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具有弹性边拱及拉杆支承的双曲扁壳(一)

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摘　要

本文利用变分原理建立了具有弹性边拱及拉杆支承的双曲扁壳的平衡方程式及相应的边界条件和角点条件．这里假定边拱只在其本身平面内有刚度，边拱的扭转刚度和垂直于其平面的弯曲刚度都略去不计．本文研究了不许自由外伸的角点铰支条件，以及能够自由外伸的角点简支条件，前者相当于周边有拉杆限制角点外伸位移的情况，后者相当于周边无拉杆的情况．对于前者而言，本文近似地假定边拱沿弧方向的抗拉伸刚度为无穷大，亦即假定扁壳的边界切向位移为零，边拱只通过其垂直于扁壳平面的弯曲来产生弹性支承的作用．这些支承条件是近似地符合当前双曲扁壳屋盖的设计条件的．

本文利用双三角级数解法求得具有弹性边拱及拉杆支承的方形底球面扁壳在自重载荷下的正确解．其特点在于先将边界条件积分处理使先满足角点条件，然后求解平面应力微分方程使满足积分后的边界条件．本文的结果直接给出拉杆中的拉力，对于具体设计问题是有用的．

本文提出的积分形式的边界条件方法，对于弹性支承的边界问题在板壳方面的应用中是有它的普遍实用意义的．

本文还给出了具有弹性边拱支承的方形底扁球壳的数值结果．角点为铰支或简支的．选取的参数值为 $\lambda=11.5936$．计算结果表明级数收敛很快．并得出了边拱的弹性变形对壳体内力、内力矩及挠度分布规律的影响．

一、引　　言

有关矩形扁壳在四边简支条件下的解业已有很多工作，其中主要者有B．З．Власов的双三角级数解法[1]，和B．B．Дикович的单三角级数解法[2]．Власов解法虽然简单，但其缺点在于收敛缓慢．何广乾、张维岳[4]及胡海昌[5]等提出一些简化计算的方法，这些计算结果和根据Дикович的计算结果[3]比较，都证明是有效的．

但是，在实际屋盖设计中，很难满足简支边界条件，在我国一系列著名的双曲扁壳的屋盖结构中，如北京车站、北京网球和山东体育馆等，都附有边拱和限制角点外移的周边拉杆[6]，这种边界条件实质上是弹性支承的边界条件．

有关弹性支承的扁壳计算是比较困难的，尤其是四角铰支的扁壳困难更大．E．И．Силкин[7]曾在假设边拱的挠曲变形函数的条件下，利用拉格朗日变分原理决定其待定系数，来近似地处理由于边拱弯曲所产生的壳体附加位移．他略去了边拱的承压轴向变形，亦

即假设边拱的承压变形刚度为无穷大，他只计算了边拱承弯能力所产生的影响．同时，由于假设的边拱挠曲变形函数有关的附加位移并不满足扁壳的微分方程，因此由这种附加位移产生的弯矩修正项将是极其不正确的．Силкин的解和通常的变分法的解相似，可能得到可靠的中心位移，但不能得到可靠的弯矩分布．

本文提出了双曲扁壳附有边拱及四角铰支或简支的系统理论，并提出了以位移为基础的双三角级数解法；其特点在于先将边界条件积分处理，使先满足全部角点条件，然后求解微分方程使满足全部积分后的边界条件．本文的解法，在一方面可以推广处理矩形底双曲扁壳的问题，另一方面推广处理拉杆拉力小于四角铰支所需反作用力时，四角弹性支承的情况．

这样的求解方法对于带有边拱（或边框）弹性支承的薄壳、薄板问题特别有效，作者认为它有着普遍的重要意义的．

本文最后给出了方形底带有边拱及拉杆四角铰支的，以及带有边拱四角简支的弹性支承扁球壳的内力和变形的数值解，选取参数$\lambda=11.5936$．计算结果表明级数收斂很快，给出了沿壳体对称轴线上以及边缘上的内力、内力矩和挠度的变化图．同时还与 B. B. Дикович[2] 四边简支（$\lambda\doteq 11$）的结果作了比较，从而得出了边拱的弹性变形对壳体内力、内力矩及挠度分布规律的影响．

二、具有边拱的四角铰支或简支的矩形底双曲扁壳的平衡方程、边界条件及角点条件

设双曲扁壳厚度h较扁壳广度相比很小，设壳体矢高比扁壳广度相比也很小，如果选座标（x，y，z）如图1所示，位移（u，v，w）顺轴向为正．设x轴向的曲率为k_1，y轴向的曲率为k_2，于是中曲面的应变分量和曲率变化均分别可以用位移表示如下：

$$\left.\begin{aligned}\varepsilon_x&=\frac{\partial u}{\partial x}-k_1w,\\ \varepsilon_y&=\frac{\partial v}{\partial y}-k_2w,\\ \gamma_{xy}&=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\end{aligned}\right\}\qquad(1)$$

$$\chi_x=\frac{\partial^2 w}{\partial x^2},\quad \chi_y=\frac{\partial^2 w}{\partial y^2},\quad \omega=2\frac{\partial^2 w}{\partial x\partial y}\qquad(2)$$

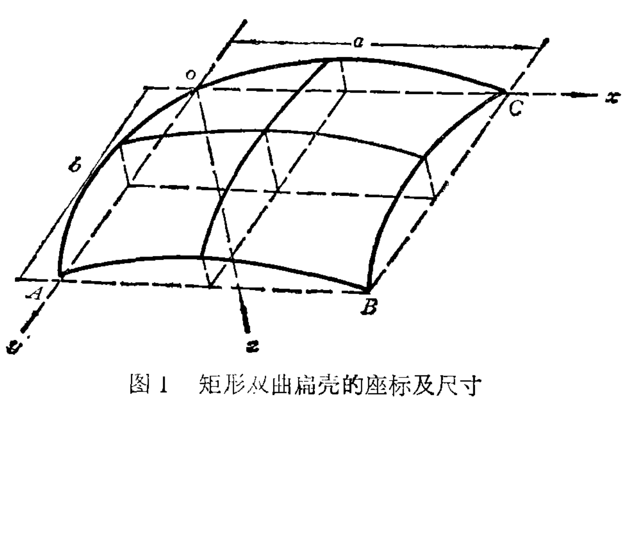

图1 矩形双曲扁壳的座标及尺寸

于是中曲面的变形能为

$$V_1=\frac{1}{2}\iint dxdy\left\{\frac{Eh}{1-\mu^2}\left[\varepsilon_x^2+\varepsilon_y^2+2\mu\varepsilon_x\varepsilon_y+\frac{1}{2}(1-\mu)\gamma_{xy}^2\right]\right.$$
$$\left.+D\left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2+\left(\frac{\partial^2 w}{\partial y^2}\right)^2+2\mu\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}+2(1-\mu)\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2\right]\right\} \quad (3)$$

其中 E 为杨氏模量，μ 为泊桑系数，D 为抗弯曲刚度 $\frac{1}{12}\frac{Eh^3}{1-\mu^2}$，积分遍及整个扁壳平面．设该扁壳受有分布横向载荷$q$作用，本文主要只讨论均布载荷，即$q$为常数，于是$q$的位能为

$$V^2=-\iint qw\,dx\,dy \quad (4)$$

设该双曲扁壳四边镶有边拱，当扁壳变形时，边拱亦随着变形．设壳体中面截线高出于边拱轴线（图2的 AB 为扁壳上 $y=b$ 的边界）之上，其值为 t_1．设边拱为一矩形截面，其高为 H_1，厚为C_1，于是边拱上各点的轴向拉应力为

$$Ee_x^*=E\left[\frac{\partial u}{\partial x}-k_1w-z\frac{\partial^2 w}{\partial x^2}\right] \quad (5)$$

其中第一第二项为壳体中面的伸缩变形所产生的应力，第三项为弯曲应力．这里业已假定 H_1k_1 很小，正应力可按直线分布处理，并略去了横向剪力的应变能，于是边拱 AB 的应变能为

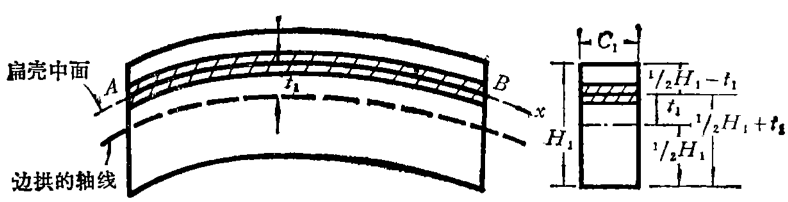

图2　边拱$AB(y=b)$的尺寸

$$\frac{1}{2}\int_0^a dx\cdot C_1\int_{-(1/2H_1-t_1)}^{1/2H_1+t_1}E\left[\frac{\partial u}{\partial x}-k_1w-z\frac{\partial^2 w}{\partial x^2}\right]^2 dz$$
$$=\frac{E}{2}\int_0^a dx\cdot C_1\left\{H_1\left(\frac{\partial u}{\partial x}-k_1w\right)^2-2H_1t_1\frac{\partial^2 w}{\partial x^2}\left(\frac{\partial u}{\partial x}-k_1w\right)\right.$$
$$\left.+\frac{1}{12}H_1^3\left(\frac{\partial^2 w}{\partial x^2}\right)^2\right\} \quad (6)$$

其中业已略去了t_1/H_1的高次项．$u,\frac{\partial u}{\partial x},w,\frac{\partial^2 w}{\partial x^2}$为壳体的中面边界上的有关各值．同样我们还有其它三边拱的应变能．如果取

$$\frac{1}{12}H_i^3C_i=J_i,\ C_iH_i=F_i,\ i=1,2 \quad (6A)$$

并假定AB和OC的边拱尺寸相同（都用$i=1$来表示），OA和BC边拱的尺寸相同（都用$i=2$来表示），则所有边拱应变能的总和为

$$V_3=\frac{1}{2}\left\{\int_{AB}+\int_{OC}\right\}\left\{EF_1\left(\frac{\partial u}{\partial x}-k_1w\right)^2-2EF_1t_1\frac{\partial^2 w}{\partial x^2}\left(\frac{\partial u}{\partial x}-k_1w\right)\right.$$

$$\left.+EJ_1\left(\frac{\partial^2 w}{\partial x^2}\right)^2\right\}dx+\frac{1}{2}\left\{\int_{CB}+\int_{OA}\right\}\left\{EF_2\left(\frac{\partial v}{\partial y}-k_2w\right)^2\right.$$

$$\left.-2EF_2t_2\frac{\partial^2 w}{\partial y^2}\left(\frac{\partial v}{\partial y}-k_2w\right)+EJ_2\left(\frac{\partial^2 w}{\partial y^2}\right)^2\right\}dy \tag{7}$$

最后，边拱自重的位能为

$$V_4=-\left\{\int_{AB}+\int_{OC}\right\}\gamma F_1wdx-\left\{\int_{CB}+\int_{OA}\right\}\gamma F_2wdy \tag{8}$$

其中γ为材料的每单位体积的重量．于是总能量为

$$V=V_1+V_2+V_3+V_4 \tag{9}$$

平衡条件为

$$\delta V=\delta V_1+\delta V_2+\delta V_3+\delta V_4=0 \tag{10}$$

通过变分，得

$$\iint\left\{N_x\delta\varepsilon_x+N_y\delta\varepsilon_y+N_{xy}\delta\gamma_{xy}-M_x\frac{\partial^2\delta w}{\partial x^2}-M_y\frac{\partial^2\delta w}{\partial y^2}-2M_{xy}\frac{\partial^2\delta w}{\partial x\partial y}-q\delta w\right\}dxdy$$

$$+\left\{\int_{AB}+\int_{OC}\right\}\left\{EF_1\left(\frac{\partial u}{\partial x}-k_1w-t_1\frac{\partial^2 w}{\partial x^2}\right)\delta\varepsilon_x+\left[EJ_1\frac{\partial^2 w}{\partial x^2}-EF_1t_1\left(\frac{\partial u}{\partial x}\right.\right.\right.$$

$$\left.\left.\left.-k_1w\right)\right]\frac{\partial^2\delta w}{\partial x^2}\right\}dx+\left\{\int_{CB}+\int_{OA}\right\}\left\{EF_2\left(\frac{\partial v}{\partial y}-k_2w-t_2\frac{\partial^2 w}{\partial y^2}\right)\delta\varepsilon_y+\left[EJ_2\frac{\partial^2 w}{\partial y^2}\right.\right.$$

$$\left.\left.-EF_2t_2\left(\frac{\partial v}{\partial y}-k_2w\right)\right]\frac{\partial^2\delta w}{\partial y^2}\right\}dy-\left\{\int_{AB}+\int_{OC}\right\}\gamma F_1\delta wdx-\left\{\int_{CB}+\int_{OA}\right\}\gamma F_2\delta wdy=0$$

$$\tag{11}$$

其中N_x，N_y，N_{xy}为薄膜内力，M_x，M_y，M_{xy}为弯矩及扭矩，其定义如下：

$$\left.\begin{aligned}&N_x=\frac{Eh}{1-\mu^2}(\varepsilon_x+\mu\varepsilon_y),\qquad N_y=\frac{Eh}{1-\mu^2}(\varepsilon_y+\mu\varepsilon_x),\qquad N_{xy}=\frac{Eh}{2(1+\mu)}\gamma_{xy}\\&M_x=-D\left(\frac{\partial^2 w}{\partial x^2}+\mu\frac{\partial^2 w}{\partial y^2}\right),\ M_y=-D\left(\frac{\partial^2 w}{\partial y^2}+\mu\frac{\partial^2 w}{\partial x^2}\right),\ M_{xy}=-(1-\mu)D\frac{\partial^2 w}{\partial x\partial y}\end{aligned}\right\}$$

$$\tag{12}$$

通过分部积分，得

$$-\iint\left\{\frac{\partial^2 M_x}{\partial x^2}+2\frac{\partial^2 M_{xy}}{\partial x\partial y}+\frac{\partial^2 M_y}{\partial y^2}+k_1N_x+k_2N_y+q\right\}\delta wdxdy$$

$$-\iint\left\{\frac{\partial N_x}{\partial x}+\frac{\partial N_{xy}}{\partial y}\right\}\delta udxdy-\iint\left\{\frac{\partial N_{xy}}{\partial x}+\frac{\partial N_y}{\partial y}\right\}\delta vdxdy$$

$$+\int_{AB}dx\left\{\left[N_{xy}-EF_1\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}-k_1w-t_1\frac{\partial^2 w}{\partial x^2}\right)\right]\delta u+N_y\delta v-M_y\frac{\partial\delta w}{\partial y}\right.$$

$$+\left[\frac{\partial M_y}{\partial y}+2\frac{\partial M_{xy}}{\partial x}-EF_1k_1\left(\frac{\partial u}{\partial x}-k_1w-t_1\frac{\partial^2 w}{\partial x^2}\right)+EJ_1\frac{\partial^4 w}{\partial x^4}\right.$$

$$-EF_1t_1\frac{\partial^2}{\partial x^2}\left(\frac{\partial u}{\partial x}-k_1w\right)-\gamma F_1\Big]\delta w\Big\}+\int_{OC}dx\Big\{\Big[-N_{xy}-EF_1\frac{\partial}{\partial x}\Big(\frac{\partial u}{\partial x}$$

$$-k_1w-t_1\frac{\partial^2 w}{\partial x^2}\Big)\Big]\delta u-N_y\delta v+M_y\frac{\partial\delta w}{\partial y}+\Big[-\frac{\partial M_y}{\partial y}-2\frac{\partial M_{xy}}{\partial x}-EF_1k_1\Big(\frac{\partial u}{\partial x}$$

$$-k_1w-t_1\frac{\partial^2 w}{\partial x^2}\Big)+EJ_1\frac{\partial^4 w}{\partial x^4}-EF_1t_1\frac{\partial^2}{\partial x^2}\Big(\frac{\partial u}{\partial x}-k_1w\Big)-\gamma F_1\Big]\delta w\Big\}$$

$$+\int_{CB}dy\Big\{N_x\delta u+\Big[N_{xy}-EF_2\frac{\partial}{\partial y}\Big(\frac{\partial v}{\partial y}-k_2w-t_2\frac{\partial^2 w}{\partial y^2}\Big)\Big]\delta v-M_x\frac{\partial\delta w}{\partial x}$$

$$+\Big[\frac{\partial M_x}{\partial x}+2\frac{\partial M_{xy}}{\partial y}-EF_2k_2\Big(\frac{\partial v}{\partial y}-k_2w-t_2\frac{\partial^2 w}{\partial y^2}\Big)+EJ_2\frac{\partial^4 w}{\partial y^4}$$

$$-EF_2t_2\frac{\partial^2}{\partial y^2}\Big(\frac{\partial v}{\partial y}-k_2w\Big)-\gamma F_2\Big]\delta w\Big\}+\int_{OA}dy\Big\{-N_x\delta u+\Big[-EF_2\frac{\partial}{\partial y}\Big(\frac{\partial v}{\partial y}$$

$$-k_2w-t_2\frac{\partial^2 w}{\partial y^2}\Big)-N_{xy}\Big]\delta v+M_x\frac{\partial\delta w}{\partial x}+\Big[-\frac{\partial M_x}{\partial x}-2\frac{\partial M_{xy}}{\partial y}-EF_2k_2\Big(\frac{\partial v}{\partial y}$$

$$-k_2w-t_2\frac{\partial^2 w}{\partial y^2}\Big)+EJ_2\frac{\partial^4 w}{\partial y^4}-EF_2t_2\frac{\partial^2}{\partial y^2}\Big(\frac{\partial v}{\partial y}-k_2w\Big)-\gamma F_2\Big]\delta w\Big\}$$

$$-\Big[EJ_1\frac{\partial^2 w}{\partial x^2}-EF_1t_1\Big(\frac{\partial u}{\partial x}-k_1w\Big)\Big]\frac{\partial\delta w}{\partial x}\Big|_A+\Big[EJ_2\frac{\partial^2 w}{\partial y^2}-EF_2t_2\Big(\frac{\partial v}{\partial y}$$

$$-k_2w\Big)\Big]\frac{\partial\delta w}{\partial y}\Big|_A+\Big\{\frac{\partial}{\partial x}\Big[EJ_1\frac{\partial^2 w}{\partial x^2}-EF_1t_1\Big(\frac{\partial u}{\partial x}-k_1w\Big)\Big]\frac{\partial}{\partial y}\Big[EJ_2\frac{\partial^2 w}{\partial y^2}$$

$$-EF_2t_2\Big(\frac{\partial v}{\partial y}-k_2w\Big)\Big]+2M_{xy}\Big\}\delta w\Big|_A-EF_1\Big(\frac{\partial u}{\partial x}-k_1w-t_1\frac{\partial^2 w}{\partial x^2}\Big)\delta u\Big|_A$$

$$+EF_2\Big(\frac{\partial v}{\partial y}-k_2w-t_2\frac{\partial^2 w}{\partial y^2}\Big)\delta v\Big|_A+\Big[EJ_1\frac{\partial^2 w}{\partial x^2}-EF_1t_1\Big(\frac{\partial u}{\partial x}-k_1w\Big)\Big]\frac{\partial\delta w}{\partial x}\Big|_B$$

$$+\Big[EJ_2\frac{\partial^2 w}{\partial y^2}-EF_2t_2\Big(\frac{\partial v}{\partial y}-k_2w\Big)\Big]\frac{\partial\delta w}{\partial y}\Big|_B-\Big\{\frac{\partial}{\partial x}\Big[EJ_1\frac{\partial^2 w}{\partial x^2}-EF_1t_1\Big(\frac{\partial u}{\partial x}$$

$$-k_1w\Big)\Big]+\frac{\partial}{\partial y}\Big[EJ_2\frac{\partial^2 w}{\partial y^2}-EF_2t_2\Big(\frac{\partial v}{\partial y}-k_2w\Big)\Big]+2M_{xy}\Big\}\delta w\Big|_B+EF_1\Big(\frac{\partial u}{\partial x}$$

$$-k_1w-t_1\frac{\partial^2 w}{\partial x^2}\Big)\delta u\Big|_B+EF_2\Big(\frac{\partial v}{\partial y}-k_2w-t_2\frac{\partial^2 w}{\partial y^2}\Big)\delta v\Big|_B+\Big[EJ_1\frac{\partial^2 w}{\partial x^2}$$

$$-EF_1t_1\Big(\frac{\partial u}{\partial x}-k_1w\Big)\Big]\frac{\partial\delta w}{\partial x}\Big|_C-\Big[EJ_2\frac{\partial^2 w}{\partial y^2}-EF_2t_2\Big(\frac{\partial v}{\partial y}-k_2w\Big)\Big]\frac{\partial\delta w}{\partial y}\Big|_C$$

$$-\Big\{\frac{\partial}{\partial x}\Big[EJ_1\frac{\partial^2 w}{\partial x^2}-EF_1t_1\Big(\frac{\partial u}{\partial x}-k_1w\Big)\Big]-\frac{\partial}{\partial y}\Big[EJ_2\frac{\partial^2 w}{\partial y^2}-EF_2t_2\Big(\frac{\partial v}{\partial y}-k_2w\Big)\Big]$$

$$-2M_{xy}\Big\}\delta w\Big|_C+EF_1\Big(\frac{\partial u}{\partial x}-k_1w-t_1\frac{\partial^2 w}{\partial x^2}\Big)\delta u\Big|_C-EF_2\Big(\frac{\partial v}{\partial y}-k_2w-t_2\frac{\partial^2 w}{\partial y^2}\Big)\delta v\Big|_C$$

$$-\left[EJ_1\frac{\partial^2 w}{\partial x^2}-EF_1t_1\left(\frac{\partial u}{\partial x}-k_1w\right)\right]\frac{\partial \delta w}{\partial x}\Big|_O-\left[EJ_2\frac{\partial^2 w}{\partial y^2}-EF_2t_2\left(\frac{\partial v}{\partial y}\right.\right.$$

$$\left.\left.-k_2w\right)\right]\frac{\partial \delta w}{\partial y}\Big|_O+\left\{\frac{\partial}{\partial x}\left[EJ_1\frac{\partial^2 w}{\partial x^2}-EF_1t_1\left(\frac{\partial u}{\partial x}-k_1w\right)\right]+\frac{\partial}{\partial y}\left[EJ_2\frac{\partial^2 w}{\partial y^2}\right.\right.$$

$$\left.\left.-EF_2t_2\left(\frac{\partial v}{\partial y}-k_2w\right)-2M_{xy}\right\}\delta w\Big|_O-EF_1\left(\frac{\partial u}{\partial x}-k_1w-t_1\frac{\partial^2 w}{\partial x^2}\right)\delta u\Big|_O$$

$$-EF_2\left(\frac{\partial v}{\partial y}-k_2w-t_2\frac{\partial^2 w}{\partial y^2}\right)\delta v\Big|_O=0 \tag{13}$$

所有这些变分，除了在边界上和角点上可能受到某种变形约束限制外，其它都是独立的.

对于简支的角点问题而言，我们的变形约束条件为四角的w等于零，其它别无限制. 因此，有下列微分方程、边界条件和角点条件：

（Ⅰ） 在矩形域($0\leqslant x\leqslant a$, $0\leqslant y\leqslant b$)内：

$$\left.\begin{aligned}&(a)\quad \frac{\partial^2 M_x}{\partial x^2}+2\frac{\partial^2 M_{xy}}{\partial x\partial y}+\frac{\partial^2 M_y}{\partial y^2}+k_1N_x+k_2N_y+q=0\\&(b)\quad \frac{\partial N_x}{\partial x}+\frac{\partial N_{xy}}{\partial y}=0,\qquad (c)\quad \frac{\partial N_y}{\partial y}+\frac{\partial N_{xy}}{\partial x}=0\end{aligned}\right\} \tag{14}$$

（Ⅱ） 在边界OA上，即$x=0$时

$$\left.\begin{aligned}&(a)\quad N_x=0,\quad (b)\, N_{xy}+EF_2\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}-k_2w-t_2\frac{\partial^2 w}{\partial y^2}\right)=0,\quad (c)\quad M_x=0\\&(d)\quad -\frac{\partial M_x}{\partial x}-2\frac{\partial M_{xy}}{\partial y}-EF_2k_2\left(\frac{\partial v}{\partial y}-k_2w-t_2\frac{\partial^2 w}{\partial y^2}\right)+EJ_2\frac{\partial^4 w}{\partial y^4}\\&\qquad +EF_2t_2\frac{\partial^2}{\partial y^2}\left(\frac{\partial v}{\partial y}-k_2w\right)-\gamma F_2=0\end{aligned}\right\} \tag{15}$$

在边界OC上，即$y=0$时

$$\left.\begin{aligned}&(a)\quad N_{xy}+EF_1\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}-k_1w-t_1\frac{\partial^2 w}{\partial x^2}\right)=0,\quad (b)\quad N\ =0,\quad (c)\quad M_y=0\\&(d)\quad -\frac{\partial M_y}{\partial y}-2\frac{\partial M_{xy}}{\partial x}-EF_1k_1\left(\frac{\partial u}{\partial x}-k_1w-t_1\frac{\partial^2 w}{\partial x^2}\right)+EJ_1\frac{\partial^4 w}{\partial x^4}\\&\qquad -EF_1t_1\frac{\partial^2}{\partial x^2}\left(\frac{\partial u}{\partial x}-k_1w\right)-\gamma F_1=0\end{aligned}\right\} \tag{16}$$

在边界BC上，即$x=a$时

$$\left.\begin{aligned}&(a)\quad N_x=0,\quad (b)\quad N_{xy}-EF_2\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}-k_2w-t_2\frac{\partial^2 w}{\partial y^2}\right)=0,\quad (c)\quad M_x=0\\&(d)\quad \frac{\partial M_x}{\partial x}+2\frac{\partial M_{xy}}{\partial y}-EF_2k_2\left(\frac{\partial v}{\partial y}-k_2w-t_2\frac{\partial^2 w}{\partial y^2}\right)+EJ_2\frac{\partial^4 w}{\partial y^4}\\&\qquad -EF_2t_2\frac{\partial^2}{\partial y^2}\left(\frac{\partial v}{\partial y}-k_2w\right)-\gamma F_2=0\end{aligned}\right\} \tag{17}$$

在边界AB上，即$y=b$时

$$
\left.\begin{aligned}
&(a)\quad N_{xy}-EF_1\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}-k_1w-t_1\frac{\partial^2 w}{\partial x^2}\right)=0,\ (b)\quad N_y=0,\ (c)\quad M_y=0\\
&(d)\quad \frac{\partial M_y}{\partial y}+2\frac{\partial M_{xy}}{\partial x}-EF_1k_1\left(\frac{\partial u}{\partial x}-k_1w-t_1\frac{\partial^2 w}{\partial x^2}\right)+EJ_1\frac{\partial^4 w}{\partial x^4}\\
&\qquad -EF_1t_1\frac{\partial^2}{\partial x^2}\left(\frac{\partial u}{\partial x}-k_1w\right)-\gamma F_1=0
\end{aligned}\right\}
\tag{18}
$$

（Ⅲ） 在简支角点$O(0,\ 0)$，$A(0,\ b)$，$B(a,\ b)$，$c(a,\ 0)$上都满足相同的角点条件

$$
\left.\begin{aligned}
&(a)\ EJ_1\frac{\partial^2 w}{\partial x^2}-EF_1t_1\left(\frac{\partial u}{\partial x}-k_1w\right)=0,\ (b)\ EJ_2\frac{\partial^2 w}{\partial y^2}-EF_2t_2\left(\frac{\partial v}{\partial y}-k_2w\right)=0\\
&(c)\ w=0,\ (d)\ \frac{\partial u}{\partial x}-k_1w-t_1\frac{\partial^2 w}{\partial x^2}=0\ (e)\ \frac{\partial v}{\partial y}-k_2w-t_2\frac{\partial^2 w}{\partial y^2}=0
\end{aligned}\right\}
\tag{19}
$$

在这里，我们假设两邻边的边拱在角点上并不固定连结在一起（这个假设和略去边拱抗扭刚度意义相同），因此，$\frac{\partial w}{\partial x}$，$\frac{\partial w}{\partial y}$在角点上不受任何限制.

如果四个角点都是铰支的，亦即如在四边拉了拉杆限制角点的外伸位移，则四角有位移约束条件$w=u=v=0$，于是角点条件不再是(19)式，而应写成

（Ⅳ） 在铰支角点$O(0,\ 0)$，$A(0,\ b)$，$B(a,\ b)$，$C(a,\ 0)$上都满足相同角点条件

$$
\left.\begin{aligned}
&(a)\quad EJ_1\frac{\partial^2 w}{\partial x^2}-EF_1t_1\left(\frac{\partial u}{\partial x}-k_1w\right)=0,\ (b)\ EJ_2\frac{\partial^2 w}{\partial y^2}-EF_2t_2\left(\frac{\partial v}{\partial y}-k_2w\right)=0\\
&(c)\quad w=0,\quad (d)\,u=0,\quad (e)\,v=0
\end{aligned}\right\}
\tag{20}
$$

所有这些方程都可以进一步简化，如以四角简支的问题为例，可以引进应力函数$\varphi(x,y)$，从(14b)，(14c)式，我们有

$$
N_x=\frac{\partial^2\varphi}{\partial y^2},\quad N_y=\frac{\partial^2\varphi}{\partial x^2},\quad N_{xy}=-\frac{\partial^2\varphi}{\partial x\partial y}
\tag{21}
$$

把(1)式代入(12)，然后利用(21)，得

$$
\left.\begin{aligned}
&\frac{\partial^2\varphi}{\partial x^2}=\frac{Eh}{1-\mu^2}\left[\frac{\partial v}{\partial y}+\mu\frac{\partial u}{\partial x}-(k_1\mu+k_2)w\right]\\
&\frac{\partial^2\varphi}{\partial y^2}=\frac{Eh}{1-\mu^2}\left[\frac{\partial u}{\partial x}+\mu\frac{\partial v}{\partial y}-(k_2\mu+k_1)w\right]\\
&\frac{\partial^2\varphi}{\partial x\partial y}=-\frac{Eh}{2(1+\mu)}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)
\end{aligned}\right\}
\tag{22}
$$

从上式消去u，v，则得协调方程

$$
\nabla^2\nabla^2\varphi+Eh\left(k_1\frac{\partial^2 w}{\partial y^2}+k_2\frac{\partial^2 w}{\partial x^2}\right)=0
\tag{23}
$$

而(14a)式可以写成

$$D\nabla^2\nabla^2 w - k_1\frac{\partial^2\varphi}{\partial y^2} - k_2\frac{\partial^2\varphi}{\partial x^2} = q \tag{24}$$

其中 $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$，(23)，(24)即为有名的双曲扁壳方程.

对于对 $x=\frac{1}{2}a$，$y=\frac{1}{2}b$都对称的载荷而言，如均布载荷，u 对$x=\frac{1}{2}a$ 反对称而对 $y=\frac{1}{2}b$对称，v 对 $x=\frac{a}{2}$ 对称而对 $y=\frac{b}{2}$反对称，w和φ则对$x=\frac{a}{2}$，$y=\frac{b}{2}$都对称.

因此，边界条件只要写出OA边（或CB边）和OC边（或AB边）上的条件就足够了，只要解满足上述对称性质，其它两边根据对称条件自然满足．于是，利用了(21)，(22)后有下列边界条件：

在边界OA（即$x=0$)上：

$$\left.\begin{aligned}
&(a)\ \frac{\partial^2\varphi}{\partial y^2}=0,\ (b)\ \frac{\partial^2\varphi}{\partial x\partial y}-\frac{\partial}{\partial y}\left[\frac{F_2}{h}\left(\frac{\partial^2\varphi}{\partial x^2}-\mu\frac{\mu^2\varphi}{\partial y^2}\right)-EF_2t_2\frac{\partial^2 w}{\partial y^2}\right]=0\\
&(c)\ \frac{\partial^2 w}{\partial x^2}+\mu\frac{\partial^2 w}{\partial y^2}=0\\
&(d)\ D\left(\frac{\partial^3 w}{\partial x^3}+(2-\mu)\frac{\partial^3 w}{\partial x\partial y^2}\right)-\frac{F_2k_2}{h}\left(\frac{\partial^2\varphi}{\partial x^2}-\mu\frac{\partial^2\varphi}{\partial y^2}\right)+EF_2t_2k_2\frac{\partial^2 w}{\partial y^2}\\
&\qquad +EJ_2\frac{\partial^4 w}{\partial y^4}-\frac{F_2t_2}{h}\left(\frac{\partial^4\varphi}{\partial x^2\partial y^2}-\mu\frac{\partial^4\varphi}{\partial y^4}\right)-\gamma F_2=0
\end{aligned}\right\} \tag{25}$$

在边界OC（即$y=0$)上

$$\left.\begin{aligned}
&(a)\ \frac{\partial^2\varphi}{\partial x\partial y}-\frac{\partial}{\partial x}\left[\frac{F_1}{h}\left(\frac{\partial^2\varphi}{\partial y^2}-\mu\frac{\partial^2\varphi}{\partial x^2}\right)-EF_1t_1\frac{\partial^2 w}{\partial x^2}\right]=0\\
&(b)\ \frac{\partial^2\varphi}{\partial x^2}=0\quad (c)\ \frac{\partial^2 w}{\partial y^2}+\mu\frac{\partial^2 w}{\partial x^2}=0\\
&(d)\ D\left(\frac{\partial^3 w}{\partial y^3}+(2-\mu)\frac{\partial^3 w}{\partial y\partial x^2}\right)-\frac{F_1k_1}{h}\left(\frac{\partial^2\varphi}{\partial y^2}-\mu\frac{\partial^2\varphi}{\partial x^2}\right)+EF_1k_1t_1\frac{\partial^2 w}{\partial x^2}\\
&\qquad +EJ_1\frac{\partial^4 w}{\partial x^4}-\frac{F_1t_1}{h}\left(\frac{\partial^4\varphi}{\partial x^2\partial y^2}-\mu\frac{\partial^4\varphi}{\partial x^4}\right)-\gamma F_1=0
\end{aligned}\right\} \tag{26}$$

同样，四角的角点条件只要写出O点的条件就足够，只要解满足上述对称性质，其它各角点的条件自然满足

在简支角点O（即$x=0$，$y=0$)上

$$\left.\begin{aligned}&(a)\ EJ_1\frac{\partial^2 w}{\partial x^2}-\frac{F_1t_1}{h}\left(\frac{\partial^2\varphi}{\partial y^2}-\mu\frac{\partial^2\varphi}{\partial x^2}\right)=0\\&(b)\ EJ_2\frac{\partial^2 w}{\partial y^2}-\frac{F_2t_2}{h}\left(\frac{\partial^2\varphi}{\partial x^2}-\mu\frac{\partial^2\varphi}{\partial y^2}\right)=0\\&(c)\ w=0,\quad (d)\frac{F_1}{h}\left(\frac{\partial^2\varphi}{\partial y^2}-\mu\frac{\partial^2\varphi}{\partial x^2}\right)-EF_1t_1\frac{\partial^2 w}{\partial x^2}=0\\&(e)\frac{F_2}{h}\left(\frac{\partial^2\varphi}{\partial x^2}-\mu\frac{\partial^2\varphi}{\partial y^2}\right)-EF_2t_2\frac{\partial^2 w}{\partial y^2}=0\end{aligned}\right\}\qquad(27)$$

有关铰支角点的条件，也可作类似的规定.

我们现在还可以进一步简化这些条件：例如(25a)中指出在$x=0$上，$\frac{\partial^2\varphi}{\partial y^2}=0$，因此(25$d$)中的同项$\frac{\partial^2\varphi}{\partial y^2}$也必为零，(25$b$)式中$\frac{\partial^3\varphi}{\partial y^3}$，和(25$d$)式中$\frac{\partial^4\varphi}{\partial y^4}$必也等于零．于是，对于这类等截面的边拱而言，(25)，(26)可以写成

在边界OA(即$x=0$)上

$$\left.\begin{aligned}&(a)\ \frac{\partial^2\varphi}{\partial y^2}=0,\ (b)\ \frac{\partial^2\varphi}{\partial x\partial y}-\frac{F_2}{h}\frac{\partial^3\varphi}{\partial x^2\partial y}+EF_2t_2\frac{\partial^3 w}{\partial y^3}=0\\&(c)\ \frac{\partial^2 w}{\partial x^2}+\mu\frac{\partial^2 w}{\partial y^2}=0\\&(d)\ D\left(\frac{\partial^3 w}{\partial x^3}+(2-\mu)\frac{\partial^3 w}{\partial x\partial y^2}\right)-\frac{F_2k_2}{h}\frac{\partial^2\varphi}{\partial x^2}+EF_2t_2k_2\frac{\partial^2 w}{\partial y^2}\\&\qquad+EJ_2\frac{\partial^4 w}{\partial y^4}-\frac{F_2t_2}{h}\frac{\partial^4\varphi}{\partial x^2\partial y^2}-\gamma F_2=0\end{aligned}\right\}\qquad(28)$$

在边界OC(即$y=0$)上

$$\left.\begin{aligned}&(a)\ \frac{\partial^2\varphi}{\partial x\partial y}-\frac{F_1}{h}\frac{\partial^3\varphi}{\partial x\partial y^2}+EF_1t_1\frac{\partial^3 w}{\partial x^3}=0,\ (b)\frac{\partial^2\varphi}{\partial x^2}=0\\&(c)\ \frac{\partial^2 w}{\partial y^2}+\mu\frac{\partial^2 w}{\partial x^2}=0\\&(d)\ D\left(\frac{\partial^3 w}{\partial y^3}+(2-\mu)\frac{\partial^3 w}{\partial x^2\partial y}\right)-\frac{F_1k_1}{h}\frac{\partial^2\varphi}{\partial y^2}+EF_1t_1k_1\frac{\partial^2 w}{\partial x^2}\\&\qquad+EJ_1\frac{\partial^4 w}{\partial x^4}-\frac{F_1t_1}{h}\frac{\partial^4\varphi}{\partial x^2\partial y^2}-\gamma F_1=0\end{aligned}\right\}\qquad(29)$$

我们很易证明，对于简支角点条件而言，只要(28)在$x=0$上成立，(29)在$y=0$上成立，则简支角点条件(27)中，除了(c)外，也必自然适合，所以，简支角点条件归化为一个条件：

在简支角点$O(0,\ 0)$上， $w=0$ (30)

(28a)表示壳体边界上法向应力为零，(28b)表示边拱轴向力的平衡，(28c)表示壳内边界弯矩为零，(28d)表示边拱在自己的平面内的弯曲平衡条件．(28a)，(28c)是略去了边拱横向的抗弯刚度和抗扭刚度的必然结果．(29)式中各式也可以作同样的解释．

(23)，(24)，(28)，(29)，(30)构成解对称载荷四角简支的带有边拱的双曲扁壳的全部

微分方程，边界条件和角点条件．

对于对称载荷四角铰支的问题而言，微分方程(23)，(24)，和边界条件(28)，(29)和四角简支的问题完全相同，但角点条件(20a，b)自然适合，其余角点条件为

$$(a)\ w=0,\qquad (b)\ u=0,\qquad (c)\ v=0 \tag{31}$$

其中u，v无法用φ来表示，在这种情况下，我们必须先利用(22)式解出u，v，然后满足(31)式．

现在让我们把全部方程式化为无量纲形式：

引进无量纲变量$\psi(\xi,\ \eta)$及$W(\xi,\ \eta)$．

$$\left.\begin{aligned}&\varphi=\frac{qa^4}{h\pi^4}\sqrt{12(1-\mu^2)}\,\psi(\xi,\ \eta),\qquad w=\frac{qa^4}{D\pi^4}W(\xi,\ \eta)\\&x=\frac{\xi a}{\pi},\quad y=\frac{\eta b}{\pi}\quad(0\leqslant\xi,\ \eta\leqslant\pi)\end{aligned}\right\} \tag{32}$$

其中ξ，η为无量纲座标，于是微分方程(23)，(24)可以写成

$$\left.\begin{aligned}&\frac{\partial^4\psi}{\partial\xi^4}+2\alpha^2\frac{\partial^4\psi}{\partial\xi^2\partial\eta^2}+\alpha^4\frac{\partial^4\psi}{\partial\eta^4}+\lambda^2\left(r\alpha^2\frac{\partial^2W}{\partial\eta^2}+\frac{\partial^2W}{\partial\xi^2}\right)=0\\&\frac{\partial^4W}{\partial\xi^4}+2\alpha^2\frac{\partial^4W}{\partial\xi^2\partial\eta^2}+\alpha^4\frac{\partial^4W}{\partial\eta^4}-\lambda^2\left(r\alpha^2\frac{\partial^2\psi}{\partial\eta^2}+\frac{\partial^2\psi}{\partial\xi^2}\right)=0\end{aligned}\right\} \tag{33}$$

其中α，r，λ为无量纲参量

$$\alpha=\frac{a}{b},\qquad r=\frac{k_1}{k_2},\qquad \lambda^2=\frac{k_2a^2}{h\pi^2}\sqrt{12(1-\mu^2)} \tag{34}$$

同样，边界条件可以化成下列各式：

(Ⅰ)　在边界OA上，即当$\xi=0$时：

$$\left.\begin{aligned}&(a)\ \frac{\partial^2\psi}{\partial\eta^2}=0,\quad (b)\ \frac{\partial^2\psi}{\partial\xi\partial\eta}-f_2\frac{\partial^3\psi}{\partial\xi^2\partial\eta}+f_2\tau_2\alpha^2\frac{\partial^3W}{\partial\eta^3}=0\\&(c)\ \frac{\partial^2W}{\partial\xi^2}+\mu\alpha^2\frac{\partial^2W}{\partial\eta^2}=0\\&(d)\ \frac{\partial^2W}{\partial\xi^2}+(2-\mu)\alpha^2\frac{\partial^3W}{\partial\xi\partial\eta^2}-f_2\lambda^2\frac{\partial^2\psi}{\partial\xi^2}+f_2\lambda^2\tau_2\alpha^2\frac{\partial^2W}{\partial\eta^2}-\tau_2f_2\alpha^2\frac{\partial^4\psi}{\partial\eta^2\partial\xi^2}\\&\qquad+G_2\frac{\partial^4W}{\partial\eta^4}-S_2=0\end{aligned}\right\} \tag{35}$$

(Ⅱ)　在边界OC上，即当$\eta=0$时

$$\left.\begin{aligned}&(a)\ \alpha\frac{\partial^2\psi}{\partial\xi\partial\eta}-f_1\alpha^2\frac{\partial^3\psi}{\partial\xi\partial\eta^2}+f_1\tau_1\frac{\partial^3W}{\partial\xi^3}=0,\quad (b)\ \frac{\partial^2\psi}{\partial\xi^2}=0\\&(c)\ \alpha^2\frac{\partial^2\psi}{\partial\eta^2}+\mu\frac{\partial^2W}{\partial\xi^2}=0\\&(d)\ \alpha^3\frac{\partial^3W}{\partial\eta^3}+(2-\mu)\alpha^2\frac{\partial^3W}{\partial\xi^2\partial\eta}-rf_1\alpha^2\lambda^2\frac{\partial^2\psi}{\partial\eta^2}+rf_1\tau_1\lambda\frac{\partial^2W}{\partial\xi^2}\\&\qquad-\tau_1f_1\alpha^2\frac{\partial^4\psi}{\partial\xi^2\partial\eta^2}+G_1\frac{\partial^4W}{\partial\xi^4}-S_1=0\end{aligned}\right\} \tag{36}$$

其中f_1，f_2，τ_1，τ_2，G_1，G_2，S_1，S_2为下列无量纲参数：

$$\left.\begin{aligned} &f_1=\frac{F_1\pi}{ah},\quad \tau_1=\frac{t_1}{h}\sqrt{12(1-\mu^2)},\quad G_1=\frac{EJ_1\pi}{aD},\quad S_1=\frac{F_1\gamma\pi}{aq} \\ &f_2=\frac{F_2\pi}{ah},\quad \tau_2=\frac{t_2}{h}\sqrt{12(1-\mu^2)},\quad G_2=\frac{EJ_2\pi}{aD},\quad S_2=\frac{F_2\gamma\pi}{aq}=S_1\frac{f_2}{f_1} \end{aligned}\right\} \tag{37}$$

(34)，(37)中共有10个独立参数，如果边拱的材料和壳体的材料完全相同，同时如果q完全是自重的作用，则

$$q=\gamma h \tag{38}$$

于是S_1，S_2可以简化为

$$S_1=\frac{F_1\gamma\pi}{aq}=\frac{F_1\pi}{ah}=f_1,\quad S_2=\frac{F_2\pi}{ah}=f_2 \tag{39}$$

如果$a=b$，$k_1=k_2$，则矩形双曲扁壳化为方形球面扁壳，在这种条件下：

$$\left.\begin{aligned} &\alpha=1,\quad r=1,\quad \lambda^2=\frac{ka^2}{h\pi^2}\sqrt{12(1-\mu^2)} \\ &f_1=f_2=f=\frac{F\pi}{ah},\quad \tau_1=\tau_2=\tau=\frac{t}{h}\sqrt{12(1-\mu^2)} \\ &G_1=G_2=G=\frac{EJ\pi}{aD},\quad S_1=S_2=S=\frac{F\gamma\pi}{aq} \end{aligned}\right\} \tag{40}$$

这里业已假定了四根边拱也有相同尺寸．于是只剩下λ，f，τ，G，S五个独立参数，如果q是自重载荷，则$S=f$，可以化为四个独立参数，这四个无量纲参数分别代表：（1）拱的截面和壳的横截面面积之比(f)，（2）拱与壳中面的差距和壳的厚度之比(τ)，（3）壳的矢高$\frac{1}{8}a^2k$和壳的厚度之比(λ^2)，（4）拱的抗弯刚度和壳的抗弯刚度之比(G)．

对于方形球面扁壳而言，微分方程及边界条件可以化为下列各式：

在方形球面扁壳内（$0\leqslant\xi$，$\eta\leqslant\pi$）

$$\left.\begin{aligned} &\frac{\partial^4\psi}{\partial\xi^4}+2\frac{\partial^4\psi}{\partial\xi^2\partial\eta^2}+\frac{\partial^4\psi}{\partial\eta^4}+\lambda^2\left(\frac{\partial^2 W}{\partial\xi^2}+\frac{\partial^2 W}{\partial\eta^2}\right)=0 \\ &\frac{\partial^4 W}{\partial\xi^4}+2\frac{\partial^4 W}{\partial\xi^2\partial\eta^2}+\frac{\partial^4 W}{\partial\eta^4}-\lambda^2\left(\frac{\partial^2\psi}{\partial\xi^2}+\frac{\partial^2\psi}{\partial\eta^2}\right)=1 \end{aligned}\right\} \tag{41}$$

在边界OA上，即当$\xi=0$时

$$\left.\begin{aligned} &(a)\quad \frac{\partial^2\psi}{\partial\eta^2}=0 \\ &(b)\quad \frac{\partial^2\psi}{\partial\xi\partial\eta}-f\frac{\partial^3\psi}{\partial\xi^2\partial\eta}+f\tau\frac{\partial^3 W}{\partial\eta^3}=0 \\ &(c)\quad \frac{\partial^2 W}{\partial\xi^2}+\mu\frac{\partial^2 W}{\partial\eta^2}=0 \\ &(d)\quad \frac{\partial^3 W}{\partial\xi^3}+(2-\mu)\frac{\partial^3 W}{\partial\xi\partial\eta^2}-f\lambda^2\frac{\partial^2\psi}{\partial\xi^2}+f\tau\lambda^2\frac{\partial^2 W}{\partial\eta^2}-\tau f\frac{\partial^4\psi}{\partial\eta^2\partial\xi^2} \\ &\qquad +G\frac{\partial^4 W}{\partial\eta^4}-S=0 \end{aligned}\right\} \tag{42}$$

在边界OC上，即当$\eta=0$时

$$\left.\begin{aligned}&(a)\quad \frac{\partial^2\psi}{\partial\xi^2}=0\\&(b)\quad \frac{\partial^2\psi}{\partial\xi\partial\eta}-f\frac{\partial^3\psi}{\partial\xi\partial\eta^2}+f\tau\frac{\partial^3 W}{\partial\xi^3}=0\\&(c)\quad \frac{\partial^2 W}{\partial\eta^2}+\mu\frac{\partial^2 W}{\partial\xi^2}=0\\&(d)\quad \frac{\partial^3 W}{\partial\eta^3}+(2-\mu)\frac{\partial^3 W}{\partial\eta\partial\xi^2}-f\lambda^2\frac{\partial^2\psi}{\partial\eta^2}+f\tau\lambda^2\frac{\partial^2 W}{\partial\xi^2}-\tau f\frac{\partial^4\psi}{\partial\xi^2\partial\eta^2}\\&\qquad +G\frac{\partial^4 W}{\partial\xi^4}-S=0\end{aligned}\right\}\tag{43}$$

在简支角点的条件下，角点上

$$W=0\tag{44}$$

(41)，(42)，(43)，(44) 为解方形球面对称扁壳四角简支问题的全部微分方程及求解条件.

对于铰支角点而言，微分方程及边界条件相同，但是四角点的条件应该改用 (31) 式. 设引用无量纲位移$U(\xi,\ \eta)$，$V(\xi,\ \eta)$.

$$u=\frac{qa^3}{Eh^2\pi^2}\sqrt{12(1-\mu^2)}\,U(\xi,\ \eta),\qquad v=\frac{qa^3}{Eh^2\pi^2}\sqrt{12(1-\mu^2)}\,V(\xi,\ \eta)\tag{45}$$

(22)式可以写成

$$\left.\begin{aligned}&\frac{\partial U}{\partial\xi}=\frac{\partial^2\psi}{\partial\eta^2}-\mu\frac{\partial^2\psi}{\partial\xi^2}+\lambda^2 W\\&\frac{\partial V}{\partial\eta}=\frac{\partial^2\psi}{\partial\xi^2}-\mu\frac{\partial^2\psi}{\partial\eta^2}+\lambda^2 W\\&\frac{\partial U}{\partial\eta}+\frac{\partial V}{\partial\xi}=-2\,(1+\mu)\frac{\partial^2\psi}{\partial\eta\partial\xi}\end{aligned}\right\}\tag{46}$$

而铰支角点条件为

$$W=0,\ U=0,\ V=0\ (\text{在角点上})\tag{47}$$

(41)，(42)，(43)，(46)，(47)为求解铰支角点的方形球面对称扁壳的全部微分方程及边界条件.

三、具有边拱的四角铰支或简支的方形扁球壳的双三角级数解

现在让我们在 (42)，(43)，(44) 的条件下，求得(41)式的解，从(41)式中消去W，得

$$\nabla^2\nabla^2\nabla^2\psi+\lambda^4\nabla^2\psi=-\lambda^2\tag{48}$$

其中 $\nabla^2=\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}$. 设$\psi$可以分解为两部份

$$\psi(\xi,\ \eta)=\psi^{(1)}+\psi^{(2)}\tag{49}$$

其中

$$\psi^{(1)}=\sum_{n=1,3,5\cdots}^{\infty}\psi_n^{(1)}(\eta)\sin n\xi,\quad \psi^{(2)}=\sum_{n=1,3,5\cdots}^{\infty}\psi_m^{(2)}(\xi)\sin m\eta\tag{50}$$

并设$\psi^{(1)}$和$\psi^{(2)}$分别为下列两方程之解

$$\left.\begin{aligned}\nabla^2\nabla^2\nabla^2\psi^{(1)}+\lambda^4\nabla^2\psi^{(1)}=-\frac{1}{2}\lambda^2\\ \nabla^2\nabla^2\nabla^2\psi^{(2)}+\lambda^4\nabla^2\psi^{(2)}=-\frac{1}{2}\lambda^2\end{aligned}\right\}\tag{51}$$

把两式相加，确可证明(49)为(48)式之解，同时还满足对称条件.

先求(51)中第一式之解，将$\frac{1}{2}\lambda^2$展开为 $\sin n\xi$的富氏级数，得

$$\frac{1}{2}\lambda^2=\frac{2\lambda^2}{\pi}\sum_{n=1,3,5\cdots}^{\infty}\frac{1}{n}\sin n\xi\tag{52}$$

把(50)的第一式及(52)式代入(51)式的第一式，比较等式两边的三角级数，得

$$\left(\frac{d^2}{d\eta^2}-n^2\right)^3\psi_n^{(1)}+\lambda^4\left(\frac{d^2}{d\eta^2}-n^2\right)\psi_n^{(1)}=-\frac{2\lambda^2}{n\pi}\tag{53}$$

这个方程的齐次式的特征方程为

$$(k_n^2-n^2)^3+\lambda^4(k_n^2-n^2)=0\tag{54}$$

其特征根为

$$\pm n,\ P_n\pm iQ_n,\ -P_n\pm iQ_n\tag{55}$$

其中

$$P_n=\frac{1}{\sqrt{2}}\sqrt{\sqrt{n^4+\lambda^4}+n^2},\qquad Q_n=\frac{1}{\sqrt{2}}\sqrt{\sqrt{n^4+\lambda^4}-n^2}\tag{56}$$

因此，对 $\eta=\frac{\pi}{2}$对称的解可以写成

$$\begin{aligned}\psi_n^{(1)}(\eta)=&\frac{2\lambda^2}{n^3\pi(\lambda^4+n^4)}+A_n\operatorname{ch}P_n\left(\eta-\frac{\pi}{2}\right)\cos Q_n\left(\eta-\frac{\pi}{2}\right)\\&+B_n\operatorname{sh}P_n\left(\eta-\frac{\pi}{2}\right)\sin Q_n\left(\eta-\frac{\pi}{2}\right)+C_n\operatorname{ch}n\left(n-\frac{\pi}{2}\right)\end{aligned}\tag{57}$$

而$\psi^{(1)}(\xi,\ \eta)$，即(51)式第一式的通解可以写成

$$\begin{aligned}\psi^{(1)}=&\sum_{n=1,3,5\cdots}^{\infty}\frac{2\lambda^2}{n^3\pi(\lambda^4+n^4)}\sin n\xi+\sum_{n=1,3,5\cdots}^{\infty}\left\{A_n\operatorname{ch}P_n\left(\eta-\frac{\pi}{2}\right)\cos Q_n\left(\eta-\frac{\pi}{2}\right)\right.\\&\left.+B_n\operatorname{sh}P_n\left(\eta-\frac{\pi}{2}\right)\sin Q_n\left(\eta-\frac{\pi}{2}\right)+C_n\operatorname{ch}n(\eta-\pi/2)\right\}\sin n\xi\end{aligned}\tag{58}$$

和这个特解相关的$W^{(1)}(\xi,\ \eta)$的解，根据(41)式的第一式可以写成

$$W^{(1)}(\xi,\ \eta)=\sum_{n=1,3,5\cdots}^{\infty}D_n\operatorname{ch}n\left(\eta-\frac{\pi}{2}\right)\sin n\xi-\frac{1}{\lambda^2}\nabla^2\psi^{(1)}\tag{59}$$

式中右端第一项为一调和函数，D_n为一待定积分常数，把(58)代入(59)式，并注意到关系

$$\left.\begin{aligned}P_n^2-Q_n^2=n^2\\ 2P_nQ_n=\lambda^2\end{aligned}\right\}\tag{60}$$

得到

$$W^{(1)}=\sum_{n=1,3,5\cdots}^{\infty}\frac{2}{n\pi(\lambda^4+n^4)}\sin n\xi+\sum_{n=1,3,5\cdots}^{\infty}\Big\{-B_n\,\mathrm{ch}\,P_n\Big(\eta-\frac{\pi}{2}\Big)\cos Q_n\Big(\eta-\frac{\pi}{2}\Big)$$

$$+A_n\,\mathrm{sh}\,P_n\Big(\eta-\frac{\pi}{2}\Big)\sin Q_n\Big(\eta-\frac{\pi}{2}\Big)+D_n\,\mathrm{ch}\,n\Big(\eta-\frac{\pi}{2}\Big)\Big\}\sin n\xi \tag{61}$$

同样(51)式的第二式的解也可以写成

$$\psi^{(2)}(\xi,\ \eta)=\sum_{m=1,3,5\cdots}^{\infty}\frac{2\lambda^2}{m^3\pi(\lambda^4+n^4)}\sin m\eta+\sum_{m=1,3,5\cdots}^{\infty}\Big\{A_m\,\mathrm{ch}\,P_m\Big(\xi-\frac{\pi}{2}\Big)$$

$$\cdot\cos Q_n\Big(\xi-\frac{\pi}{2}\Big)+B_m\,\mathrm{sh}\,P_m\Big(\xi-\frac{\pi}{2}\Big)\sin Q_m\Big(\xi-\frac{\pi}{2}\Big)$$

$$+C_m\,\mathrm{ch}\,m\Big(\xi-\frac{\pi}{2}\Big)\Big\}\sin m\eta \tag{62}$$

$$W^{(2)}(\xi,\ \eta)=\sum_{m=1,3,5\cdots}^{\infty}\frac{2}{m\pi(m^4+\lambda^4)}\sin m\eta$$

$$+\sum_{m=1,3,5\cdots}^{\infty}\Big\{-B_m\,\mathrm{ch}\,P_m\Big(\xi-\frac{\pi}{2}\Big)\cos Q_m\Big(\xi-\frac{\pi}{2}\Big)$$

$$+A_m\,\mathrm{sh}\,P_m\Big(\xi-\frac{\pi}{2}\Big)\sin Q_m\Big(\xi-\frac{\pi}{2}\Big)+D_m\,\mathrm{ch}\,m\Big(\xi-\frac{\pi}{2}\Big)\Big\}\sin m\eta \tag{63}$$

其中，根据对称条件，(62)，(63) 的待定常数 A_m，B_m，C_m，D_m，只要 $n=m$，应该分别等于(58)，(61)中的待定常数A_n，B_n，C_n，D_n.

于是，(41)式的对称解可以写成

$$\psi=\psi^{(1)}+\psi^{(2)}=\sum_{n=1,3,5\cdots}^{\infty}\Big\{\frac{2\lambda^2}{n^3\pi(n^4+\lambda^4)}+A_n\,\mathrm{ch}P_n\Big(\eta-\frac{\pi}{2}\Big)\cos Q_n\Big(\eta-\frac{\pi}{2}\Big)$$

$$+B_n\,\mathrm{sh}\,P_n\Big(\eta-\frac{\pi}{2}\Big)\sin Q_n\Big(\eta-\frac{\pi}{2}\Big)+C_n\,\mathrm{ch}\,n\Big(\eta-\frac{\pi}{2}\Big)\Big\}\sin n\xi$$

$$+\sum_{m=1,3,5\cdots}^{\infty}\Big\{\frac{2\lambda^2}{m^3\pi(m^4+\lambda^4)}+A_m\,\mathrm{ch}\,P_m\Big(\xi-\frac{\pi}{2}\Big)\cos Q_m\Big(\xi-\frac{\pi}{2}\Big)$$

$$+B_m\,\mathrm{sh}\,P_m\Big(\xi-\frac{\pi}{2}\Big)\sin Q_m\Big(\xi-\frac{\pi}{2}\Big)+C_m\,\mathrm{ch}\,m\Big(\xi-\frac{\pi}{2}\Big)\Big\}\sin m\eta \tag{64}$$

$$W=W^{(1)}+W^{(2)}=\sum_{n=1,3,5\cdots}^{\infty}\Big\{\frac{2}{n\pi(n^4+\lambda^4)}-B_n\,\mathrm{ch}\,P_n\Big(\eta-\frac{\pi}{2}\Big)\cos Q_n\Big(\eta-\frac{\pi}{2}\Big)$$

$$+A_n \operatorname{sh} P_n\left(\eta-\frac{\pi}{2}\right)\sin Q_n\left(\eta-\frac{\pi}{2}\right)+D_n \operatorname{ch} n\left(\eta-\frac{\pi}{2}\right)\Big\}\sin n\xi$$

$$+\sum_{m=1,3,5\cdots}^{\infty}\left\{\frac{2}{m\pi(m^4+\lambda^4)}-B_m \operatorname{ch} P_m\left(\xi-\frac{\pi}{2}\right)\cos Q_m\left(\xi-\frac{\pi}{2}\right)\right.$$

$$\left.+A_m \operatorname{sh} P_m\left(\xi-\frac{\pi}{2}\right)\sin Q_m\left(\xi-\frac{\pi}{2}\right)+D_m \operatorname{ch} m\left(\xi-\frac{\pi}{2}\right)\right\}\sin m\eta \tag{65}$$

这个解不仅对于ξ，η都对称，而且对ξ，η互换时也对称．因此，它适用于方形球面扁壳在均匀分布载荷下的解．很易验证，这个解是符合简支的角点条件$W=0$的．A_n，B_n，C_n，D_n由四个边界条件(42)式决定．在满足了(42)式以后，根据对称条件，互换ξ，η，即可证明同样也适合于其它各边的边界条件．

先可以利用(42a，c)来消去两个常数，从(42a，c)得

$$\frac{2\lambda^2}{m^3\pi(m^4+\lambda^4)}+A_m \operatorname{ch}\frac{P_m\pi}{2}\cos\frac{Q_m\pi}{2}+B_m \operatorname{sh}\frac{P_m\pi}{2}\sin\frac{Q_m\pi}{2}$$

$$+C_m \operatorname{ch}\frac{m\pi}{2}=0 \tag{66}$$

$$-\frac{2m\mu}{\pi(m^4+\lambda^4)}+[\lambda^2A_m-(1-\mu)m^2B_m]\operatorname{ch}\frac{P_m\pi}{2}\cos\frac{Q_m\pi}{2}$$

$$+[(1-\mu)m^2A_m+\lambda^2B_m]\operatorname{sh}\frac{P_m\pi}{2}\sin\frac{Q_m\pi}{2}$$

$$+D_m(1-\mu)m^2\operatorname{ch}\frac{m\pi}{2}=0 \tag{67}$$

解之，得C_m及D_m的表达式

$$\left.\begin{aligned}C_m=&-\left\{\frac{2\lambda^2}{m^3\pi(m^4+\lambda^4)}+A_m \operatorname{ch}\frac{P_m\pi}{2}\cos\frac{Q_m\pi}{2}\right.\\&\left.+B_m \operatorname{sh}\frac{P_m\pi}{2}\sin\frac{Q_m\pi}{2}\right\}\operatorname{sech}\frac{m\pi}{2}\\D_m=&\left\{\frac{2\mu}{\pi m(1-\mu)(m^4+\lambda^4)}+\left[B_m-\frac{\lambda^2A_m}{m^2(1-\mu)}\right]\operatorname{ch}\frac{P_m\pi}{2}\cos\frac{Q_m\pi}{2}\right.\\&\left.-\left[A_m+\frac{\lambda^2B_m}{m^2(1-\mu)}\right]\operatorname{sh}\frac{P_m\pi}{2}\sin\frac{Q_m\pi}{2}\right\}\operatorname{sech}\frac{m\pi}{2}\end{aligned}\right\} \tag{68}$$

这样就可以消去常数C_m，D_m，于是(64)，(65)可以写成

$$\psi(\xi,\eta)=\sum_{n=1,3,5\cdots}^{\infty}\left\{\frac{2\lambda^2}{n^3\pi(n^4+\lambda^4)}\left[1-\frac{\operatorname{ch} n\left(\eta-\frac{\pi}{2}\right)}{\operatorname{ch}\frac{n\pi}{2}}\right]\right.$$

$$+A_n\left[\operatorname{ch} P_n\left(\eta-\frac{\pi}{2}\right)\cos Q_n\left(\eta-\frac{\pi}{2}\right)\right.$$

$$-\operatorname{ch}\frac{P_n\pi}{2}\cos\frac{Q_n\pi}{2}\frac{\operatorname{ch} n\left(\eta-\frac{\pi}{2}\right)}{\operatorname{ch}\frac{n\pi}{2}}\Bigg]$$

$$+B_n\left[\operatorname{sh}P_n\left(\eta-\frac{\pi}{2}\right)\sin Q_n\left(\eta-\frac{\pi}{2}\right)\right.$$

$$\left.-\operatorname{sh}\frac{P_n\pi}{2}\sin\frac{Q_n\pi}{2}\frac{\operatorname{ch} n\left(\eta-\frac{\pi}{2}\right)}{\operatorname{ch}\frac{n\pi}{2}}\right]\Bigg\}\sin n\xi$$

$$+\sum_{m=1,3,5\cdots}^{\infty}\left\{\frac{2\lambda^2}{m^3\pi(m^4+\lambda^4)}\left[1-\frac{\operatorname{ch} m\left(\xi-\frac{\pi}{2}\right)}{\operatorname{ch}\frac{m\pi}{2}}\right]\right.$$

$$+A_m\left[\operatorname{ch}P_m\left(\xi-\frac{\pi}{2}\right)\cos Q_m\left(\xi-\frac{\pi}{2}\right)\right.$$

$$\left.-\operatorname{ch}\frac{P_m\pi}{2}\cos\frac{Q_m\pi}{2}\frac{\operatorname{ch} m\left(\xi-\frac{\pi}{2}\right)}{\operatorname{ch}\frac{m\pi}{2}}\right]$$

$$+B_m\left[\operatorname{sh}P_m\left(\xi-\frac{\pi}{2}\right)\sin Q_m\left(\xi-\frac{\pi}{2}\right)\right.$$

$$\left.\left.-\operatorname{sh}\frac{P_m\pi}{2}\sin\frac{Q_m\pi}{2}\frac{\operatorname{ch} m\left(\xi-\frac{\pi}{2}\right)}{\operatorname{ch}\frac{m\pi}{2}}\right]\right\}\sin m\eta \tag{69}$$

$$W(\xi,\ \eta)=\sum_{n=1,3,5\cdots}^{\infty}\left\{\frac{2}{n\pi(n^4+\lambda^4)}\left[1+\frac{\mu}{1-\mu}\frac{\operatorname{ch} n\left(\eta-\frac{\pi}{2}\right)}{\operatorname{ch}\frac{n\pi}{2}}\right]\right.$$

$$+A_n\left[\operatorname{sh}P_n\left(\eta-\frac{\pi}{2}\right)\sin Q_n\left(\eta-\frac{\pi}{2}\right)\right.$$

$$-\operatorname{sh}\frac{P_n\pi}{2}\sin\frac{Q_n\pi}{2}\frac{\operatorname{ch} n\left(\eta-\frac{\pi}{2}\right)}{\operatorname{ch}\frac{n\pi}{2}}$$

$$\left.-\frac{\lambda^2}{(1-\mu)n^2}\operatorname{ch}\frac{P_n\pi}{2}\cos\frac{Q_n\pi}{2}\frac{\operatorname{ch} n\left(\eta-\frac{\pi}{2}\right)}{\operatorname{ch}\frac{n\pi}{2}}\right]$$

$$
\begin{aligned}
&-B_n\left[\operatorname{ch}P_n\left(\eta-\frac{\pi}{2}\right)\cos Q_n\left(\eta-\frac{\pi}{2}\right)\right.\\
&-\operatorname{ch}\frac{P_n\pi}{2}\cos\frac{Q_n\pi}{2}\,\frac{\operatorname{ch}n\left(\eta-\frac{\pi}{2}\right)}{\operatorname{ch}\frac{n\pi}{2}}\\
&\left.\left.+\frac{\lambda^2}{(1-\mu)n^2}\operatorname{sh}\frac{P_n\pi}{2}\sin\frac{Q_n\pi}{2}\,\frac{\operatorname{ch}n\left(\eta-\frac{\pi}{2}\right)}{\operatorname{ch}\frac{n\pi}{2}}\right]\right\}\sin n\xi\\
&+\sum_{m=1,3,5\cdots}^{\infty}\left\{\frac{2}{m\pi(m^4+\lambda^4)}\left[1+\frac{\mu}{1-\mu}\,\frac{\operatorname{ch}m\left(\xi-\frac{\pi}{2}\right)}{\operatorname{ch}\frac{m\pi}{2}}\right]\right.\\
&+A_m\left[\operatorname{sh}P_m\left(\xi-\frac{\pi}{2}\right)\sin Q_m\left(\xi-\frac{\pi}{2}\right)\right.\\
&-\operatorname{sh}\frac{P_m\pi}{2}\sin\frac{Q_m\pi}{2}\,\frac{\operatorname{ch}m\left(\xi-\frac{\pi}{2}\right)}{\operatorname{ch}\frac{m\pi}{2}}\\
&\left.-\frac{\lambda^2}{(1-\mu)m^2}\operatorname{ch}\frac{P_m\pi}{2}\cos\frac{Q_m\pi}{2}\,\frac{\operatorname{ch}m\left(\xi-\frac{\pi}{2}\right)}{\operatorname{ch}\frac{m\pi}{2}}\right]\\
&-B_m\left[\operatorname{ch}P_m\left(\xi-\frac{\pi}{2}\right)\cos Q_m\left(\xi-\frac{\pi}{2}\right)\right.\\
&-\operatorname{ch}\frac{P_m\pi}{2}\cos\frac{Q_m\pi}{2}\,\frac{\operatorname{ch}m\left(\xi-\frac{\pi}{2}\right)}{\operatorname{ch}\frac{m\pi}{2}}\\
&\left.\left.+\frac{\lambda^2}{(1-\mu)m^2}\operatorname{sh}\frac{P_m\pi}{2}\sin\frac{Q_m\pi}{2}\,\frac{\operatorname{ch}m\left(\xi-\frac{\pi}{2}\right)}{\operatorname{ch}\frac{m\pi}{2}}\right]\right\}\sin m\eta \qquad (70)
\end{aligned}
$$

或

$$
\psi(\xi,\ \eta)=\sum_{n=1,3,5\cdots}^{\infty}\left\{\frac{2\lambda^2}{n^3\pi(n^4+\lambda^4)}\left[1-\frac{\operatorname{ch}n\left(\eta-\frac{\pi}{2}\right)}{\operatorname{ch}\frac{n\pi}{2}}\right]+A_n\Phi_n^1(\eta)+B_n\Phi_n^2(\eta)\right\}\sin n\xi
$$

$$+\sum_{m=1,3,5\cdots}^{\infty}\left\{\frac{2\lambda^2}{m^3\pi(m^4+\lambda^4)}\left[1-\frac{\operatorname{ch} n\left(\xi-\frac{\pi}{2}\right)}{\operatorname{ch}\frac{n\pi}{2}}\right]+A_m\Phi_m^1(\xi)+B_m\Phi_m^2(\xi)\right\}\sin m\eta$$

(71A)

$$W(\xi,\ \eta)=\sum_{n=1,3,5\cdots}^{\infty}\left\{\frac{2}{n\pi(n^4+\lambda^4)}\left[1+\frac{\mu}{1-\mu}\frac{\operatorname{ch} n\left(\eta-\frac{\pi}{2}\right)}{\operatorname{ch}\frac{n\pi}{2}}\right]+A_n\Phi_n^3(\eta)\right.$$

$$\left.-B_n\Phi_n^4(\eta)\right\}\sin n\xi+\sum_{m=1,3,5\cdots}^{\infty}\left\{\frac{2}{m\pi(m^4+\lambda^4)}\left[1+\frac{\mu}{1-\mu}\frac{\operatorname{ch} m\left(\xi-\frac{\pi}{2}\right)}{\operatorname{ch}\frac{m\pi}{2}}\right]\right.$$

$$\left.+A_m\Phi_m^3(\xi)-B_m\Phi_m^4(\xi)\right\}\sin m\eta \tag{71B}$$

其中函数$\Phi_m^1(\xi)$，$\Phi_m^2(\xi)$，$\Phi_m^3(\xi)$，$\Phi_m^4(\xi)$的表达式为

$$\left.\begin{aligned}
\Phi_m^1(\xi)&=\operatorname{ch} P_m\left(\xi-\frac{\pi}{2}\right)\cos Q_m\left(\xi-\frac{\pi}{2}\right)-\operatorname{ch}\frac{P_m\pi}{2}\cos\frac{Q_m\pi}{2}\frac{\operatorname{ch} m\left(\xi-\frac{\pi}{2}\right)}{\operatorname{ch}\frac{m\pi}{2}}\\
\Phi_m^2(\xi)&=\operatorname{sh} P_m\left(\xi-\frac{\pi}{2}\right)\sin Q_m\left(\xi-\frac{\pi}{2}\right)-\operatorname{sh}\frac{P_m\pi}{2}\sin\frac{Q_m\pi}{2}\frac{\operatorname{ch} m\left(\xi-\frac{\pi}{2}\right)}{\operatorname{ch}\frac{m\pi}{2}}\\
\Phi_m^3(\xi)&=\operatorname{sh} P_m\left(\xi-\frac{\pi}{2}\right)\sin Q_m\left(\xi-\frac{\pi}{2}\right)-\left(\operatorname{sh}\frac{P_m\pi}{2}\sin\frac{Q_m\pi}{2}\right.\\
&\quad\left.+\frac{\lambda^2}{(1-\mu)m^2}\operatorname{ch}\frac{P_m\pi}{2}\cos\frac{Q_m\pi}{2}\right)\frac{\operatorname{ch} m\left(\xi-\frac{\pi}{2}\right)}{\operatorname{ch}\frac{m\pi}{2}}\\
\Phi_m^4(\xi)&=\operatorname{ch} P_m\left(\xi-\frac{\pi}{2}\right)\cos Q_m\left(\xi-\frac{\pi}{2}\right)-\left(\operatorname{ch}\frac{P_m\pi}{2}\cos\frac{Q_m\pi}{2}\right.\\
&\quad\left.-\frac{\lambda^2}{(1-\mu)m^2}\operatorname{sh}\frac{P_m\pi}{2}\sin\frac{Q_m\pi}{2}\right)\frac{\operatorname{ch} m\left(\xi-\frac{\pi}{2}\right)}{\operatorname{ch}\frac{m\pi}{2}}
\end{aligned}\right\} \tag{72}$$

将m，ξ和n，η置换，即得$\Phi_n^1(\eta)$，$\Phi_n^2(\eta)$，$\Phi_n^3(\eta)$，$\Phi_n^4(\eta)$的表达式.

现在让我们通过(46)式，积分求解U，V. 从(46)式有

$$\left.\begin{aligned}
&\frac{\partial U}{\partial\xi}=\frac{\partial^2\psi}{\partial\eta^2}-\mu\frac{\partial^2\psi}{\partial\xi^2}+\lambda^2W,\quad \frac{\partial V}{\partial\eta}=\frac{\partial^2\psi}{\partial\xi^2}-\mu\frac{\partial^2\psi}{\partial\eta^2}+\lambda^2W\\
&\frac{\partial U}{\partial\eta}+\frac{\partial V}{\partial\xi}=-2(1+\mu)\frac{\partial^2\psi}{\partial\eta\partial\xi}
\end{aligned}\right\} \tag{73}$$

把(71)式代入(73)式，得

$$\frac{\partial U}{\partial \xi}=\sum_{n=1,3,5\cdots}^{\infty}\left\{\frac{2\lambda^2}{n\pi(n^4+\lambda^4)}\left[(1+\mu)-\left(1+\mu-\frac{\mu}{1-\mu}\right)\frac{\operatorname{ch} m\left(\eta-\frac{\pi}{2}\right)}{\operatorname{ch}\frac{n\pi}{2}}\right]\right.$$

$$+A_n[\Phi_n^5(\eta)+\mu n^2\Phi_n^1(\eta)+\lambda^2\Phi_n^3(\eta)]+B_n[\Phi_n^6(\eta)+\mu n^2\Phi_n^2(\eta)$$

$$-\lambda^2\Phi_n^4(\eta)]\sin n\xi\sum_{m=1,3,5\cdots}^{\infty}\left\{\frac{2\lambda^2}{m\pi(m^4+\lambda^4)}\left[(1+\mu)+\frac{\mu}{1-\mu}\right]\frac{\operatorname{ch} m\left(\xi-\frac{\pi}{2}\right)}{\operatorname{ch}\frac{m\pi}{2}}\right.$$

$$+A_m[-m^2\Phi_m^1(\xi)-\mu\Phi_m^5(\xi)+\lambda^2\Phi_m^3(\xi)]+B_m[-m^2\Phi_m^2(\xi)-\mu\Phi_m^6(\xi)$$

$$\left.-\lambda^2\Phi_m^4(\xi)]\right\}\sin m\eta \tag{74A}$$

$$\frac{\partial V}{\partial \eta}=\sum_{n=1,3,5\cdots}^{\infty}\left\{\frac{2\lambda^2}{n\pi(n^4+\lambda^4)}\left[(1+\mu)+\frac{\mu}{1-\mu}\right]\frac{\operatorname{ch} m\left(\eta-\frac{\pi}{2}\right)}{\operatorname{ch}\frac{n\pi}{2}}\right.$$

$$+A_n[-n^2\Phi_n^1(\eta)-\mu\Phi_n^5(\eta)+\lambda^2\Phi_n^3(\eta)]+B_n[-n^2\Phi_n^2(\eta)-\mu\Phi_n^6(\eta)-\lambda^2\Phi_n^4(\eta)]\sin n\xi$$

$$+\sum_{m=1,3,5\cdots}^{\infty}\left\{\frac{2\lambda^2}{m\pi(m^4+\lambda^4)}\left[(1+\mu)-\left(1+\mu-\frac{\mu}{1-\mu}\right)\frac{\operatorname{ch} m\left(\xi-\frac{\pi}{2}\right)}{\operatorname{ch}\frac{m\pi}{2}}\right]\right.$$

$$\left.+A_m[\Phi_m^5(\xi)+\mu m^2\Phi_m^1(\xi)+\lambda^2\Phi_m^3(\xi)]+B_m[\Phi_m^6(\xi)+\mu m^2\Phi_m^2(\xi)-\lambda^2\Phi_m^4(\xi)]\right\}\sin m\eta \tag{74B}$$

$$\frac{\partial U}{\partial \eta}+\frac{\partial V}{\partial \xi}=-2(1+\mu)\sum_{n=1,3,5\cdots}^{\infty}n\left\{\frac{-2\lambda^2}{n^2\pi(n^4+\lambda^4)}\frac{\operatorname{sh} n\left(\eta-\frac{\pi}{2}\right)}{\operatorname{ch}\frac{n\pi}{2}}+A_n\Phi_n^7(\eta)\right.$$

$$\left.+B_n\Phi_n^8(\eta)\right\}\cos n\xi-2(1+\mu)\sum_{m=1,3,5\cdots}^{\infty}m\left\{\frac{-2\lambda^2}{m^2\pi(m^4+\lambda^4)}\frac{\operatorname{sh} m\left(\xi-\frac{\pi}{2}\right)}{\operatorname{ch}\frac{m\pi}{2}}\right.$$

$$\left.+A_m\Phi_m^7(\xi)+B_m\Phi_m^8(\xi)\right\}\cos m\eta \tag{74C}$$

其中

$$\Phi_m^5(\xi)=\frac{\partial^2\Phi_m^1(\xi)}{\partial\xi^2}=m^2\operatorname{ch}P_m\left(\xi-\frac{\pi}{2}\right)\cos Q_m\left(\xi-\frac{\pi}{2}\right)$$

$$-\lambda^2\operatorname{sh}P_m\left(\xi-\frac{\pi}{2}\right)\sin Q_m\left(\xi-\frac{\pi}{2}\right)-m^2\operatorname{ch}\frac{P_m\pi}{2}\cos\frac{Q_m\pi}{2}\frac{\operatorname{ch} m\left(\xi-\frac{\pi}{2}\right)}{\operatorname{ch}\frac{m\pi}{2}} \tag{75A}$$

$$\Phi_m^6(\xi)=\frac{\partial^2\Phi_m^2(\xi)}{\partial\xi^2}=m^2\operatorname{sh}P_m\left(\xi-\frac{\pi}{2}\right)\sin Q_m\left(\xi-\frac{\pi}{2}\right)$$

$$+\lambda^2\operatorname{ch}P_m\left(\xi-\frac{\pi}{2}\right)\cos Q_m\left(\xi-\frac{\pi}{2}\right)-m^2\operatorname{sh}\frac{P_m\pi}{2}\sin\frac{Q_m\pi}{2}\frac{\operatorname{ch}m\left(\xi-\frac{\pi}{2}\right)}{\operatorname{ch}\frac{m\pi}{2}} \tag{75B}$$

$$\Phi_m^7(\xi)=\frac{\partial\Phi_m^1(\xi)}{\partial\xi}=P_m\operatorname{sh}P_m\left(\xi-\frac{\pi}{2}\right)\cos Q_m\left(\xi-\frac{\pi}{2}\right)$$

$$-Q_m\operatorname{ch}P_m\left(\xi-\frac{\pi}{2}\right)\sin Q_m\left(\xi-\frac{\pi}{2}\right)-m\operatorname{ch}\frac{P_m\pi}{2}\cos\frac{Q_m\pi}{2}\frac{\operatorname{sh}m\left(\xi-\frac{\pi}{2}\right)}{\operatorname{ch}\frac{m\pi}{2}} \tag{75C}$$

$$\Phi_m^8(\xi)=\frac{\partial\Phi_m^2(\xi)}{\partial\xi}=P_m\operatorname{ch}P_m\left(\xi-\frac{\pi}{2}\right)\sin Q_m\left(\xi-\frac{\pi}{2}\right)$$

$$+Q_m\operatorname{sh}P_m\left(\xi-\frac{\pi}{2}\right)\cos Q_m\left(\xi-\frac{\pi}{2}\right)-m\operatorname{sh}\frac{P_m\pi}{2}\sin\frac{Q_m\pi}{2}\frac{\operatorname{sh}m\left(\xi-\frac{\pi}{2}\right)}{\operatorname{ch}\frac{m\pi}{2}} \tag{75D}$$

$\Phi_n^5(\eta)$, $\Phi_n^6(\eta)$, $\Phi_n^7(\eta)$, $\Phi_n^8(\eta)$可以通过n, η和 m, ξ的互换从(75A, B, C, D)中求得.

积分(74A, B),并利用积分公式

$$\Phi_m^{13}(\xi)=\int\Phi_m^1(\xi)d\xi=\frac{P_m}{\sqrt{m^4+\lambda^4}}\operatorname{sh}P_m\left(\xi-\frac{\pi}{2}\right)\cos Q_m\left(\xi-\frac{\pi}{2}\right)$$

$$+\frac{Q_m}{\sqrt{m^4+\lambda^4}}\operatorname{ch}P_m\left(\xi-\frac{\pi}{2}\right)\sin Q_m\left(\xi-\frac{\pi}{2}\right)$$

$$-\frac{1}{m}\operatorname{ch}\frac{P_m\pi}{2}\cos\frac{Q_m\pi}{2}\frac{\operatorname{sh}m\left(\xi-\frac{\pi}{2}\right)}{\operatorname{ch}\frac{m\pi}{2}} \tag{76A}$$

$$\Phi_m^{14}(\xi)=\int\Phi_m^2(\xi)d\xi=\frac{-Q_m}{\sqrt{m^4+\lambda^4}}\operatorname{sh}P_m\left(\xi-\frac{\pi}{2}\right)\cos Q_m\left(\xi-\frac{\pi}{2}\right)$$

$$+\frac{P_m}{\sqrt{m^4+\lambda^4}}\operatorname{ch}P_m\left(\xi-\frac{\pi}{2}\right)\sin Q_m\left(\xi-\frac{\pi}{2}\right)$$

$$-\frac{1}{m}\operatorname{sh}\frac{P_m\pi}{2}\sin\frac{Q_m\pi}{2}\frac{\operatorname{sh}m\left(\xi-\frac{\pi}{2}\right)}{\operatorname{ch}\frac{m\pi}{2}} \tag{76B}$$

$$\Phi_m^{15}(\xi)=\int\Phi_m^3(\xi)d\xi=\frac{-Q_m}{\sqrt{m^4+\lambda^4}}\operatorname{sh}P_m\left(\xi-\frac{\pi}{2}\right)\cos Q_m\left(\xi-\frac{\pi}{2}\right)$$

$$+\frac{P_m}{\sqrt{m^4+\lambda^4}}\text{ch}\,P_m\left(\xi-\frac{\pi}{2}\right)\sin Q_m\left(\xi-\frac{\pi}{2}\right)$$

$$-\frac{1}{m}\left(\text{sh}\frac{P_m\pi}{2}\sin\frac{Q_m\pi}{2}+\frac{\lambda^2}{(1-\mu)m^2}\text{ch}\frac{P_m\pi}{2}\cos\frac{Q_m\pi}{2}\right)\frac{\text{sh}\,m\left(\xi-\frac{\pi}{2}\right)}{\text{ch}\frac{m\pi}{2}} \tag{76C}$$

$$\Phi_m^{16}(\xi)=\int\Phi_m^4(\xi)d\xi=\frac{p_m}{\sqrt{m^4+\lambda^4}}\text{sh}\,P_m\left(\xi-\frac{\pi}{2}\right)\cos Q_m\left(\xi-\frac{\pi}{2}\right)$$

$$+\frac{Q_m}{\sqrt{m^4+\lambda^4}}\text{ch}\,P_m\left(\xi-\frac{\pi}{2}\right)\sin Q_m\left(\xi-\frac{\pi}{2}\right)$$

$$-\frac{1}{m}\left(\text{ch}\frac{P_m\pi}{2}\cos\frac{Q_m\pi}{2}-\frac{\lambda^2}{(1-\mu)m^2}\text{sh}\frac{P_m\pi}{2}\sin\frac{Q_m\pi}{2}\right)\frac{\text{sh}\,m\left(\xi-\frac{\pi}{2}\right)}{\text{ch}\frac{m\pi}{2}} \tag{76D}$$

从(74A，B)中积分求得

$$U(\xi,\ \eta)=U_0(\eta)-\sum_{n=1,3,5\cdots}^{\infty}\frac{1}{n}\left\{\frac{2\lambda^2}{n\pi(n^4+\lambda^4)}\left[(1+\mu)\right.\right.$$

$$\left.-\left(1+\mu-\frac{\mu}{1-\mu}\right)\frac{\text{ch}\,n\left(\eta-\frac{\pi}{2}\right)}{\text{ch}\frac{n\pi}{2}}\right]+A_n[\Phi_n^5(\eta)+\mu n^2\Phi_n^1(\eta)$$

$$\left.+\lambda^2\Phi_n^3(\eta)]+B_n[\Phi_n^6(\eta)+\mu n^2\Phi_n^2(\eta)-\lambda^2\Phi_n^4(\eta)]\right\}\cos n\xi$$

$$+\sum_{m=1,3,5\cdots}^{\infty}\left\{\frac{2\lambda^2}{m^2\pi(m^4+\lambda^4)}\left[1+\mu+\frac{\mu}{1-\mu}\right]\frac{\text{sh}\,m\left(\xi-\frac{\pi}{2}\right)}{\text{ch}\frac{m\pi}{2}}\right.$$

$$+A_m[-m^2\Phi_m^{13}(\xi)-\mu\Phi_m^7(\xi)+\lambda^2\Phi_m^{15}(\xi)$$

$$\left.+B_m[-m^2\Phi_m^{14}(\xi)-\mu\Phi_m^8(\xi)-\lambda^2\Phi_m^{16}(\xi)]\right\}\sin m\eta \tag{77A}$$

$$V(\xi,\ \eta)=V_0(\xi)+\sum_{n=1,3,5\cdots}^{\infty}\left\{\frac{2\lambda^2}{n^2\pi(n^4+\lambda^4)}\left[1+\mu+\frac{\mu}{1-\mu}\right]\frac{\text{sh}\,n\left(\eta-\frac{\pi}{2}\right)}{\text{ch}\frac{n\pi}{2}}\right.$$

$$+A_n[-n^2\Phi_n^{13}(\eta)-\mu\Phi_n^7(\eta)+\lambda^2\Phi_n^{15}(\eta)]$$

$$\left.+B_n[-n^2\Phi_n^{14}(\eta)-\mu\Phi_n^8(\eta)-\lambda^2\Phi_n^{16}(\eta)]\right\}\sin n\xi$$

$$-\sum_{m=1,3,5\cdots}^{\infty}\frac{1}{m}\left\{\frac{2\lambda^2}{m\pi(m^4+\lambda^4)}\left[1+\mu-\left(1+\mu-\frac{\mu}{1-\mu}\right)\frac{\text{ch}\,m\left(\xi-\frac{\pi}{2}\right)}{\text{ch}\frac{m\pi}{2}}\right]\right.$$

$$+A_m[\Phi_m^5(\xi)+\mu m^2\Phi_m^1(\xi)+\lambda^2\Phi_m^3(\xi)]$$
$$\left.+B_m[\Phi_m^6(\xi)+\mu m^2\Phi_m^2(\xi)-\lambda^2\Phi_m^4(\xi)]\right\}\cos m\eta \tag{77B}$$

其中$U_0(\eta)$，$V_0(\xi)$分别为积分函数．将(77A，B)代入(74C)即得

$$\frac{dU_0(\eta)}{d\eta}=\frac{-dV_0(\xi)}{d\xi} \tag{78}$$

式左端为η的函数，右端为ξ的函数，两者恒等，必为一常数，设这一常数为α，分别积分，得

$$U_0(\eta)=\alpha\eta+\beta,\qquad V_0(\xi)=-\alpha\xi+\gamma \tag{79}$$

由于U，V，ξ，η的对称性质，及其互换对称性质，即可证明

$$\alpha=0,\qquad \beta=\gamma=0 \tag{80}$$

或即

$$U_0(\eta)=0,\qquad V_0(\xi)=0 \tag{81}$$

(71A，B)，和(77A，B)为业已满足了边界条件(42a，c)和(43a，c)的解．$A_n(A_m)$及B_n(B_m)将由其余的边界条件及角点条件决定．

我们引进无量纲薄膜内力

$$\left.\begin{aligned}
N_\xi&=\frac{\partial^2\psi}{\partial\eta^2}=\frac{1}{1-\mu^2}\left\{\left(\frac{\partial U}{\partial\xi}-\lambda^2W\right)+\mu\left(\frac{\partial V}{\partial\eta}-\lambda^2W\right)\right\}\\
N_\eta&=\frac{\partial^2\psi}{\partial\xi^2}=\frac{1}{1-\mu^2}\left\{\left(\frac{\partial V}{\partial\eta}-\lambda^2W\right)+\mu\left(\frac{\partial U}{\partial\xi}-\lambda^2W\right)\right\}\\
N_{\xi\eta}&=-\frac{\partial^2\psi}{\partial\xi\partial\eta}=\frac{1}{2(1+\mu)}\left(\frac{\partial U}{\partial\eta}+\frac{\partial V}{\partial\xi}\right)
\end{aligned}\right\} \tag{82}$$

和无量纲弯矩以及扭矩

$$\left.\begin{aligned}
&M_\xi=-\left(\frac{\partial^2W}{\partial\xi^2}+\mu\frac{\partial^2W}{\partial\eta^2}\right),\qquad M_\eta=-\left(\frac{\partial^2W}{\partial\eta^2}+\mu\frac{\partial^2W}{\partial\xi^2}\right)\\
&M_{\xi\eta}=-(1-\mu)\frac{\partial^2W}{\partial\xi\partial\eta}
\end{aligned}\right\} \tag{83}$$

它们和有量钢的内力素N_x、N_y、N_{xy}、M_x、M_y及M_{xy}的关系为

$$\left.\begin{aligned}
&N_x=\frac{qa^2}{h\pi^2}\sqrt{12(1-\mu^2)}N_\xi,\quad N_y=\frac{qa^2}{h\pi^2}\sqrt{12(1-\mu^2)}N_\eta,\\
&N_{xy}=\frac{-qa^2}{h\pi^2}\sqrt{12(1-\mu^2)}N_{\xi\eta}\qquad M_x=\frac{qa^2}{\pi^2}M_\xi,\\
&M_y=\frac{qa^2}{\pi^2}M_\eta,\quad M_{xy}=-\frac{qa^2}{\pi^2}M_{\xi\eta}
\end{aligned}\right\} \tag{84}$$

这些无量纲内力素和Дикович[2]所用符号有相同的意义，但仅相差一正负号．

于是，边界条件(42)和(43)可以写成：

（1）在边界OA上，即当$\xi=0$时

$$\left.\begin{aligned}
&(a)\quad N_{\xi}=0\\
&(b)\quad N_{\xi\eta}+f\frac{\partial}{\partial\eta}\left(\frac{\partial V}{\partial\eta}-\lambda^2W-\tau\frac{\partial^2W}{\partial\eta^2}\right)=0\\
&(c)\quad M_{\xi}=0\\
&(d)\quad -\frac{\partial M_{\xi}}{\partial\xi}-2\frac{\partial M_{\xi\eta}}{\partial\eta}-f\lambda^2\left[\frac{\partial V}{\partial\eta}-\lambda^2W-\tau\frac{\partial^2W}{\partial\eta^2}\right]\\
&\qquad +G\frac{\partial^4W}{\partial\eta^4}-\tau f\frac{\partial^2}{\partial\eta^2}\left(\frac{\partial V}{\partial\eta}-\lambda^2W\right)-f=0
\end{aligned}\right\}\tag{85}$$

（2）在边界OC上，即当$\eta=0$时

$$\left.\begin{aligned}
&(a)\quad N_{\eta}=0\\
&(b)\quad N_{\xi\eta}+f\frac{\partial}{\partial\xi}\left(\frac{\partial U}{\partial\xi}-\lambda^2W-\tau\frac{\partial^2W}{\partial\xi^2}\right)=0\\
&(c)\quad M_{\eta}=0\\
&(d)\quad -\frac{\partial M_{\eta}}{\partial\eta}-2\frac{\partial M_{\xi\eta}}{\partial\xi}-f\lambda^2\left[\frac{\partial U}{\partial\xi}-\lambda^2W-\tau\frac{\partial^2W}{\partial\xi^2}\right]\\
&\qquad +G\frac{\partial^4W}{\partial\xi^4}-\tau f\frac{\partial^2}{\partial\xi^2}\left(\frac{\partial U}{\partial\xi}-\lambda^2W\right)-f=0
\end{aligned}\right\}\tag{86}$$

在$AB(\eta=\pi)$，$CD(\xi=\pi)$上，可以根据对称条件求得相应的边界条件．

角点条件只要写出O角$(\xi=\eta=0)$的条件，其它角点上，条件相同．

在角点O上，即$\xi=\eta=0$上，有铰支条件

$$\left.\begin{aligned}
&(a)\quad G\frac{\partial^2W}{\partial\xi^2}-f\tau\left(\frac{\partial U}{\partial\xi}-\lambda^2W\right)=0\\
&(b)\quad G\frac{\partial^2W}{\partial\eta^2}-f\tau\left(\frac{\partial V}{\partial\eta}-\lambda^2W\right)=0\\
&(c)\quad W=0,\ (d)\quad U=0,\ (e)\quad V=0
\end{aligned}\right\}\tag{87}$$

如果在角点上允许向外平移ΔU，ΔV，则$(87d, e)$改写为$U=-\Delta U$，$V=-\Delta V$．角点上的反作用力的水平分量T_x，T_y为

$$\left.\begin{aligned}
T_x&=\frac{qa^3}{h\pi^3}\sqrt{12(1-\mu)}\,f\left(\frac{\partial U}{\partial\xi}-\lambda^2W-\tau\frac{\partial^2W}{\partial\xi^2}\right)\\
T_y&=\frac{qa^3}{h\pi^3}\sqrt{12(1-\mu)}\,f\left(\frac{\partial V}{\partial\eta}-\lambda^2W-\tau\frac{\partial^2W}{\partial\eta^2}\right)
\end{aligned}\right\}(\xi=\eta=0)\tag{88}$$

在我们这个问题中，由于对称性

$$T_y=T_x=\frac{qa^3}{h\pi^3}\sqrt{12(1-\mu)}\,T_0(0,\ 0)\tag{89A}$$

$$\begin{aligned}
T_0(0,\ 0)=T_0&=f\left(\frac{\partial U}{\partial\xi}-\lambda^2W-\tau\frac{\partial^2W}{\partial\xi^2}\right)_{\xi=\eta=0}\\
&=f\left(\frac{\partial V}{\partial\eta}-\lambda^2W-\tau\frac{\partial^2W}{\partial\eta^2}\right)_{\xi=\eta=0}
\end{aligned}\tag{89B}$$

现在让我们讨论一下(85)式中四个边界条件的物理意义，$N_{\xi\eta}$，N_{ξ}，$Q_{\xi}^*=\dfrac{\partial M_{\xi}}{\partial\xi}$

$+2\frac{\partial M_{\xi\eta}}{\partial\eta}$为通过壳的边界作用在拱上的水平剪力，轴向力和广义横向剪力，M_ξ为边拱所受的扭矩．由于我们略去了水平方向的抗弯刚度和它的抗扭刚度，因此，$N_\xi=0, M_\xi=0$.这就是说，(85B)式为拱的轴向力的平衡方程，(85D)式为拱的横向力的平衡方程．我们很容易从(88)中看到(85B)，(85D)中的$f\left(\frac{\partial V}{\partial\eta}-\lambda^2W-\tau\frac{\partial^2W}{\partial\eta^2}\right)$代表拱的轴向内力，$G\frac{\partial^2W}{\partial\eta^2}-f\tau\left(\frac{\partial V}{\partial\eta}-\lambda^2W\right)$代表拱内所受的弯矩．为了易于认识，我们以后称

$$\text{在}\xi=0\text{上}\quad \left.\begin{aligned}T_\eta(\eta)&=f\left(\frac{\partial V}{\partial\eta}-\lambda^2W-\tau\frac{\partial^2W}{\partial\eta^2}\right)\\ G_\eta(\eta)&=G\frac{\partial^2W}{\partial\eta^2}-f\tau\left(\frac{\partial V}{\partial\eta}-\lambda^2W\right)\end{aligned}\right\}\tag{90}$$

于是边界条件(85)式可以写成

在$\xi=0$上，

$$\left.\begin{aligned}&(a)\quad N_\xi=0,\quad (b)\quad N_{\xi\eta}+\frac{dT_\eta}{d\eta}=0,\quad (c)\quad M_\xi=0\\ &(d)\quad -\frac{dM_\xi}{d\xi}-2\frac{\partial M_{\xi\eta}}{\partial\eta}-\lambda^2T_\eta+\frac{d^2G_\eta}{d\eta^2}-f=0\end{aligned}\right\}\tag{91}$$

同样如果引进

$$\text{在}\eta=0\text{上，}\quad \left.\begin{aligned}T_\xi(\xi)&=f\left(\frac{\partial U}{\partial\xi}-\lambda^2W-\tau\frac{\partial^2W}{\partial\xi^2}\right)\\ G_\xi(\xi)&=G\frac{\partial^2W}{\partial\xi^2}-f\tau\left(\frac{\partial U}{\partial\xi}-\lambda^2W\right)\end{aligned}\right\}\tag{92}$$

则边界条件(86)可以写成

在$\eta=0$上

$$\left.\begin{aligned}&(a)\ N_\eta=0,\quad (b)\ N_{\xi\eta}+\frac{dT_\xi}{d\xi}=0,\ (c)\ M_\eta=0\\ &(d)\ -\frac{dM_\eta}{d\eta}-2\frac{\partial M_{\xi\eta}}{\partial\xi}-\lambda^2T_\xi+\frac{d^2G_\xi}{d\xi^2}-f=0\end{aligned}\right\}\tag{93}$$

而在角点O($\xi=\eta=0$上)，或在其它角点上，有条件：

对铰支角点的情形

$$G_\xi=0,\ G_\eta=0,\ W=0,\ U=0,\ V=0\tag{94}$$

对简支角点的情形

$$G_\xi=0,\ G_\eta=0,\ W=0,\ T_\xi=0,\ T_\eta=0\tag{95}$$

在下面，我们将利用这个壳内的解在边界上的下列各量的表达式：(一) $U(\xi,0)$，$W(\xi,0)$，(二) $V(0,\eta)$，$W(0,\eta)$，(三) $N_{\xi\eta}(\xi,0)$，$Q^*_\eta(\xi,0)$，(四) $N_{\xi\eta}(0,\eta)$，$Q^*_\xi(0,\eta)$．从(70)，(77A，B)，我们有

$$U(\xi,0)=-\sum_{n=1,3,5\cdots}^{\infty}\frac{1}{n}\left\{\frac{\mu}{1-\mu}\frac{2\lambda^2}{n\pi(n^4+\lambda^4)}-A_n\left[\lambda^2\,\mathrm{sh}\frac{P_n\pi}{2}\sin\frac{Q_n\pi}{2}\right.\right.$$

$$+\frac{\lambda^4}{n^2(1-\mu^2)}\text{ch}\frac{P_n\pi}{2}\cos\frac{Q_n\pi}{2}\Big]+B_n\Big[\lambda^2\,\text{ch}\frac{P_n\pi}{2}\cos\frac{Q_n\pi}{2}$$

$$-\frac{\lambda^4}{n^2(1-\mu^2)}\text{sh}\frac{P_n\pi}{2}\sin\frac{Q_n\pi}{2}\Big]\Big\}\cos n\xi \tag{96A}$$

$$W(\xi,0)=\sum_{n=1,3,5\cdots}^{\infty}\Big\{\frac{2}{(1-\mu)}\frac{1}{n\pi(n^4+\lambda^4)}-A_n\Big[\frac{\lambda^2}{n^2(1-\mu^2)}\Big]\text{ch}\frac{P_n\pi}{2}\cos\frac{Q_n\pi}{2}$$

$$-B_n\Big[\frac{\lambda^2}{n^2(1-\mu^2)}\text{sh}\frac{P_n\pi}{2}\sin\frac{Q_n\pi}{2}\Big\}\sin n\xi \tag{96B}$$

$$V(0,\eta)=-\sum_{m=1,3,5\cdots}^{\infty}\frac{1}{m}\Big\{\frac{\mu}{1-\mu}\frac{2\lambda^2}{m\pi(m^4+\lambda^4)}-A_m\Big[\lambda^2\,\text{sh}\frac{P_m\pi}{2}\sin\frac{Q_m\pi}{2}$$

$$+\frac{\lambda^4}{m^2(1-\mu^2)}\text{ch}\frac{P_m\pi}{2}\cos\frac{Q_m\pi}{2}\Big]+B_m\Big[\lambda^2\,\text{ch}\frac{P_m\pi}{2}\cos\frac{Q_m\pi}{2}$$

$$-\frac{\lambda^4}{m^2(1-\mu^2)}\text{sh}\frac{P_m\pi}{2}\sin\frac{Q_m\pi}{2}\Big]\Big\}\cos m\eta \tag{96C}$$

$$W(0,\eta)=\sum_{m=1,3,5\cdots}^{\infty}\Big\{\frac{2}{(1-\mu)}\frac{1}{m\pi(m^4+\lambda^4)}-A_m\Big[\frac{\lambda^2}{m^2(1-\mu^2)}\Big]\text{ch}\frac{P_m\pi}{2}\cos\frac{Q_m\pi}{2}$$

$$-B_m\Big[\frac{\lambda^2}{m^2(1-\mu^2)}\Big]\text{sh}\frac{P_m\pi}{2}\sin\frac{Q_m\pi}{2}\Big\}\sin m\eta \tag{96D}$$

同样，根据定义(82)、(83)，我们可以计算 $N_{\xi\eta}$，Q_ξ^*、Q_η^* 在边界上的值。

$$N_{\xi\eta}(\xi,0)=-\sum_{n=1,3,5\cdots}^{\infty}n\Big\{\frac{2\lambda^2}{n^2\pi(n^4+\lambda^4)}+h\frac{n\pi}{2}+A_n\Phi_n^7(0)+B_n\Phi_n^8(0)\Big\}\cos n\xi$$

$$-\sum_{m=1,3,5\cdots}^{\infty}m\left\{\frac{-2\lambda^2}{m^2\pi(m^4+\lambda^4)}\frac{\text{sh}\,m\left(\xi-\frac{\pi}{2}\right)}{\text{ch}\frac{m\pi}{2}}+A_m\Phi_m^7(\xi)+B_m\Phi_m^8(\xi)\right\} \tag{97A}$$

$$Q_\eta^*(\xi,0)=\left(\frac{\partial M_\eta}{\partial\eta}+2\frac{\partial M_{\xi\eta}}{\partial\xi}\right)_{\eta=0}=-\frac{\partial^3W}{\partial\eta^3}+(2-\mu)\frac{\partial^3W}{\partial\eta\partial\xi^2}\Big]_{\eta=0}$$

$$=-\sum_{n=1,3,5\cdots}^{\infty}\Big\{\frac{2n^2\mu}{n(n^4+\lambda^4)}+h\frac{n\pi}{2}+\alpha_n^1A_n-\beta_n^1B_n\Big\}\sin n\xi$$

$$+\sum_{m=1,3,5\cdots}^{\infty}m\left\{\frac{2m}{m(m^4+\lambda^4)}\left[1-\mu\frac{\text{ch}\,m\left(\xi-\frac{\pi}{2}\right)}{\text{ch}\frac{m\pi}{2}}\right]\right.$$

$$\left.+A_m\Phi_m^{17}(\xi)-B_m\Phi_m^{18}(\xi)\right\} \tag{97B}$$

$$N_{\xi\eta}(0,\ \eta)=-\sum_{n=1,3,5\cdots}^{\infty}n\left\{\frac{-2\lambda^2}{n^2\pi(n^4+\lambda^4)}\frac{\operatorname{sh}n\left(\eta-\frac{\pi}{2}\right)}{\operatorname{ch}\frac{n\pi}{2}}+A_n\Phi_n^7(\eta)+B_n\Phi_n^8(\eta)\right\}$$

$$-\sum_{m=1,3,5\cdots}^{\infty}m\left\{\frac{2\lambda^2}{m^2\pi(m^4+\lambda^4)}\operatorname{th}\frac{m\pi}{2}+A_m\Phi_m^7(0)+B_m\Phi_m^8(0)\right\}\cos m\eta \tag{97C}$$

$$Q_\xi^*(0,\ \eta)=\left(\frac{\partial M_\xi}{\partial\xi}+2\frac{\partial M_{\xi\eta}}{\partial\eta}\right)_{\xi=0}=-\left[\frac{\partial^3W}{\partial\xi^3}+(2-\mu)\frac{\partial^3W}{\partial\xi\partial\eta^2}\right]_{\xi=0}$$

$$=\sum_{n=1,3,5\cdots}^{\infty}n\left\{\frac{2n}{n(n^4+\lambda^4)}\left[1-\mu\frac{\operatorname{ch}n\left(\eta-\frac{\pi}{2}\right)}{\operatorname{ch}\frac{n\pi}{2}}\right]+A_n\Phi_n^{17}(\eta)-B_n\Phi_n^{18}(\eta)\right\}$$

$$-\sum_{m=1,3,5\cdots}^{\infty}\left\{\frac{2m^2\mu}{m(m^4+\lambda^4)}\operatorname{th}\frac{m\pi}{2}+\alpha_m^1A_m-\beta_m^1B_m\right\}\sin m\eta \tag{97D}$$

其中

$$\alpha_n^1=[(1-\mu)n^2P_n+\lambda^2Q_n]\operatorname{ch}\frac{P_n\pi}{2}\sin\frac{Q_n\pi}{2}+[(1-\mu)n^2Q_n-\lambda^2P_n]\operatorname{sh}\frac{P_n\pi}{2}\cos\frac{Q_n\pi}{2}$$

$$-n\left[n^2(1-\mu)\operatorname{sh}\frac{P_n\pi}{2}\sin\frac{Q_n\pi}{2}+\lambda^2\operatorname{ch}\frac{P_n\pi}{2}\cos\frac{Q_n\pi}{2}\right]\operatorname{th}\frac{n\pi}{2} \tag{98A}$$

$$\beta_n^1=[(1-\mu)n^2P_n+\lambda^2Q_n]\operatorname{sh}\frac{P_n\pi}{2}\cos\frac{Q_n\pi}{2}-[(1-\mu)n^2Q_n-\lambda^2P_n]\operatorname{ch}\frac{P_n\pi}{2}\sin\frac{Q_n\pi}{2}$$

$$-n\left[n^2(1-\mu)\operatorname{ch}\frac{P_n\pi}{2}\cos\frac{Q_n\pi}{2}-\lambda^2\operatorname{sh}\frac{P_n\pi}{2}\sin\frac{Q_n\pi}{2}\right]\operatorname{th}\frac{n\pi}{2} \tag{98B}$$

$$\Phi_m^{17}(\xi)=-\left\{m^2(1-\mu)\operatorname{sh}P_m\left(\xi-\frac{\pi}{2}\right)\sin Q_m\left(\xi-\frac{\pi}{2}\right)\right.$$

$$+(2-\mu)\lambda^2\operatorname{ch}P_m\left(\xi-\frac{\pi}{2}\right)\cos Q_m\left(\xi-\frac{\pi}{2}\right)-m^2(1-\mu)\left[\operatorname{sh}\frac{P_n\pi}{2}\sin\frac{Q_m\pi}{2}\right.$$

$$\left.\left.+\frac{\lambda^2}{(1-\mu)m^2}\operatorname{ch}\frac{P_m\pi}{2}\cos\frac{Q_m\pi}{2}\right]\frac{\operatorname{ch}m\left(\xi-\frac{\pi}{2}\right)}{\operatorname{ch}\frac{m\pi}{2}}\right\} \tag{98C}$$

$$\Phi_m^{18}(\xi)=-\left\{m^2(1-\mu)\operatorname{ch}P_m\left(\xi-\frac{\pi}{2}\right)\cos Q_m\left(\xi-\frac{\pi}{2}\right)\right.$$

$$-(2-\mu)\lambda^2\operatorname{sh}P_m\left(\xi-\frac{\pi}{2}\right)\sin Q_m\left(\xi-\frac{\pi}{2}\right)-m^2(1-\mu)\left[\operatorname{ch}\frac{P_m\pi}{2}\cos\frac{Q_m\pi}{2}\right.$$

$$-\frac{\lambda^2}{(1-\mu)m^2}\operatorname{sh}\frac{P_m\pi}{2}\quad\sin\frac{Q_m\pi}{2}\Bigg]\frac{\operatorname{ch} m\left(\xi-\frac{\pi}{2}\right)}{\operatorname{ch}\frac{m\pi}{2}}\Bigg\} \tag{98D}$$

α_m^1, β_m^1, $\Phi_n^{17}(\eta)$, $\Phi_n^{18}(\eta)$可以通过n, η和m, ξ的互换求得.

(本文未完待续)

Doubly Curved Shallow Shells with the Rectangular Bases Elastically Supported by Edge Arch Beams and Tie-Rods(I)

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Abstract

The equations of equilibrium of shallow shells with rectangular bases supported elastically with edge arch beams are obtained through the variational principle together with corresponding boundary conditions and corner conditions. It is assumed that the edge arch beams are of narrow plate form, so that only the bending rigidities in their own planes are taken into consideration, torsional rigidities and bending rigidities out of their own plane are neglected. In this paper, two cases of corner conditions are discussed. First of these is pinned corner conditions. Second of these is simply supported corner conditions, such that the corners can be moved freely in horizontal directions. The former corresponds to the conditions of these with heavy tension beams, in which the tension rigidities of the rods can be assumed infinite. The later corresponds to the conditions of elastically supported beams without tension rods. In the former case, the edge tangential displacement of shallow shells is assumed to be zero everywhere, so that the vertical displacement of the edge arch beams gives the only elastic supporting forces. This kind of supporting conditions is a good approximation for most practical roof design.

In this paper, the result of the problem of shallow spherical shell of square base supported elastically by edge arch beams and tie-rods under the above said restricted corner conditions, are calculated by the method of double trigonometric series. The edge conditions are integrated along their respective edge, and the conditions of corners are satisfied by proper choices of integration constants. The integrated edge conditions are then used to determine the unknown constants in the double trigonometric series. The result of this paper gives the tension in the tie-rods directly, which is an important quantity in the shell roof design practice.

The numerical results will be given in the second part of this paper.