)]\end{aligned}$$

$$+(\Omega_1^2(0)+\Omega_2^2(0))(S_2T_8)+(\Omega_1(0)\bar{\Omega}_1(0)-\Omega_2(0)\bar{\Omega}_2(0))(S_2T_6)$$
$$+(\Omega_1(0)\bar{\Omega}_2(0)+\Omega_2(0)\bar{\Omega}_1(0))(S_2T_5)$$
$$\Delta_4=S_1[\Omega_1(0)(T_2T_7-T_8T_6)+\Omega_2(0)(T_1T_7-T_3T_5)+\bar{\Omega}_1(0)(T_1T_6-T_2T_5)]$$
$$-(\Omega_1^2(0)+\bar{\Omega}_1^2(0))(S_2T_7)+(\Omega_1(0)\bar{\Omega}_1(0)-\Omega_2(0)\bar{\Omega}_2(0))(S_2T_5)$$
$$-(\Omega_1(0)\bar{\Omega}_2(0)+\Omega_2(0)\bar{\Omega}_1(0))(S_2T_6)$$

而

$$T_1=\left(\frac{d\Omega_2(\varphi)}{d\varphi}\right)_{\varphi=0}-\alpha(1+\nu)\Omega_2(0)$$

$$T_2=\left(\frac{d\Omega_1(\varphi)}{d\varphi}\right)_{\varphi=0}-\alpha(1+\nu)\Omega_1(0)$$

$$T_3=\left(\frac{d\bar{\Omega}_2(\psi)}{d\psi}\right)_{\psi=0}+\alpha(1+\nu)\bar{\Omega}_2(0)$$

$$T_4=\left(\frac{d\bar{\Omega}_1(\psi)}{d\psi}\right)_{\psi=0}+\alpha(1+\nu)\bar{\Omega}_1(0)$$

$$T_5=\left(\frac{d\Omega_1(\varphi)}{d\varphi}\right)_{\varphi=0}-\alpha(1-\nu)\Omega_1(0)$$

$$T_6=\left(\frac{d\Omega_2(\varphi)}{d\varphi}\right)_{\varphi=0}+\alpha(1-\nu)\Omega_2(0)$$

$$T_7=-\left(\frac{d\bar{\Omega}_1(\psi)}{d\psi}\right)_{\varphi=0}-\alpha(1-\nu)\bar{\Omega}_1(0)$$

$$T_8=\left(\frac{d\bar{\Omega}_2(\psi)}{d\psi}\right)_{\psi=0}+\alpha(1-\nu)\bar{\Omega}_2(0)$$

$$S_1=2\kappa(0)+\alpha$$

$$S_2=2\left[\left(\frac{d\kappa(\varphi)}{d\varphi}\right)_{\varphi=0}-\alpha(1+\nu)\kappa(0)\right]-\alpha^2(1+\nu)$$

为了与Turner-Ford实验结果和有限元法计算结果[4]相比较，按 Turner-Ford 实验模型尺寸计算了轴向应力、环向应力及变形.

Turner-Ford实验模型尺寸如下表：

模　型	a(cm)	R(cm)	h(cm)
B	4.9530	17.6022	0.1397
C	5.0038	17.6022	0.4318
D	2.4892	17.6022	0.1372

计算得参数

$$\mu=\sqrt{3(1-\nu^2)}\,\frac{a^2}{Rh},\quad \alpha=\frac{a}{R}$$

值如下表：

模 型	B	C	D
μ	16.48	5.44	4.24
α	0.281	0.284	0.141

不同理论与实验所得到的单位变形值

$\left(\delta \Big/ \frac{4\mu RP}{\pi ahE}\right)$如下表：

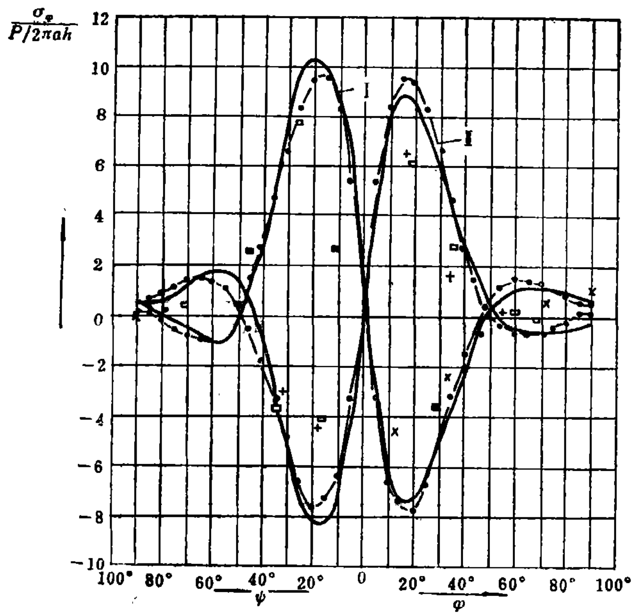
模 型	粗环壳理论	细环壳理论	有限元法	实 验
B	0.604	0.592	0.606	0.412
C	0.685	0.662	0.699	0.671
D	0.702	0.695	0.705	0.615

三种模型的轴向应力(σ_φ)及环向应力分布由图3-A、B，4-A、B和5-A、B给出，由于所得结果与有限元法很一致，故未标出，可参看文[4].

从单位变形结果看来，各种理论与实验所得结果比较一致.（B模型疑结果有误）.

从轴向应力及环向应力结果看来，无论是细壳或实验值，最大峰值一般略低，应力分布一般较为符合.

以上结果证明了文[3]所指出的当$\alpha=a/R=0.3$时，细环壳理论仍基本可用.



B 模 型
I 外表面
II 内表面
■× Turner-Ford外表面实验点
□+ Turner-Ford内表面实验点
-·- 细壳理论
$\mu=16.48$ $\alpha=a/R=0.281$

图3A 半圆弧波纹管单位在轴向力作用下的轴向应力(σ_φ)分布，Turner-Ford 实验点和理论曲线的比较

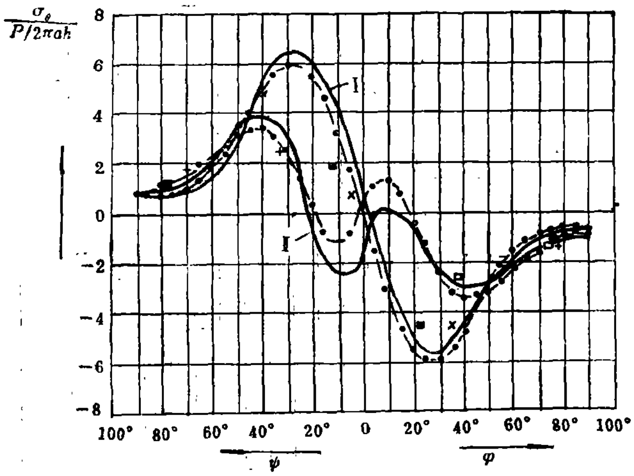
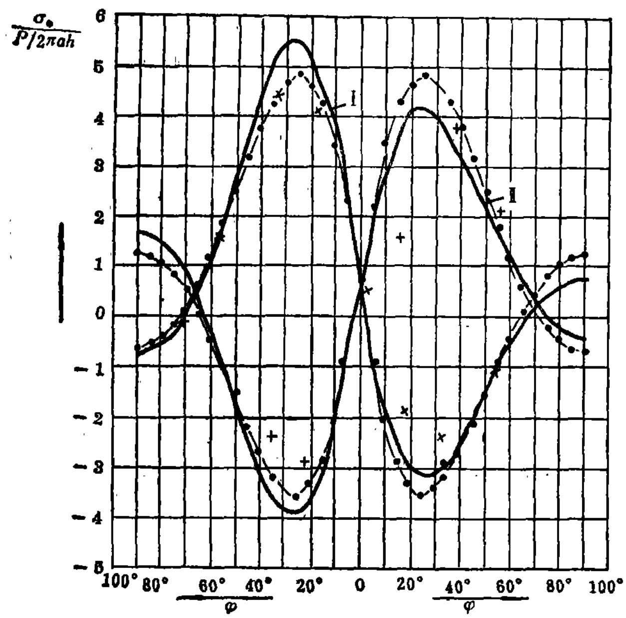


图 3B 半圆弧波纹管单元在轴向力作用下的环向应力(σ_θ)分布，Turner-Ford 实验点和理论曲线的比较



C 模型
I 外表面
II 内表面
× Turner-Ford外表面实验点
+ Turner-Ford内表面实验点
-·- 细壳理论
$\mu=5.44$ $\alpha=a/R=0.284$

图 4A 半圆弧波纹管单元在轴向力作用下的轴向应力(σ_φ)分布，Turner-Ford实验点和理论曲线的比较

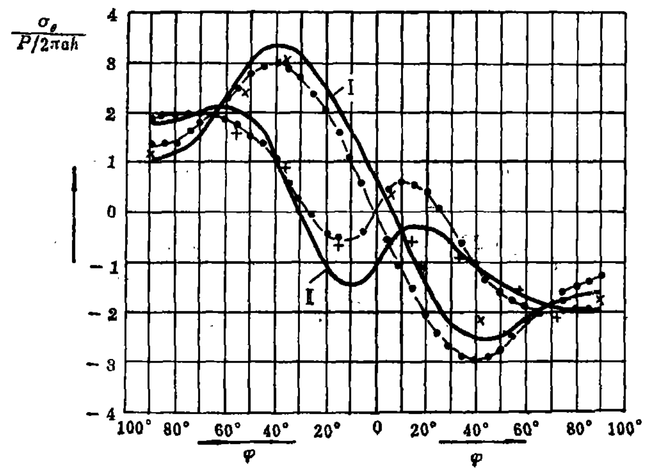
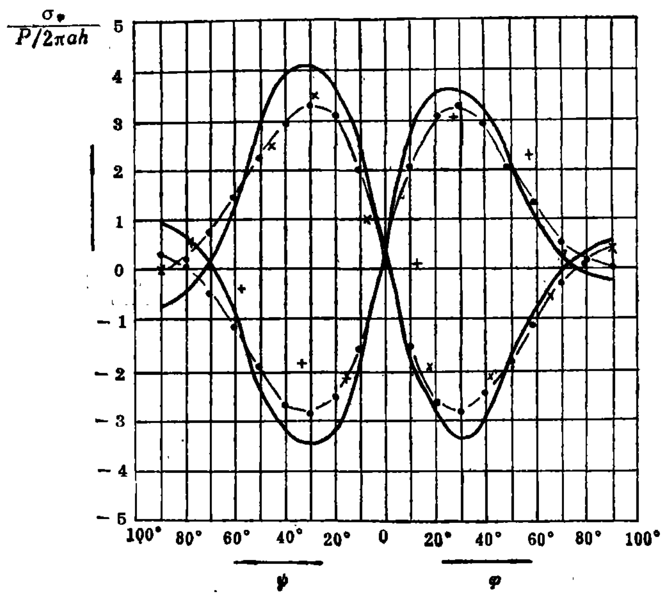


图 4B 半圆弧波纹管单元在轴向力作用下的环向应力(σ_θ)分布，Turner-Ford 实验点和理论曲线的比较



D 模型
Ⅰ 外表面
Ⅱ 内表面
× Turner-Ford外表面实验点
+ Turner-Ford内表面实验点
-·- 细壳理论
$\mu=4.24$ $\alpha=a/R=0.141$

图 5A 半圆弧波纹管单元在轴向力作用下的轴向应力(σ_φ)分布，Turner-Ford实验点和理论曲线的比较

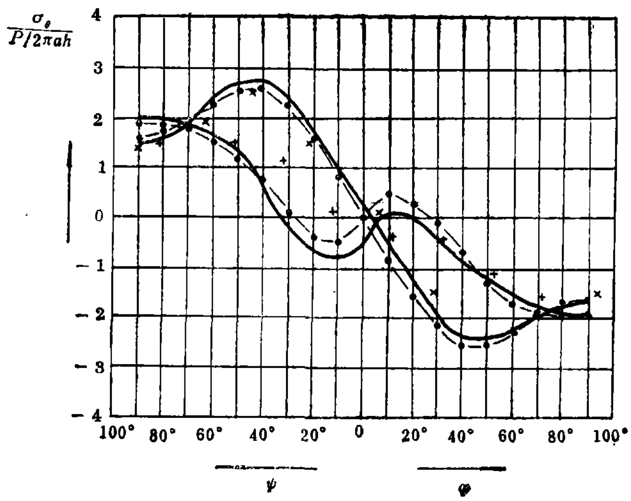


图 5B 半圆弧波纹管单元在轴向力作用下的环向应力(σ_θ)分布，Turner-Ford实验点和理论曲线的比较

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Calculations for Semi-Circular Arc Type Corrugated Tube—— Applications of General Solutions of Ring Shell Eguation

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Abstract

In this paper, the deformation and stress distribution of semi-circular arc type corrugated tube under the actions of axial compression are calculated by means of the general solutions of ring shell theory given in a previous paper[1]. The results of calculation fit fairly well with experimental data given by C. E. Turner-H. Ford(1957).