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双曲-双曲奇异摄动问题的指数型拟合差分格式

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摘　要

本文讨论带有关于x的一阶导数项的双曲奇异摄动初边值问题，在较弱的相容性条件下构造了问题的渐近解并证明了解的一致有效性．然后我们对原问题构造一个指数型拟合差分格式并建立了离散能量不等式．最后我们证明差分问题的解一致收敛于原问题的精确解．

关键词　双曲型方程　奇异摄动　指数型拟合　差分格式　初边值问题

一、引　言

近年来我们对双曲型偏微分方程奇异摄动问题的差分解法作了一些探讨，例如[1]，[2]和[3]，根据问题的特性提出了不同类型的差分格式，对相应差分问题的解建立了先验估计，据此估计证明了这些格式在l_2意义下的一致收敛性．但在这些工作中所讨论的微分方程都不含关于空间变量x的一阶导数项．本文将讨论下面具有一阶导数项$\partial u/\partial x$的双曲型方程初、边值问题：

$$L_\varepsilon u=\varepsilon\left(\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}\right)+a(x,t)\frac{\partial u}{\partial t}+b(x,t)\frac{\partial u}{\partial x}+c(x,t)u=f(x,t)$$

$$((x,t)\in G=\{0<x<l,0<t\leqslant T\}) \tag{1.1}$$

$$u(x,0)=\varphi(x),\ \frac{\partial u}{\partial t}(x,0)=\psi(x)\quad(0\leqslant x\leqslant l) \tag{1.2}$$

$$u(0,t)=0,\ u(l,t)=0,\qquad(0\leqslant t\leqslant T) \tag{1.3}$$

假定 $a(x,t)$，$b(x,t)$，$c(x,t)$，$f(x,t)$，$\varphi(x)$，$\psi(x)$ 是充分光滑函数；$b(x,t)$，$f(x,t)$，$\varphi(x)$，$\psi(x)$满足相容性条件：

C_1：$\varphi(0)=0$，$\psi(0)=0$，$\varphi(l)=0$，$\psi(l)=0$；

C_2：$\varphi''(0)=0$，$b(0,0)\varphi'(0)=f(0,0)$，$\varphi''(l)=0$，$b(l,0)\varphi'(l)=f(l,0)$；

函数$a(a,t)\geqslant a_0>0$；退化算子

$$L_0 w=a(x,t)\frac{\partial w}{\partial t}+b(x,t)\frac{\partial w}{\partial x}+c(x,t)w \tag{1.4}$$

的次特征线关于方程(1.1)的特征方向 timelike：

* 创刊十周年暨一百期纪念特刊(Ⅱ)论文．

$$|b(x,t)|/a(x,t)<1 \qquad ((x,t)\in\bar{G}) \tag{1.5}$$

为确定起见，我们假定$b(x,t)\geqslant b_0>0$，因此条件(1.5)变为

$$b(x,t)<a(x,t) \quad ((x,t)\in\bar{G}) \tag{1.5'}$$

相应的退化问题为：

$$L_0u_0=f(x,t) \tag{1.6}$$

$$u_0(x,0)=\varphi(x),\ u_0(0,t)=0 \tag{1.7}$$

其中 L_0定义如(1.4)。

因此摄动问题(1.1)～(1.3)将在$t=0$和$x=l$附近出现边界层。退化问题(1.6)，(1.7)是半有界域$D=\{x,t),x\geqslant0,t\geqslant0\}$上一阶双曲型偏微分方程初、边值问题，按[4]，在相容性条件C_1，C_2下其解有一阶连续偏导数，但二阶导数是间断的。为了保证渐近解的构造过程得以实现，需要二阶导数连续，即$u_0(x,t)\in C^2(\bar{D})$，势必要增加更高阶的相容性条件，这未免限制过多。

本文首先建立了摄动问题(1.1)～(1.3)的能量不等式，然后在比较弱的相容性条件下构造了渐近解，并证明了此解在能量范数意义下的一致有效性。在第三节对摄动问题(1.1)～(1.3)提出了指数型拟合差分格式并建立了离散的能量不等式，最后一节证明了差分问题解在能量范数意义下的一致收敛性。

二、解的先验估计

下面的定理给出问题(1.1)～(1.3)的解的能量估计。

定理2.1 设$u(x,t)$是问题(1.1)～(1.3)的解且方程的系数、右端及初始函数满足上节的条件。那么存在与ε无关的正常数C，使下式当ε充分小时成立：

$$\|u\|+\varepsilon\left\|\frac{\partial u}{\partial t}\right\|+\varepsilon\left\|\frac{\partial u}{\partial x}\right\|\leqslant CK(G,\varepsilon) \tag{2.1}$$

其中

$$\|u\|=\left[\int_0^l u^2(x,t)dx\right]^{1/2},$$

$$K(G,\varepsilon)=\|f\|_{\bar{G}}+\|\varphi\|_{[0,l]}+\varepsilon\|\varphi'\|_{[0,l]}+\varepsilon\|\psi(x)\|_{[0,l]},$$

$$\|f\|_{\bar{G}}=\left[\iint_{\bar{G}}f^2(x,t)dxdt\right]^{1/2},\quad \|v\|_{[0,l]}=\left[\int_0^l v^2dx\right]^{1/2}.$$

证 以$2\varepsilon a^{-1}\partial u/\partial t+u$乘方程(1.1)两端，再按区域$G_t=\{(x,s)\,|\,0\leqslant x\leqslant l,\ 0<s\leqslant t\}$积分，则有

$$\begin{aligned}
&\frac{1}{2}\int_0^l au^2dx+\varepsilon^2\int_0^l a^{-1}\left[\left(\frac{\partial u}{\partial t}\right)^2+\left(\frac{\partial u}{\partial x}\right)^2\right]dx+\varepsilon\int_0^l\int_0^t\left[\left(\frac{\partial u}{\partial t}\right)^2\right.\\
&\left.+2a^{-1}b\frac{\partial u}{\partial t}\frac{\partial u}{\partial x}+\left(\frac{\partial u}{\partial x}\right)^2\right)dxds=\int_0^l\int_0^t ufdxds+2\varepsilon\int_0^l\int_0^t a^{-1}\frac{\partial u}{\partial t}fdxds\\
&+\frac{1}{2}\int_0^l\int_0^t\left(\frac{\partial a}{\partial t}+\frac{\partial b}{\partial x}-2c\right)u^2dxds-2\varepsilon\int_0^l\int_0^t a^{-1}cu\frac{\partial u}{\partial t}dxds\\
&+\int_0^l\int_0^t\frac{\partial a^{-1}}{\partial t}\left[\left(\frac{\partial u}{\partial t}\right)^2+\left(\frac{\partial u}{\partial x}\right)^2\right]dxds-2\varepsilon^2\int_0^l\int_0^t\frac{\partial a^{-1}}{\partial x}\frac{\partial u}{\partial t}\frac{\partial u}{\partial x}dxds
\end{aligned}$$

$$+\varepsilon^2\int_0^l a^{-1}(x,0)\psi^2dx+\varepsilon^2\int_0^l a^{-1}(x,0)\varphi'^2dx-\varepsilon\int_0^l u\frac{\partial u}{\partial t}dx$$

$$+\varepsilon\int_0^l \varphi(x)\psi(x)dx+\frac{1}{2}\int_0^l a(x,0)\varphi^2dx \tag{2.2}$$

由timelike条件(1.5)′，我们有

$$\int_0^l\int_0^t\left[\left(\frac{\partial u}{\partial t}\right)^2+2a^{-1}b\frac{\partial u}{\partial t}\frac{\partial u}{\partial x}+\left(\frac{\partial u}{\partial x}\right)^2\right]dxds\geqslant\frac{\omega}{2}\int_0^l\int_0^t\left[\left(\frac{\partial u}{\partial t}\right)^2+\left(\frac{\partial u}{\partial x}\right)^2\right]dxds$$

其中 $\omega=\min\limits_{\bar{G}}[1-(a^{-1}b)^2]$。再利用不等式

$$2|pq|\leqslant cp^2+\frac{1}{c}q^2,$$

通过对常数c的适当选取，(2.2)式成为

$$m\int_0^l\left[u^2+\varepsilon^2\left(\frac{\partial u}{\partial t}\right)^2+\varepsilon^2\left(\frac{\partial u}{\partial x}\right)^2\right]dx\leqslant M\int_0^l\int_0^t\left[u^2+\varepsilon^2\left(\frac{\partial u}{\partial t}\right)^2+\varepsilon^2\left(\frac{\partial u}{\partial x}\right)^2\right]dxds$$

$$+M\left[\int_0^l\int_0^T f^2dxds+\int_0^l\varphi^2dx+\varepsilon^2\int_0^l\psi^2dx+\varepsilon^2\int_0^l\varphi'^2dx\right]$$

其中 m，M为正常数。由Gronwall不等式即得

$$\int_0^l\left[u^2+\varepsilon^2\left(\frac{\partial u}{\partial t}\right)^2+\varepsilon^2\left(\frac{\partial u}{\partial x}\right)^2\right]dx\leqslant C\cdot K(G,a)$$

其中 $C=m^{-1}M\exp(MT/m)$。此式即所要证的结论。

三、渐近解的构造及余项估计

在相容性条件C_1、C_2下退化问题(1.6)、(1.7)的解$u_0(x,t)\in C^1(D)$，但它的二阶导数是间断的。为了保证迭代过程的实现，需要二阶导数连续，这势必要加更高阶的相容性条件。根据相容性条件的方程，我们知道，若初始条件和边界条件是齐次的，并且方程右端$f(x,t)$在原点邻域内恒等于0，则所有相容性条件都是满足的，现对退化问题(1.6)，(1.7)作如下处理。

设$u_0(x,t)=w_0(x,t)+\varphi(x)$，并由$\varphi(0)=0$，知

$$L_0w_0=F(x,t),\quad w_0(x,0)=0,\quad w_0(0,t)=0 \tag{3.1}$$

其中 算子L_0定义如(1.4)，$F(x,t)=f(x,t)-L_0\varphi$。设$w(y)\in C^\infty$，并具有以下性质：

$$\omega(y)=\begin{cases}0 & \left(0\leqslant y\leqslant\frac{1}{2}\right)\\ 1 & (y\geqslant 1)\end{cases}$$

$0\leqslant\omega(y)\leqslant 1$。设 $\bar{F}(x,t)=\omega(t/\delta)F(x,t)$，则

$$F(x,t)-\bar{F}(x,t)=\left(1-\omega\left(\frac{t}{\delta}\right)\right)F(x,t)=\begin{cases}F(x,t) & (0\leqslant t\leqslant\delta/2)\\ 0 & (t\geqslant\delta)\end{cases}$$

因此$\bar{F}(x,t)$与$F(x,t)$只在$\{0\leqslant t\leqslant\delta,\ 0\leqslant x\leqslant 1\}$有差别。现以$\bar{F}(x,t)$为右端，将问题(3.1)修改为

$$\left.\begin{aligned}&L_0\overline{w}_0=\overline{F}(x,t)=\omega\left(\frac{t}{\delta}\right)(f(x,t)-L_0\varphi)\\&\overline{w}_0(x,0)=0,\ \ \overline{w}_0(0,t)=0\end{aligned}\right\}\tag{3.2}$$

则问题(3.2)对任意$\delta>0$满足相容性条件．由[5]的附录，不难得到(3.2)的解 $\overline{w}_0$ 有如下估计

$$\left.\begin{aligned}&\overline{w}_0=O(1),\quad \frac{\partial\overline{w}_0}{\partial x}=O(1),\ \frac{\partial\overline{w}_0}{\partial t}=O(1)\\&\frac{\partial^2\overline{w}_0}{\partial x^2}=O(\delta^{-1}),\ \ \frac{\partial^2\overline{w}_0}{\partial t^2}=O(\delta^{-1}),\ \ \frac{\partial^2\overline{w}_0}{\partial x\partial t}=O(\delta^{-1})\end{aligned}\right\}\tag{3.3}$$

设 $\bar{f}(x,t)=\omega\left(\frac{t}{\delta}\right)f(x,t)+\left(1-\omega\left(\frac{t}{\delta}\right)\right)L_0\varphi$，将退化问题(1.6)、(1.7)修改为

$$\left.\begin{aligned}&L_0\bar{u}_0=\bar{f}(x,t)\\&\bar{u}_0(x,0)=\varphi(x),\ \ \bar{u}_0(0,t)=0\end{aligned}\right\}\tag{3.4}$$

与退化问题(1.6)、(1.7)相减，得

$$\left.\begin{aligned}&L_0(u_0-\bar{u}_0)=f(x,t)-\bar{f}(x,t)=\left(1-\omega\left(\frac{t}{\delta}\right)\right)F(x,t)\\&u_0(x,0)-\bar{u}_0(x,0)=0,\ \ u_0(0,t)-\bar{u}_0(0,t)=0\end{aligned}\right\}\tag{3.5}$$

不难验证$\bar{u}_0=\overline{w}_0+\varphi$，从而由(3.3)得

$$\left.\begin{aligned}&\bar{u}_0=O(1),\ \frac{\partial\bar{u}_0}{\partial x}=O(1),\ \frac{\partial\bar{u}_0}{\partial t}=O(1)\\&\frac{\partial^2\bar{u}_0}{\partial x^2}=O(\delta^{-1}),\ \frac{\partial^2\bar{u}_0}{\partial t^2}=O(\delta^{-1}),\ \frac{\partial^2\bar{u}_0}{\partial x\partial t}=O(\delta^{-1})\end{aligned}\right\}\tag{3.6}$$

并由相容性条件C_1、C_2知问题(3.5)满足相容性条件，则由[5]的附录可得

$$u_0-\bar{u}_0=O(\delta)\tag{3.7}$$

将问题(1.1)～(1.3)修改为

$$\left.\begin{aligned}&L_\varepsilon\bar{u}=\bar{f}(x,t)=\omega\left(\frac{t}{\delta}\right)f(x,t)+\left(1-\omega\left(\frac{t}{\delta}\right)\right)L_0\varphi\\&\bar{u}(0,x)=\varphi(x),\frac{\partial\bar{u}}{\partial t}(0,x)=\psi(x),\bar{u}(0,t)=\bar{u}(l,t)=0\end{aligned}\right\}\tag{3.8}$$

则问题(3.4)恰为问题(3.8)的退化问题．记$v(x,t)=u(x,t)-\bar{u}(x,t)$，则$v(x,t)$满足

$$L_\varepsilon v=\left(1-\omega\left(\frac{t}{\delta}\right)\right)F(x,t)$$

$$v(x,0)=0,\quad \frac{\partial v}{\partial t}(x,0)=0,\quad v(0,t)=v(l,t)=0$$

由能量不等式(2.1)立刻得

$$\|v\|=\|u-\bar{u}\|=O(\delta^{1/2})\qquad(0\leqslant t\leqslant T)\tag{3.9}$$

现构造问题(3.8)的渐近解为如下形式：

$$\bar{u}(x,t)=\bar{u}_0(x,t)+\varepsilon v_0^{(0)}(x,\ \tau)+v_0^{(1)}(\xi,\ t)+\varepsilon v_1^{(1)}(\xi,t)+z$$

记 $\tilde{u}(x,t)=\bar{u}_0+\varepsilon v_0^{(0)}+v_0^{(1)}+\varepsilon v_1^{(1)}$，其中$\tau=t/\varepsilon$。$\xi=(l-x)/\varepsilon$，$\bar{u}_0$是问题(3.4)的解；$v_0^{(0)}$ 满足

$$\frac{\partial^2 v_0^{(0)}}{\partial\tau^2}+a(x,0)\frac{\partial v_0^{(0)}}{\partial\tau}=0$$

$$\frac{\partial v_0^{(0)}}{\partial \tau}(x,0)+\frac{\partial \bar{u}_0}{\partial t}(x,0)=\psi(x), \qquad \lim_{\tau\to\infty} v_0^{(0)}=0$$

不难解出 $v_0^{(0)}=\frac{1}{a(x,0)}\left(\frac{\partial \bar{u}_0}{\partial t}(x,0)-\psi(x)\right)\exp(-a(x,0)\tau)$；$v_0^{(1)}$满足

$$\frac{\partial^2 v_0^{(1)}}{\partial \xi^2}+b(l,t)\frac{\partial v_0^{(1)}}{\partial \xi}=0$$

$$v_0^{(1)}(0,t)+\bar{u}_0(l,t)=0, \qquad \lim_{\xi\to\infty} v_0^{(1)}(\xi,t)=0$$

不难解出 $v_0^{(1)}(\xi,t)=-\bar{u}_0(l,t)\exp(-b(l,t)\xi)$；$v_1^{(1)}$满足

$$\frac{\partial^2 v_1^{(1)}}{\partial \xi^2}+b(l,t)\frac{\partial v_1^{(1)}}{\delta \xi}=\xi\frac{\partial b}{\partial x}(l,t)\frac{\partial v_0^{(1)}}{\partial \xi}+a(l,t)\frac{\partial v_0^{(1)}}{\partial t}+c(l,t)v_0^{(1)}$$

$$v_1^{(1)}(0,t)=0, \lim_{\xi\to\infty} v_1^{(1)}(\xi,t)=0,$$

不难解出 $v_1^{(1)}(\xi,t)=-\left[\frac{\alpha(t)}{2b(l,t)}\xi^2+\frac{\alpha(t)+\beta(t)b(l,t)}{b^2(l,t)}\xi\right]\exp(-b(l,t)\xi)$，其中 $\alpha(t)=\bar{u}_0(l,t)\left[a(l,t)\frac{\partial b}{\partial t}(l,t)+b(l,t)\frac{\partial b}{\partial x}(l,t)\right]$，$\beta(t)=-\bar{u}_0(l,t)c(l,t)-\frac{\partial \bar{u}_0}{\partial t}(l,t)a(l,t)$。$z$满足下面方程及初、边值条件

$$\left.\begin{aligned}
&L_\varepsilon z=-\varepsilon r(x,t,\varepsilon)+\varepsilon^2\left(\frac{\partial^2 v_0^{(0)}}{\partial x^2}-\frac{\partial^2 v_1^{(1)}}{\partial t^2}\right)=\Phi(x,t,\varepsilon)\\
&z|_{t=0}=-\varepsilon[v_1^{(1)}(\xi,0)+v_0^{(0)}(x,0)]=\varphi_1(x)\\
&\left.\frac{\partial z}{\partial t}\right|_{t=0}=-\frac{\partial v_0^{(1)}}{\partial t}(\xi,0)-\varepsilon\frac{\partial v_1^{(1)}}{\partial t}(\xi,0)=\psi_1(x)\\
&z|_{x=0}=-v_0^{(0)}(l/\varepsilon,t)-\varepsilon v_1^{(1)}(l/\varepsilon,t)=\omega_0(t)\\
&z|_{x=l}=-\varepsilon v_0^{(0)}(l,\tau)=\omega_l(t)
\end{aligned}\right\}\tag{3.10}$$

其中 $r(x,t,\varepsilon)=\frac{\partial^2 \bar{u}_0}{\partial t^2}-\frac{\partial^2 \bar{u}_0}{\partial x^2}+b(x,t)\frac{\partial v_0^{(0)}}{\partial x}+\tau a_{01}(x,t_1)\frac{\partial v_0^{(0)}}{\partial \tau}+c(x,t)v_0^{(0)}$

$$+\frac{1}{2}\xi^2 b_{20}(x_2,t)\frac{\partial v_0^{(1)}}{\partial \xi}-\xi a_{10}(x_1,t)\frac{\partial v_0^{(1)}}{\partial t}-\xi d_{10}(x_3,t)v_0^{(1)}+\frac{\partial^2 v_0^{(1)}}{\partial t^2}$$

$$+\xi b_{10}(x_4,t)\frac{\partial v_1^{(1)}}{\partial \xi}+a(x,t)\frac{\partial v_1^{(1)}}{\partial t}+c(x,t)v_1^{(1)}.$$

根据估计式(3.6)及边界层函数的表达式，我们有

$$r(x,t,\varepsilon)=O(\delta^{-1}), \quad \frac{\partial^2 v_0^{(0)}}{\partial x^2}=O(\delta^{-2}), \quad \frac{\partial^2 v_1^{(1)}}{\partial t^2}=O(\delta^{-2})$$

于是 $\Phi(x,t,\varepsilon)=O(\varepsilon\delta^{-1}+\varepsilon^2\delta^{-2})$。我们还有

$$\varphi_1(x)=O(\varepsilon), \ \varphi_1'(x)=O(\varepsilon^{1/2}+\varepsilon\delta^{-1}), \ \psi_1(x)=O(\varepsilon^{1/2}\delta^{-1}+\varepsilon^{3/2}\delta^{-1}),$$

$$\omega_0(t)=O(\varepsilon^n), \ \omega_l(t)=O(\varepsilon), \ \omega_l'(t)=O(\varepsilon^{1/2}), \ \omega_l''(t)=O(1)$$

作变换将(3.10)的边界条件化为齐次，利用能量不等式(2.1)不难得

$$\|z\|=\|\bar{u}(x,t)-\tilde{u}(x,t)\|\leqslant C(\delta^{1/2}+\varepsilon^{1/2}+\varepsilon\delta^{-1}+\varepsilon^2\delta^{-2})\tag{3.11}$$

记$\tilde{u}_1(x,t)=u_0(x,t)+\varepsilon v_0^{(0)}(x,\tau)+v_0^{(1)}(\xi,t)+\varepsilon v_1^{(1)}(\xi,t)$，由(3.7)、(3.9)及(3.11)，并取

$\delta=\varepsilon^{2/3}$立刻得

$$\|u(x,t)-\tilde{u}_1(x,t)\|\leqslant C\varepsilon^{1/3} \tag{3.12}$$

上面出现的范数的意义同定理2.1.

注3.1 如果问题(1.1)～(1.3)的系数、右端及初值函数满足足够的相容性条件使解 $u(x,t)\in C^3(G)$，则不需引入函数$\omega(t/\delta)$，即可构造出渐近解，且形式与$\tilde{u}_1(x,t)$完全一样，只需将 $v_0^{(0)}$、$v_0^{(1)}$及 $v_1^{(1)}$表达式中的$\bar{u}_0$换成u_0，并有如下余项估计

$$|u(x,t)-\tilde{u}_1(x,t)|\leqslant C\varepsilon^{1/2} \tag{3.13}$$

四、差分格式及离散能量不等式

我们在x方向及t方向均采用等距网格，设步长分别为h和k，且$Nh=l$。则离散区域$G_d=\{(x_i,t_j),\ i=0,1,\cdots,N,\ j=0,1,\cdots,[T/k],\ x_i=ih,\ t_j=jk\}$。用 $u^d(x,t)$ 表示 $u(x,t)$ 的近似值，定义差分算子：

$$L_\varepsilon^{(h,k)}u^d(x,t)\equiv\gamma_1(x,t,k)u^d_{\bar{t}t}(x,t)-\gamma_2(x,t,h)u^d_{x\bar{x}}+a(x,t)u^d_{\bar{t}}$$

$$+b(x,t)u^d_{\bar{x}}+c(x,t)u^d=f(x,t)\qquad((x,t)\in G_d) \tag{4.1}$$

$$u^d(x,0)=\varphi(x),\ \ u^d(x,k)-u^d(x,0)=k\psi(x) \tag{4.2}$$

$$u^d(0,t)=u^d(l,t)=0 \tag{4.3}$$

其中　$(x,t)=(x_i,t_j)$，$\gamma_1(x,t,k)=\dfrac{a(x,t)k\exp(-a(x,t)k/\varepsilon)}{1-\exp(-a(x,t)k/\varepsilon)}$，

$$\gamma_2(x,t,h)=\frac{b(x,t)h\exp(-b(x,t)h/\varepsilon)}{1-\exp(-b(x,t)h/\varepsilon)}.$$

差分问题(4.1)～(4.3)的解有如下的能量估计.

定理4.1 设$u^d(x,t)$是问题(4.1)～(4.3)的解，并设网格步长h,k满足 $b(x,t)h\leqslant a(x,t)k$ 对一切$(x,t)\in\bar{G}_d$成立，则当ε,h,k足够小时，有

$$\|u^d\|_s+\|\gamma_1u^d_{\bar{t}}\|_s+\|\sqrt{\gamma_1\gamma_2}\,u^d_{\bar{x}}\|_s\leqslant CK(h,k,\varepsilon) \tag{4.4}$$

其中　$K(h,k,\varepsilon)=hk\sum\limits_{j=2}^{J}\sum\limits_{i=1}^{N}f^2+\|\gamma_1u^d_{\bar{t}}\|_1+\|\sqrt{\gamma_1\tau}\,u^d_{\bar{t}}\|_1+\|\sqrt{\gamma_1\gamma_2}\,u^d_{\bar{x}}\|_1+\|u^d\|_1$，

$$\|v\|_s^2=h\sum_{i=1}^{N}v(ih,sk)^2\text{且}J=[T/k]\qquad(s=2,3,\cdots,J).$$

证　利用[2]中列出的一些差分关系，以 $2\gamma_1a^{-1}u^d_{\bar{t}}+u^d$乘方程(4.1)两边，得到（在本证明中为方便起见记$y=u^d$）：

$$\gamma_1(a^{-1}r_1y_{\bar{t}}^2)_{\bar{t}}+\gamma_1ka^{-1}\gamma_1y_{\bar{t}t}^2+\gamma_1(a^{-1}\gamma_2y_{\bar{x}}^2)_{\bar{t}}+\gamma_1ka^{-1}\gamma_2y^2_{\bar{x}t}-2\gamma_1(a^{-1}\gamma_2y_{\bar{t}}y_x)_{\bar{x}}$$

$$+2\gamma_1y_{\bar{t}}^2+2\gamma_1a^{-1}by_{\bar{x}}y_{\bar{t}}+\gamma_1(a^{-1}cy^2)_{\bar{t}}+\gamma_1ka^{-1}cy_{\bar{t}}^2+(r_1yy_{\bar{t}})_{\bar{t}}-\gamma_1y_{\bar{t}}y_{\bar{t}}(x,t-k)$$

$$-(\gamma_2yy_x)_{\bar{x}}+\gamma_2y_{\bar{x}}^2+\frac{1}{2}(ay^2)_{\bar{t}}+\frac{1}{2}\ kay_{\bar{t}}^2+\frac{1}{2}(by^2)_{\bar{x}}+\frac{1}{2}hby_{\bar{x}}^2$$

$$=f(2a^{-1}\gamma_1 y_{\bar{t}}+y)+\gamma_1(a^{-1}\gamma_1)_{\bar{t}}y_{\bar{t}}^2(x,t-k)-2\gamma_1(a^{-1}\gamma_2)_{\bar{x}}y_{\bar{t}}(x-h,t)y_{\bar{x}}$$

$$+\gamma_1(a^{-1}\gamma_2)_{\bar{t}}y_{\bar{x}}^2(x,t-k)+\gamma_1(ca^{-1})_{\bar{t}}u^2(x,t-k)+(\gamma_1)_{\bar{t}}y(x,t-k)y_{\bar{t}}(x,t-k)$$

$$-(\gamma_2)_{\bar{x}}y(x-h,t)y_{\bar{x}}+\frac{1}{2}a_{\bar{t}}y^2(x,t-k)+\frac{1}{2}b_{\bar{x}}y^2(x-h,t)-cy^2 \qquad (4.5)$$

其中 a，b，c，γ_1，$\gamma_2,y,y_{\bar{t}}$，$y_{\bar{x}}$均在(x,t)取值．注意到函数$x\exp(-x)/(1-\exp(-x))$单调递减，则在$b(x,t)h\leqslant a(x,t)k$的条件下$\gamma_1\leqslant\gamma_2$．对上式求和$hk\sum\limits_{j=2}^{S}\sum\limits_{i=1}^{N}$，利用

$$a^{-1}\gamma_1^2 y_{\bar{t}}^2+\gamma_1 y y_{\bar{t}}+\frac{1}{2}ay^2+\gamma_1 a^{-1}cy^2\geqslant m_1(\gamma_1^2 y_{\bar{t}}^2+y^2),$$

$$\gamma_1 y_{\bar{x}}^2+2\gamma_1 a^{-1}by_{\bar{x}}y_{\bar{t}}+\ \gamma_2 y_{\bar{t}}^2\geqslant m_2(\gamma_1 y_{\bar{x}}^2+\gamma_2 y_{\bar{t}}^2),$$

（其中$m_1>0$，$m_2>0$）

$$hk\sum_{j=2}^{s}\sum_{i=1}^{N}(\gamma_1 y_{\bar{t}}^2-\gamma_1 y_{\bar{t}}y_{\bar{t}}(x,t-k))\geqslant hk\sum_{j=2}^{s}\sum_{i=1}^{N}(\gamma_1 y_{\bar{t}}^2-\frac{1}{2}\gamma_1 y_{\bar{t}}^2$$

$$-\frac{1}{2}\gamma_1 y_{\bar{t}}^2(x,t-k))=-hk\sum_{i=1}^{N}\frac{1}{2}\gamma_1 y_{\bar{t}}^2|_{t=k}$$

以及
$$\gamma_1,\gamma_2,|(\gamma_1)_{\bar{t}}|,|(\gamma_1)_{\bar{t}}|\leqslant C\varepsilon,$$
$$\gamma_1+ka\geqslant m_3\varepsilon,\ \gamma_2+hb\geqslant m_3\varepsilon\ (m_3>0)$$

可以得到（ε,h,k充分小时）

$$\|y\|_s^2+\|\gamma_1 y_{\bar{t}}\|_s^2+\|\sqrt{\gamma_1\gamma_2}\,y_{\bar{t}}\|_s^2\leqslant CK^2(h,k,\varepsilon)+Ck\sum_{j=2}^{s-1}\|y\|_j^2$$

则由离散的Gronwall不等式（见[6]）立刻得(4.4)．

注4.1 条件$b(x,t)h\leqslant a(x,t)k$是由于$\partial u/\partial x$项的系数非零及采用指数型拟合格式引起．如果$b(x,t)\equiv 0$或改变差分格式类型，则该条件可以除去．

五、差分格式的收敛性

在本节中，我们假设足够的相容性条件成立以使问题(1.1)～(1.3)的解$u(x,t)\in C^2(\bar{G})$．从解的渐近表达式（参第三节）知$u(x,t)$及其导数有如下估计

$$\left|\frac{\partial^k u(y,t)}{\partial x^i\partial t^{k-i}}\right|\leqslant C(\varepsilon^{-i}+\varepsilon^{1-(k-i)}) \qquad (0\leqslant k\leqslant 3,\ 0\leqslant i\leqslant k) \qquad (5.1)$$

为方便起见，我们还设

$$c_1 k\leqslant h\leqslant c_2 k \qquad (c_1>0,\ c_2>0) \qquad (5.2)$$

由(5.1)，易有

$$L_\varepsilon^{(h,k)}(u(x,t)-u^d(x,t))=O\left(\frac{h}{\varepsilon^2}+\frac{k}{\varepsilon}\right)$$

$$(u-u^d)|_{t=0}=0,\ (u-u^d)|_{t=k}=\min(k^2/\varepsilon,k)$$

$$(u-u^d)|_{x=0}=0,\ (u-u^d)|_{x=l}=0$$

由能量估计(4.4)，立刻得古典估计

$$\|u-u^d\|_s\leqslant C\left(\frac{h}{\varepsilon^2}+\frac{k}{\varepsilon}\right) \tag{5.3}$$

现在来看非古典估计．为简单起见，考虑拟合因子为

$$\gamma_1=\frac{a(x,0)k\ \exp(-a(x,0)k/\varepsilon)}{1-\exp(-a(x,0)k/\varepsilon)},\quad \gamma_2=\frac{b(l,t)h\ \exp(-b(l,t)h/\varepsilon)}{1-\exp(-b(l,t)h/\varepsilon)}$$

则不难验证：

$$L_\varepsilon^{(h,k)}(\tilde{u}_1-u^d)=L_\varepsilon^{(h,k)}u_0+\varepsilon L_\varepsilon^{(h,k)}v_0^{(0)}+L_\varepsilon^{(h,k)}v_0^{(1)}+\varepsilon L_\varepsilon^{(h,k)}v_1^{(1)}-f$$

$$=O\left(\varepsilon+h+k+\exp\left(-m_0\frac{l-x}{\varepsilon}\right)\right)\qquad (m_0>0),$$

及

$$(\tilde{u}_1-u^d)|_{t=0}=O(\varepsilon),\quad (\tilde{u}_{1\bar{t}}-u_{\bar{t}}^d)|_{t=k}=O(1)$$

$$(\tilde{u}_1-u^d)|_{x=0}=O(\varepsilon^n),\quad (\tilde{u}_1-u^d)|_{x=l}=0,$$

其中n可取得任意大．作变换把边界条件化成齐次，再由(4.4)及(5.2)得

$$\|\tilde{u}_1-u^d\|_s\leqslant C(\sqrt{\max(\varepsilon,h)}+k+\varepsilon^n/h) \tag{5.4}$$

从而由渐近解余项估计(3.13)，得

$$\|u-u^d\|_s\leqslant C(\sqrt{\max(\varepsilon,h)}+k+\varepsilon^n/h) \tag{5.5}$$

综合(5.3)及(5.5)，可得到下面的收敛性定理．

定理5.1 设定理4.1的条件成立，问题(1.1)～(1.3)的解$u(x,t)\in C^3(\bar{G})$，且步长h，k满足(5.2)．那么差分问题(4.1)～(4.3)的解$u^d(x,t)$在能量范数意义下一致收敛于$u(x,t)$，且有估计

$$\|u-u^d\|_s\leqslant Ch^{1/5}\qquad (s=0,1,\cdots,J) \tag{5.6}$$

其中范数$\|\cdot\|_s$的意义同定理4.1．

证 当$\varepsilon^{\frac{5}{2}}\leqslant h$时，利用估计(5.5)；当$\varepsilon^{\frac{5}{2}}\geqslant h$时，利用估计(5.3)立刻得(5.6)．

注5.1 如果将第三节中对问题(1.6)，(1.7)的处理过程直接用于问题(1.1)～(1.3)，则我们可以在相容性条件C_1、C_2下($u(x,t)\in C^2(\bar{G})$)，从渐近解推断解的导数估计．类似于定理5.1可证明差分格式在能量范数意义下一致收敛．

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An Exponentially Fitted Difference Scheme for the Hyperbolic-hyperbolic Singularly Perturbed Initial-Boundary Value Problem

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Abstract

In this paper we discuss an initial-boundary value problem of hyperbolic type with first derivative with respect to x. The asymptotic solution is constructed and its uniform validity is proved under weaker compativility conditions. Then we develop an exponentially fitted difference scheme and establish discrete energy inequality. Finally, we prove that the solution of difference problem uniformly converges to the solution of the original problem.

Key words hyperbolic equation, singular perturbation, exponential fitting, difference scheme, boundary value problem