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多夹层板壳的非线性理论及应用（Ⅱ）——正交异性材料扁壳的基本方程

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摘　要

本文把文[1]建立的多夹层壳体的中小转动一阶大挠度理论，具体地运用到多夹层扁壳中去．给出了正交异性材料的多夹层扁壳的大挠度问题平衡方程和边界条件及其特例，宏观各向同性材料多夹层扁壳的大挠度方程．

关键词　多夹层扁壳　正交各向异性　基本方程

本文的第（Ⅰ）部分，根据考虑横向剪切变形的壳体几何关系，获得了多夹层壳体在小应变状态下中转动二阶大挠度理论的基本方程，以及简化的中转动、中小转动一阶大挠度理论．以壳体中面的面内位移u_α、法向位移w和与横向剪切变形有关的广义位移γ_α为基本变量，给出了张量形式表示的几何关系、广义内力、应变能密度和平衡方程、边界条件，这些关系在任意的坐标系都是成立的．但如果需要解决实际问题，还是要根据壳体的构造特点和载荷形式，选择具体的坐标系来研究．本文的这一部分以工程中广泛应用的扁壳为例，建立了正交异性材料的多夹层壳体的基本方程及其特例，各向同性材料多夹层扁壳的基本方程，以便于系统求解及研究多夹层扁壳（包括板）的特性．

一、正交异性材料多夹层扁壳的基本方程

扁壳，是指较为扁平的开口薄壳，一般地，其中面的最大矢高远小于它的底面尺寸．这里讨论的，是建筑工程中常用的，具有矩形底面的扁壳．

取直角坐标系$Oxyz$，并令xOy坐标面与壳体的底面重合．对于扁壳，一般可以假定其中面的几何特性与底面的几何特性相同，因此中面上各点的曲面坐标(α,β)可用底面坐标(x,y)来表示．那么，根据第一部分已推得的中小转动一阶理论的公式，可以获得多夹层扁壳的大挠度基本方程．

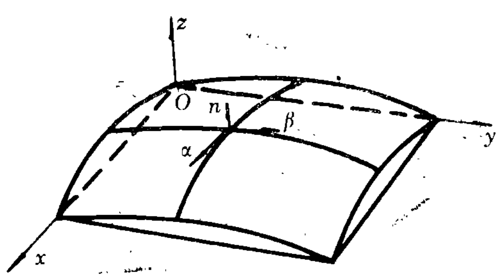

图1　扁壳的坐标

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1．位移场和几何关系

以u，v，w分别表示壳体中面位移，γ_x，γ_y为与横向剪切变形有关的广义位移。那么壳体第i层的位移

$$\left.\begin{aligned}&{}_iu_1^*=u+{}_if_1\gamma_x+\omega_xz+{}_id_1\gamma_x(z-{}_ig)\\&{}_iu_2^*=v+{}_if_2\gamma_y+\omega_yz+{}_id_2\gamma_y(z-{}_ig)\\&{}_iu_3=w\qquad(h_{i-1}\leqslant z\leqslant h_i)\end{aligned}\right\}\tag{1.1}$$

并且有

$${}_i\gamma_x={}_id_1\gamma_x,\quad {}_i\gamma_y={}_id_2\gamma_y\tag{1.2}$$

$$\left.\begin{aligned}&{}_ig=(h_i+h_{i-1})/2\\&{}_if_1=\frac{h_i-h_{i-1}}{2}{}_id_1+(h_{i-1}-h_{i-2})_{i-1}d_1+\cdots+(h_1-h_0)_1d_1+h_0\,{}_0d_1\\&{}_if_2=\frac{h_i-h_{i-1}}{2}{}_id_2+(h_{i-1}-h_{i-2})_{i-1}d_2+\cdots+(h_1-h_0)_1d_2+h_0\,{}_0d_2\end{aligned}\right\}\tag{1.3}$$

那么第i层的应变为

$$\left.\begin{aligned}&{}_i\varepsilon_x=e_x-z\chi_x-{}_i\bar f_1\lambda_x-z\,{}_id_1\lambda_x,\quad {}_i\varepsilon_y=e_y-z\chi_y-{}_i\bar f_2\lambda_y-z\,{}_id_2\lambda_y\\&{}_i\varepsilon_{xy}=e_{xy}-z\chi_{xy}-({}_i\bar f_1\lambda_{yx}+{}_i\bar f_2\lambda_{xy})/2-z({}_id_1\lambda_{yx}+{}_id_2\lambda_{xy})/2\end{aligned}\right\}\tag{1.4}$$

$${}_i\varepsilon_{xz}={}_id_1\gamma_x/2,\quad {}_i\varepsilon_{yz}={}_id_2\gamma_y/2\tag{1.5}$$

其中

$${}_i\bar f_1={}_if_1-{}_ig\,{}_id_1,\quad {}_i\bar f_2={}_if_2-{}_ig\,{}_id_2\tag{1.6}$$

$$\left.\begin{aligned}&e_x=\frac{\partial u}{\partial x}+\frac{w}{R_x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2,\quad e_y=\frac{\partial v}{\partial y}+\frac{w}{R_y}+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\\&e_{xy}=\frac{1}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right)+\frac{w}{R_{xy}}\end{aligned}\right\}\tag{1.7}$$

$$\chi_x=\frac{\partial^2 w}{\partial x^2},\quad \chi_y=\frac{\partial^2 w}{\partial y^2},\quad \chi_{xy}=\frac{\partial^2 w}{\partial x\partial y}\tag{1.8}$$

$$\left.\begin{aligned}&\lambda_x=-\frac{\partial\gamma_x}{\partial x}=-\frac{\partial\psi_x}{\partial x}-\frac{\partial^2 w}{\partial x^2},\quad \lambda_y=-\frac{\partial\gamma_y}{\partial y}=-\frac{\partial\psi_y}{\partial y}-\frac{\partial^2 w}{\partial y^2}\\&\lambda_{xy}=-\frac{\partial\gamma_y}{\partial x}=-\frac{\partial\psi_y}{\partial x}-\frac{\partial^2 w}{\partial x\partial y},\quad \lambda_{yx}=-\frac{\partial\gamma_x}{\partial y}=-\frac{\partial\psi_x}{\partial y}-\frac{\partial^2 w}{\partial x\partial y}\end{aligned}\right\}\tag{1.9}$$

$$\gamma_x=\psi_x+\partial w/\partial x,\quad \gamma_y=\psi_y+\partial w/\partial y\tag{1.10}$$

这里的R_x，R_y和R_{xy}是扁壳中面的曲率半径。

若引入应力函数，还需要用到以下的协调条件，

$$\begin{aligned}\frac{\partial^2 e_x}{\partial y^2}+\frac{\partial^2 e_y}{\partial x^2}-2\frac{\partial^2 e_{xy}}{\partial x\partial y}=&\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2-\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}+\frac{1}{R_x}\frac{\partial^2 w}{\partial y^2}\\&+\frac{1}{R_y}\frac{\partial^2 w}{\partial x^2}-\frac{2}{R_{xy}}\frac{\partial^2 w}{\partial x\partial y}\end{aligned}\tag{1.11}$$

2．本构关系和广义内力

表层是正交各向异性材料构成，并且材料的主方向与坐标轴的方向一致，那么有

$$\left.\begin{aligned}
{}_i\sigma_x&=\frac{{}_iE_x}{1-{}_i\nu_{xy}\,{}_i\nu_{yx}}{}_i\varepsilon_x+\frac{{}_i\nu_{xy}\,{}_iE_y}{1-{}_i\nu_{xy}\,{}_i\nu_{yx}}{}_i\varepsilon_y=\frac{{}_iE_1}{{}_i\mu}{}_i\varepsilon_x+\frac{{}_i\nu_{12}\,{}_iE_2}{{}_i\mu}{}_i\varepsilon_y\\
{}_i\sigma_y&=\frac{{}_i\nu_{xy}\,{}_iE_x}{1-{}_i\nu_{xy}\,{}_i\nu_{yx}}{}_i\varepsilon_x+\frac{{}_iE_y}{1-{}_i\nu_{xy}\,{}_i\nu_{yx}}{}_i\varepsilon_y=\frac{{}_i\nu_{12}\,{}_iE_1}{{}_i\mu}{}_i\varepsilon_x+\frac{{}_iE_2}{{}_i\mu}{}_i\varepsilon_y\\
{}_i\tau_{xy}&=2\,{}_iG_{xy}\,{}_i\varepsilon_{xy}=2{}_iG_{12}\,{}_i\varepsilon_{xy}
\end{aligned}\right\}\tag{1.12}$$

由于表层的${}_iG_{xz}$，${}_iG_{yz}$一般比夹心大许多，可忽略表层的横向剪切变形，即有${}_id_1={}_id_2=0$，那么此时(1.6)式变为

$$\left.\begin{aligned}
{}_i\bar{f}_1={}_if_1&=(h_{i-1}-h_{i-2})_{i-1}d_1+(h_{i-3}-h_{i-4})_{i-3}d_1+\cdots+(h_2-h_1)_2d_1+h_0\,{}_0d_1\\
{}_i\bar{f}_2={}_if_2&=(h_{i-1}-h_{i-2})_{i-1}d_2+(h_{i-3}-h_{i-4})_{i-3}d_2+\cdots+(h_2-h_1)_2d_2+h_0\,{}_0d_2
\end{aligned}\right\}\tag{1.13}$$

对于第j层夹心，只考虑横向剪切作用，那么有

$${}_j\tau_{xz}={}_jG_{xz}\cdot 2{}_j\varepsilon_{xz}={}_jG_{13}\,{}_jd_1\gamma_x,\quad {}_j\tau_{yz}={}_jG_{yz}\cdot 2{}_j\varepsilon_{yz}={}_jG_{23}\,{}_jd_2\gamma_y \tag{1.14}$$

式中

$$\left.\begin{aligned}
&{}_jd_1=G_{13}/{}_jG_{13},\quad {}_jd_2=G_{23}/{}_jG_{23}\\
&G_{13}=\sum_{\mathrm{II}}(h_j-h_{j-1})\Big/\sum_{\mathrm{II}}\frac{h_j-h_{j-1}}{{}_jG_{13}},\quad G_{23}=\sum_{\mathrm{II}}(h_j-h_{j-1})\sum_{\mathrm{II}}\Big/\frac{h_j-h_{j-1}}{{}_jG_{23}}
\end{aligned}\right\}\tag{1.15}$$

这里的求和符号$\sum\limits_{\mathrm{II}}$表示只对夹心求和。

壳体的内力、内力矩为

$$\begin{bmatrix}N\\M\\m\end{bmatrix}=\begin{bmatrix}A&F&B\\-F^T&D&H\\-B'&H'&J\end{bmatrix}\begin{bmatrix}e\\\chi\\\lambda\end{bmatrix}\tag{1.16}$$

其中

$$\left.\begin{aligned}
&N=[N_x,\ N_y,\ N_{xy}]^T,\ M=[M_x,\ M_y,\ M_{xy}]^T,\ m=[m_x,\ m_y,\ m_{xy},\ m_{yz}]^T\\
&e=[e_x,\ e_y,\ e_{xy}]^T,\ \chi=[\chi_x,\ \chi_y,\ \chi_{xy}]^T,\ \lambda=[\lambda_x,\ \lambda_y,\ \lambda_{xy},\ \lambda_{yz}]^T
\end{aligned}\right\}\tag{1.17}$$

式中的A，F，B，D，H，J，B'，H'为系数矩阵，可由文[1]的(3.5)式求得.下面以A为例，有

$$A=\begin{bmatrix}A_{11}&A_{12}&0\\A_{12}&A_{22}&0\\0&0&A_{66}\end{bmatrix}=\sum_{\mathrm{I}}\begin{bmatrix}\frac{{}_iE_1}{{}_i\mu}(h_i-h_{i-1})&\frac{{}_i\nu_{12}\,{}_iE_2}{{}_i\mu}(h_i-h_{i-1})&0\\\frac{{}_i\nu_{12}\,{}_iE_2}{{}_i\mu}(h_i-h_{i-1})&\frac{{}_iE_2}{{}_i\mu}(h_i-h_{i-1})&0\\0&0&2\,{}_iG_{12}(h_i-h_{i-1})\end{bmatrix}\tag{1.18}$$

这里的求和符号$\sum\limits_{\mathrm{I}}$表示只对表层求和。

对横向剪切力Q_x，Q_y，可推得

$$\left.\begin{aligned}
&Q_x=G_{13}\gamma_x\sum_{\mathrm{II}}(h_j-h_{j-1})=Q_{11}\gamma_x,\quad Q_y=G_{23}\gamma_y\sum_{\mathrm{II}}(h_j-h_{j-1})=Q_{22}\gamma_y\\
&Q_{11}=\sum_{\mathrm{II}}{}_jG_{13}\,{}_jd_1(h_j-h_{j-1})=G_{13}\sum_{\mathrm{II}}(h_j-h_{j-1})\\
&Q_{22}=\sum_{\mathrm{II}}{}_jG_{23}\,{}_jd_2(h_j-h_{j-1})=G_{23}\sum_{\mathrm{II}}(h_j-h_{j-1})
\end{aligned}\right\}\tag{1.19}$$

如果多夹层扁壳以中面为对称面上下对称铺层，此时有

$$F=0,\ B=0,\ B'=0\tag{1.20}$$

那么，N只与e有关，M，m只与χ，λ有关，即拉—弯是解耦的。

3．平衡方程与边界条件

由文[1]的(3.8)和(3.9)式，可得平衡方程与边界条件．平衡方程为

$$\left.\begin{aligned}&\frac{\partial N_x}{\partial x}+\frac{\partial N_{xy}}{\partial y}=0,\quad \frac{\partial N_{xy}}{\partial x}+\frac{\partial N_y}{\partial y}=0\\&\frac{\partial^2 M_x}{\partial x^2}+2\frac{\partial^2 M_{xy}}{\partial x\partial y}+\frac{\partial^2 M_y}{\partial y^2}=-q-N_x\left(\frac{\partial^2 w}{\partial x^2}-\frac{1}{R_x}\right)\\&\qquad -N_y\left(\frac{\partial^2 w}{\partial y^2}-\frac{1}{R_y}\right)-2N_{xy}\left(\frac{\partial^2 w}{\partial x\partial y}-\frac{1}{R_{xy}}\right)\\&\frac{\partial m_x}{\partial x}+\frac{\partial m_{yx}}{\partial y}=Q_x,\quad \frac{\partial m_{xy}}{\partial x}+\frac{\partial m_y}{\partial y}=Q_y\end{aligned}\right\}\tag{1.21}$$

边界条件

$$\left.\begin{aligned}&x=a_0,\ a\text{时},\ N_x=N_x^0, &&\text{或}\ \delta u=0\\&\qquad N_{xy}=N_{xy}^0, &&\delta v=0\\&\qquad \frac{\partial M_x}{\partial x}+\frac{\partial M_{xy}}{\partial y}+N_x\frac{\partial w}{\partial x}+N_{xy}\frac{\partial w}{\partial y}=0, &&\delta w=0\\&\qquad M_x-m_x=0, &&\delta(\partial w/\partial y)=0\\&\qquad M_{xy}-m_{yx}=0, &&\delta(\partial w/\partial y)=0\\&\qquad m_x=0, &&\delta\psi_x=0\\&\qquad m_{yx}=0, &&\delta\psi_y=0\end{aligned}\right\}\tag{1.22}$$

$y=b_0$，b时的边界条件同样可写出，这里略去．

这样，我们就获得了正交各向异性矩形底面的多夹层扁壳一阶大挠度理论的所有方程．若是对称铺层，平衡方程(1.21)可用u，v，w，ψ_x，ψ_y表示为

$$\left.\begin{aligned}&A_{11}\frac{\partial^2 u}{\partial x^2}+\frac{A_{66}}{2}\frac{\partial^2 u}{\partial y^2}+\left(A_{12}+\frac{A_{66}}{2}\right)\frac{\partial^2 v}{\partial x\partial y}+\left(A_{11}\frac{\partial^2 w}{\partial x^2}+\frac{A_{66}}{2}\frac{\partial^2 w}{\partial y^2}\right)\frac{\partial w}{\partial x}\\&\quad+\left(A_{12}+\frac{A_{66}}{2}\right)\frac{\partial^2 w}{\partial x\partial y}\frac{\partial w}{\partial y}+\frac{\partial}{\partial x}\left(A_{11}\frac{w}{R_x}+A_{12}\frac{w}{R_y}\right)\\&\quad+\frac{\partial}{\partial y}\left(A_{66}\frac{w}{R_{xy}}\right)=0\\&\left(A_{12}+\frac{A_{66}}{2}\right)\frac{\partial^2 u}{\partial x\partial y}+\frac{A_{66}}{2}\frac{\partial^2 v}{\partial x^2}+A_{22}\frac{\partial^2 v}{\partial y^2}+\left(A_{12}+\frac{A_{66}}{2}\right)\frac{\partial^2 w}{\partial x\partial y}\frac{\partial w}{\partial x}\\&\quad+\left(\frac{A_{66}}{2}\frac{\partial^2 w}{\partial x^2}+A_{22}\frac{\partial^2 w}{\partial y^2}\right)\frac{\partial w}{\partial y}+\frac{\partial}{\partial x}\left(A_{66}\frac{w}{R_{xy}}\right)\\&\quad+\frac{\partial}{\partial y}\left(A_{12}\frac{w}{R_x}+A_{22}\frac{w}{R_y}\right)=0\\&(D_{11}-H_{11})\frac{\partial^4 w}{\partial x^4}+(2D_{12}+2D_{66}-H_{12}-H_{21}-2H_{33}-2H_{34})\frac{\partial^4 w}{\partial x^2\partial y^2}\\&\quad+(D_{22}-H_{22})\frac{\partial^4 w}{\partial y^4}-H_{11}\frac{\partial^3\psi_x}{\partial x^3}-(H_{21}+2H_{34})\frac{\partial^3\psi_x}{\partial x\partial y^2}\\&\quad-(H_{12}+2H_{33})\frac{\partial^3\psi_y}{\partial x^2\partial y}-H_{22}\frac{\partial^3\psi_y}{\partial y^3}\end{aligned}\right\}\tag{1.23a～e}$$

$$
\begin{aligned}
&=-q-\left[A_{11}\frac{\partial u}{\partial x}+A_{12}\frac{\partial v}{\partial y}+\left(\frac{A_{11}}{R_x}+\frac{A_{12}}{R_y}\right)w+\frac{A_{11}}{2}\left(\frac{\partial w}{\partial x}\right)^2\right.\\
&\quad\left.+\frac{A_{12}}{2}\left(\frac{\partial w}{\partial y}\right)^2\right]\left(\frac{\partial^2 w}{\partial x^2}-\frac{1}{R_x}\right)-\left[A_{12}\frac{\partial u}{\partial x}+A_{22}\frac{\partial v}{\partial y}+\left(\frac{A_{12}}{R_x}+\frac{A_{22}}{R_y}\right)w\right.\\
&\quad\left.+\frac{A_{12}}{2}\left(\frac{\partial w}{\partial x}\right)^2+\frac{A_{22}}{2}\left(\frac{\partial w}{\partial y}\right)^2\right]\left(\frac{\partial^2 w}{\partial y^2}-\frac{1}{R_y}\right)-A_{66}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right.\\
&\quad\left.+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}+\frac{2w}{R_{xy}}\right)\left(\frac{\partial^2 w}{\partial x\partial y}-\frac{1}{R_{xy}}\right)\\
&(H_{11}-J_{11})\frac{\partial^3 w}{\partial x^3}+(H_{12}+H_{33}+H_{34}-J_{12}-J_{43}-J_{44})\frac{\partial^3 w}{\partial x\partial y^2}-J_{11}\frac{\partial^2\psi_x}{\partial x^2}\\
&\quad-J_{44}\frac{\partial^2\psi_x}{\partial y^2}-(J_{12}+J_{43})\frac{\partial^2\psi_y}{\partial x\partial y}=Q_{11}\left(\psi_x+\frac{\partial w}{\partial x}\right)\\
&(H_{21}+H_{33}+H_{34}-J_{21}-J_{33}-J_{34})\frac{\partial^3 w}{\partial x^2\partial y}+(H_{22}-J_{22})\frac{\partial^3 w}{\partial y^3}\\
&\quad-(J_{21}+J_{34})\frac{\partial^2\psi_x}{\partial x\partial y}-J_{33}\frac{\partial^2\psi_y}{\partial x^2}-J_{22}\frac{\partial^2\psi_y}{\partial y^2}=Q_{22}\left(\psi_y+\frac{\partial w}{\partial y}\right)
\end{aligned}
$$

若引入应力函数φ

$$N_x=\partial^2\varphi/\partial y^2,\quad N_y=\partial^2\varphi/\partial x^2,\quad N_{xy}=-\partial^2\varphi/\partial x\partial y \tag{1.24}$$

平衡方程(1.21)的前两式已经满足，但需要增加一个协调条件(1.11)式．此时平衡方程为

$$
\left.
\begin{aligned}
&A'_{11}\frac{\partial^4\varphi}{\partial y^4}+2(A'_{12}+A'_{66})\frac{\partial^4\varphi}{\partial x^2\partial y^2}+A'_{22}\frac{\partial^4\varphi}{\partial x^4}\\
&\quad=\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2-\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}+\frac{1}{R_x}\frac{\partial^2 w}{\partial y^2}+\frac{1}{R_y}\frac{\partial^2 w}{\partial x^2}-\frac{2}{R_{xy}}\frac{\partial^2 w}{\partial x\partial y}\\
&(D_{11}-H_{11})\frac{\partial^4 w}{\partial x^4}+(2D_{12}+2D_{66}-H_{12}-H_{21}-2H_{33}-2H_{34})\frac{\partial^4 w}{\partial x^2\partial y^2}\\
&\quad+(D_{22}-H_{22})\frac{\partial^4 w}{\partial y^4}-H_{11}\frac{\partial^3\psi_x}{\partial x^3}-(H_{21}+2H_{34})\frac{\partial^3\psi_x}{\partial x\partial y^2}\\
&\quad-(H_{12}+2H_{33})\frac{\partial^3\psi_y}{\partial x^2\partial y}-H_{22}\frac{\partial^3\psi_y}{\partial y^3}\\
&\quad=-q-\frac{\partial^2\varphi}{\partial y^2}\left(\frac{\partial^2 w}{\partial x^2}-\frac{1}{R_x}\right)-\frac{\partial^2\varphi}{\partial x^2}\left(\frac{\partial^2 w}{\partial y^2}-\frac{1}{R_y}\right)\\
&\quad+2\frac{\partial^2\varphi}{\partial x\partial y}\left(\frac{\partial^2 w}{\partial x\partial y}-\frac{1}{R_{xy}}\right)\\
&(H_{11}-J_{11})\frac{\partial^3 w}{\partial x^3}+(H_{12}+H_{33}+H_{34}-J_{12}-J_{43}-J_{44})\frac{\partial^3 w}{\partial x\partial y^2}-J_{11}\frac{\partial^2\psi_x}{\partial x^2}\\
&\quad-J_{44}\frac{\partial^2\psi_y}{\partial y^2}-(J_{12}+J_{43})\frac{\partial^2\psi_y}{\partial x\partial y}=Q_{11}\left(\psi_x+\frac{\partial w}{\partial x}\right)\\
&(H_{21}+H_{33}+H_{34}-J_{21}-J_{33}-J_{34})\frac{\partial^3 w}{\partial x^2\partial y}+(H_{22}-J_{22})\frac{\partial^3 w}{\partial y^3}\\
&\quad-(J_{21}+J_{34})\frac{\partial^2\psi_x}{\partial x\partial y}-J_{33}\frac{\partial^2\psi_y}{\partial x^2}-J_{22}\frac{\partial^2\psi_y}{\partial y^2}=Q_{22}\left(\psi_y+\frac{\partial w}{\partial y}\right)
\end{aligned}
\right\}\quad(1.25\text{a}\sim\text{d})
$$

其中

$$\left.\begin{aligned}&A'_{11}=A_{22}/(A_{11}A_{22}-A_{12}^2),\quad A'_{22}=A_{11}/(A_{11}A_{22}-A_{12}^2)\\&A'_{12}=-A_{12}/(A_{11}A_{22}-A_{12}^2),\quad A'_{66}=1/A_{66}\end{aligned}\right\}\tag{1.26}$$

边界条件略去，这里不再列出。

从平衡方程(1.23)、(1.25)可以看出，即使对称铺层的正交异性多夹层扁壳的基本方程也相当复杂，除去某些特别的例子，这组方程是不易求解的．下面，研究各向同性材料的多夹层扁壳．

二、各向同性材料多夹层扁壳的大挠度方程

下面研究各层材料为宏观各向同性的多夹层扁壳．由于各向同性是正交异性的特例，因此可由上节的公式很方便地推得本节所需的公式．

位移假设不变．由于是各向同性材料，那么有${}_id_1={}_id_2={}_id$，因此

$${}_i\gamma_x={}_id\gamma_x,\quad {}_i\gamma_y={}_id\gamma_y$$

$${}_if={}_if_1={}_if_2=\frac{h_i-h_{i-1}}{2}{}_id+(h_{i-1}-h_{i-2})_{i-1}d+\cdots+(h_1-h_0)_1d+h_0\,{}_0d\tag{2.1}$$

第i层的应变为

$$\left.\begin{aligned}&{}_i\varepsilon_x=e_x-z\chi_x-{}_i\bar{f}\lambda_x-z\,{}_id\lambda_x,\quad {}_i\varepsilon_y=e_y-z\chi_y-{}_i\bar{f}\lambda_y-z\,{}_id\lambda_y\\&{}_i\varepsilon_{xy}=e_{xy}-z\chi_{xy}-{}_i\bar{f}\lambda_{(xy)}-z\,{}_id\lambda_{(xy)}\end{aligned}\right\}\tag{2.2}$$

$${}_i\varepsilon_{xz}={}_id\gamma_x/2,\quad {}_i\varepsilon_{yz}={}_id\gamma_y/2\tag{2.3}$$

其中

$$\left.\begin{aligned}&{}_i\bar{f}={}_if-{}_ig\,{}_id\\&\lambda_{(xy)}=\frac{1}{2}(\lambda_{xy}+\lambda_{yx})=-\frac{1}{2}\left(\frac{\partial\gamma_y}{\partial x}+\frac{\partial\gamma_x}{\partial y}\right)=-\frac{1}{2}\left(\frac{\partial\psi_y}{\partial x}+\frac{\partial\psi_x}{\partial y}\right)-\frac{\partial^2w}{\partial x\partial y}\end{aligned}\right\}\tag{2.4}$$

第i层表层，各向同性材料的本构关系

$${}_i\sigma_x=\frac{{}_iE}{1-{}_i\nu^2}({}_i\varepsilon_x+{}_i\nu\,{}_i\varepsilon_y),\quad {}_i\sigma_y=\frac{{}_iE}{1-{}_i\nu^2}({}_i\varepsilon_y+{}_i\nu\,{}_i\varepsilon_x),\quad {}_i\tau_{xy}=\frac{{}_iE}{1+{}_i\nu}{}_i\varepsilon_{xy}\tag{2.5}$$

同样地，可认为表层有${}_id=0$，那么

$${}_i\bar{f}={}_if=(h_{i-1}-h_{i-2})_{i-1}d+(h_{i-3}-h_{i-4})_{i-3}d+\cdots+(h_2-h_1)_2d+h_0\,{}_0d\tag{2.6}$$

这里，我们对${}_if$略作修正，即把${}_{i-1}d$前的系数由$(h_{i-1}-h_{i-2})$稍放大至$(h_i+h_{i-1}-(h_{i-2}+h_{i-3}))/2$，

$$\begin{aligned}{}_if=&\frac{h_i+h_{i-1}-(h_{i-2}+h_{i-3})}{2}{}_{i-1}d+\frac{h_{i-2}+h_{i-3}-(h_{i-4}+h_{i-5})}{2}{}_{i-3}d\\&+\cdots+\frac{h_3+h_2-(h_1+h_0)}{2}{}_2d+\frac{h_1+h_0}{2}{}_0d\\=&(h_{i-1}-h_{i-2})_{i-1}d'_{i-1}d+(h_{i-3}-h_{i-4})_{i-3}d'_{i-3}d+\cdots+(h_2-h_1)_2d'\,{}_2d+h_0\,{}_0d'\,{}_0d\end{aligned}$$

$${}_{i-1}d'=[h_i+h_{i-1}-(h_{i-2}+h_{i-3})]/2(h_{i-1}-h_{i-2})\tag{2.7}$$

那么第j层夹心的应力应变关系中，${}_id$应用${}_jd'\,{}_jd$代替

$$\left.\begin{aligned}&{}_j\tau_{xz}={}_jG\cdot 2{}_j\varepsilon_{xz}={}_jG{}_j\gamma_x={}_jG{}_jd'{}_jd\gamma_x,\quad {}_j\tau_{yz}={}_jG\cdot 2{}_j\varepsilon_{yz}={}_jG{}_j\gamma_y={}_jG{}_jd'{}_jd\gamma_y\\&{}_jd=G/{}_jG,\quad G=\sum_{\mathrm{I}}(h_j-h_{j-1})\Big/\sum_{\mathrm{I}}\frac{h_j-h_{j-1}}{{}_jG}\end{aligned}\right\}\tag{2.8}$$

下面的推导限于对称铺层．对于内力N_x，N_y，N_{xy}，可推得

$$\left.\begin{aligned}&N_x=E\left\{\frac{\partial u}{\partial x}+\frac{w}{R_x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2+\nu\left[\frac{\partial v}{\partial y}+\frac{w}{R_y}+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right]\right\}\\&N_y=E\left\{\nu\left[\frac{\partial u}{\partial x}+\frac{w}{R_x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]+\frac{\partial v}{\partial y}+\frac{w}{R_y}+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right\}\\&N_{xy}=\frac{1-\nu}{2}E\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}+\frac{2w}{R_{xy}}\right)\end{aligned}\right\}\tag{2.9}$$

式中的E，ν见(2.12)．并且同样可求得M_x，M_y，M_{xy}和m_x，m_y，m_{xy}．

对于横向剪力Q_x，Q_y，由于${}_jd$作过修正，有

$$Q_x=Gh_c\gamma_x,\quad Q_y=Gh_c\gamma_y,\quad h_c=h-t_n\tag{2.10}$$

即h_c是壳厚h减去最外表层的厚度t_n．此时，当多夹层扁壳若只有单夹心层时，上式定义的剪力与已有理论一致[2]．

为了便于运用，给出应变能密度W如下

$$\left.\begin{aligned}&W=W_1+W_2\\&W_1=\frac{1}{2(1-\nu^2)E}[(N_x^2+N_y^2)-2(1+\nu)(N_xN_y-N_{xy}^2)]+\frac{1}{2}D'\left[\left(\frac{\partial\psi_x}{\partial x}\right)^2\right.\\&\quad\left.+\left(\frac{\partial\psi_y}{\partial y}\right)^2+2\nu\frac{\partial\psi_x}{\partial x}\frac{\partial\psi_y}{\partial y}+\frac{1-\nu}{2}\left(\frac{\partial\psi_x}{\partial y}+\frac{\partial\psi_y}{\partial x}\right)^2\right]+\frac{1}{2}(D_f+D_f')\left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2\right.\\&\quad\left.+\left(\frac{\partial^2 w}{\partial y^2}\right)^2+2\nu\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}+2(1-\nu)\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2\right]+D_f'\left[\frac{\partial\psi_x}{\partial x}\frac{\partial^2 w}{\partial x^2}\right.\\&\quad\left.+\frac{\partial\psi_y}{\partial y}\frac{\partial^2 w}{\partial y^2}+\nu\left(\frac{\partial\psi_x}{\partial x}\frac{\partial^2 w}{\partial y^2}+\frac{\partial\psi_y}{\partial y}\frac{\partial^2 w}{\partial x^2}\right)+(1-\nu)\left(\frac{\partial\psi_x}{\partial y}+\frac{\partial\psi_y}{\partial x}\right)\frac{\partial^2 w}{\partial x\partial y}\right]\\&W_2=\frac{1}{2}Gh'\left(\psi_x+\frac{\partial w}{\partial x}\right)^2+\frac{1}{2}Gh'\left(\psi_y+\frac{\partial w}{\partial y}\right)^2\end{aligned}\right\}\tag{2.11}$$

其中

$$\left.\begin{aligned}&\nu=\nu_i,\quad h'=\sum_{\mathrm{I}}\frac{(h_{j+1}+h_j-h_{j-1}-h_{j-2})^2}{4(h_j-h_{j-1})}\\&E=2\sum_{\mathrm{II}}\frac{{}_iE}{1-{}_i\nu^2}(h_i-h_{i-1}),\quad D=2\sum_{\mathrm{II}}\frac{{}_iE}{1-{}_i\nu^2}\frac{h_i^2-h_{i-1}^2}{2}{}_if\\&D_2=2\sum_{\mathrm{II}}\frac{{}_iE}{1-{}_i\nu^2}\frac{h_i^3-h_{i-1}^3}{3},\quad D'=2\sum_{\mathrm{II}}\frac{{}_iE}{1-{}_i\nu^2}(h_i-h_{i-1})\,{}_if^2\\&D_f=D_2-D,\quad D_f'=D'-D,\quad \sum_{\mathrm{I}}=\sum_{i=1,3,5}^{2n-1}\end{aligned}\right\}\tag{2.12}$$

若以位移u，v，w，ψ_x，ψ_y为基本变量，那么平衡方程为

$$
\left.\begin{aligned}
&\frac{\partial^2 u}{\partial x^2}+\frac{1-\nu}{2}\frac{\partial^2 u}{\partial y^2}+\frac{1+\nu}{2}\frac{\partial^2 v}{\partial x\partial y}=-\frac{\partial w}{\partial x}\left(\frac{\partial^2 w}{\partial x^2}+\frac{1-\nu}{2}\frac{\partial^2 w}{\partial y^2}\right)\\
&\quad-\frac{1+\nu}{2}\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x\partial y}-\frac{\partial}{\partial x}\left(\frac{w}{R_x}+\nu\frac{w}{R_y}\right)-(1-\nu)\frac{\partial}{\partial y}\left(\frac{w}{R_{xy}}\right)\\
&\frac{1+\nu}{2}\frac{\partial^2 u}{\partial x\partial y}+\frac{1-\nu}{2}\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=-\frac{\partial w}{\partial y}\left(\frac{1-\nu}{2}\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)\\
&\quad-\frac{1+\nu}{2}\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x\partial y}-\frac{\partial}{\partial y}\left(\nu\frac{w}{R_x}+\frac{w}{R_y}\right)-(1-\nu)\frac{\partial}{\partial x}\left(\frac{w}{R_{xy}}\right)\\
&Gh'\left(\frac{\partial\psi_x}{\partial x}+\frac{\partial\psi_y}{\partial y}+\nabla^2 w\right)-(\nabla_f+D'_f)\nabla^4 w-D'_f\left(\frac{\partial}{\partial x}\nabla^2\psi_x+\frac{\partial}{\partial y}\nabla^2\psi_y\right)\\
&=-q-E\left\{\left[\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2+\frac{w}{R_x}\right]\left[\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}-\left(\frac{1}{R_x}+\frac{\nu}{R_y}\right)\right]\right.\\
&\quad+\left[\frac{\partial v}{\partial y}+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2+\frac{w}{R_y}\right]\left[\nu\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}-\left(\frac{\nu}{R_x}+\frac{1}{R_y}\right)\right]\\
&\quad\left.+(1-\nu)\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}+\frac{2w}{R_{xy}}\right)\left(\frac{\partial^2 w}{\partial x\partial y}-\frac{1}{R_{xy}}\right)\right\}\\
&D'\left(\frac{\partial^2\psi_x}{\partial x^2}+\frac{1-\nu}{2}\frac{\partial^2\psi_x}{\partial y^2}+\frac{1+\nu}{2}\frac{\partial^2\psi_y}{\partial x\partial y}\right)+D'_f\frac{\partial}{\partial x}\nabla^2 w=Gh'\left(\psi_x+\frac{\partial w}{\partial x}\right)\\
&D'\left(\frac{1+\nu}{2}\frac{\partial^2\psi_x}{\partial x\partial y}+\frac{1-\nu}{2}\frac{\partial^2\psi_y}{\partial x^2}+\frac{\partial^2\psi_y}{\partial y^2}\right)+D'_f\frac{\partial}{\partial y}\nabla^2 w=Gh'\left(\psi_y+\frac{\partial w}{\partial y}\right)
\end{aligned}\right\}\quad(2.13)
$$

对应于平面内不可移动边界，一般有夹紧固定和铰支边界。边界条件以夹紧固定为例，有

$$
\left.\begin{aligned}
&x=a_0,\ a\text{时},\ u=0,\quad v=0,\quad w=0,\quad \psi_x=0,\quad \psi_y=0,\quad \partial w/\partial x=0\\
&y=b_0,\ b\text{时},\ u=0,\quad v=0,\quad w=0,\quad \psi_x=0,\quad \psi_y=0,\quad \partial w/\partial y=0
\end{aligned}\right\}\quad(2.14)
$$

引入应力函数φ，以φ，w，ψ_x，ψ_y为基本变量的平衡方程为

$$
\left.\begin{aligned}
&\nabla^4\varphi=(1-\nu^2)E\left[\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2-\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}+\frac{1}{R_x}\frac{\partial^2 w}{\partial y^2}+\frac{1}{R_y}\frac{\partial^2 w}{\partial x^2}-\frac{2}{R_{xy}}\frac{\partial^2 w}{\partial x\partial y}\right]\\
&Gh'\left(\frac{\partial\psi_x}{\partial x}+\frac{\partial\psi_y}{\partial y}+\nabla^2 w\right)-(D_f+D'_f)\nabla^4 w-D'_f\left(\frac{\partial}{\partial x}\nabla^2\psi_x+\frac{\partial}{\partial y}\nabla^2\psi_y\right)\\
&=-q-\left[\frac{\partial^2\varphi}{\partial y^2}\left(\frac{\partial^2 w}{\partial x^2}-\frac{1}{R_x}\right)+\frac{\partial^2\varphi}{\partial x^2}\left(\frac{\partial^2 w}{\partial y^2}-\frac{1}{R_y}\right)-2\frac{\partial^2\varphi}{\partial x\partial y}\left(\frac{\partial^2 w}{\partial x\partial y}-\frac{1}{R_{xy}}\right)\right]\\
&D'\left(\frac{\partial^2\psi_x}{\partial x^2}+\frac{1-\nu}{2}\frac{\partial^2\psi_x}{\partial y^2}+\frac{1+\nu}{2}\frac{\partial^2\psi_y}{\partial x\partial y}\right)+D'_f\frac{\partial}{\partial x}\nabla^2 w=Gh'\left(\psi_x+\frac{\partial w}{\partial x}\right)\\
&D'\left(\frac{1+\nu}{2}\frac{\partial^2\psi_x}{\partial x\partial y}+\frac{1-\nu}{2}\frac{\partial^2\psi_y}{\partial x^2}+\frac{\partial^2\psi_y}{\partial y^2}\right)+D'_f\frac{\partial}{\partial y}\nabla^2 w=Gh'\left(\psi_y+\frac{\partial w}{\partial y}\right)
\end{aligned}\right\}\quad(2.15)
$$

对应于平面内可移动边界，一般有滑动固支和简支边界．边界条件以简支为例，有

$$\left.\begin{aligned}
&x=a_0,\ a\ 时,\ \frac{\partial^2\varphi}{\partial y^2}=N_x^0,\quad \frac{\partial^2\varphi}{\partial x\partial y}=-N_{xy}^0,\quad w=0\\
&\quad D'\frac{\partial\psi_x}{\partial x}+D_f'\frac{\partial^2 w}{\partial x^2}=0,\quad \psi_y=0,\quad D_f'\frac{\partial\psi_x}{\partial x}+(D_f+\nabla_f')\frac{\partial^2 w}{\partial x^2}=0\\
&y=b_0,\ b\ 时,\ \frac{\partial^2\varphi}{\partial x^2}=N_y^0,\quad \frac{\partial^2\varphi}{\partial x\partial y}=-N_{xy}^0,\quad w=0,\quad \psi_x=0\\
&\quad D'\frac{\partial\psi_y}{\partial y}+D_f'\frac{\partial^2 w}{\partial y^2}=0,\quad D_f'\frac{\partial\psi_y}{\partial y}+(D_f+D_f')\frac{\partial^2 w}{\partial y^2}=0
\end{aligned}\right\}\tag{2.16}$$

当夹心层材料相同时，有${}_jd=1$，那么${}_if=(h_i+h_{i-1})/2$，由(2.12)式得

$$D'=D,\ D_f'=0,\ D_f=2\sum_{\mathrm{I}}\frac{{}_iE}{1-{}_i\nu^2}\ \frac{(h_i-h_{i-1})^3}{2}\tag{2.17}$$

那么可知D_f与D'之比$\varepsilon'=D_f/D'$为一小量.

为了系统地求解多夹层扁壳的大挠度问题，对平衡方程和边界条件进行无量纲化，以便于求解. 同时，假设(2.17)式成立，即$D_f'=0$. 现引入下列无量纲参数

$$\left.\begin{aligned}
&\lambda=\frac{a}{b},\quad \xi=\frac{x}{a},\quad \eta=\frac{y}{b},\quad \Psi_\xi=\frac{a}{\bar h}\psi_x,\quad \Psi_\eta=\frac{a}{\bar h}\psi_y\\
&W=\frac{w}{\bar h},\quad U=\frac{a}{\bar h^2}u,\quad V=\frac{a}{\bar h^2}v\\
&\varepsilon=\frac{D'}{Gh'a^2},\quad \varepsilon_0=\frac{E\bar h^2}{D'},\quad \varepsilon_f=\frac{D_f}{Gh'a^2}\\
&Q=\frac{a^4}{D'\bar h}q,\quad \Phi=\frac{\varphi}{D'}+\frac{P}{2\lambda^2}(\eta^2+\rho\lambda^2\xi^2),\quad P=\frac{a^2}{D'}N_x^0,\quad \rho=\frac{N_y^0}{N_x^0}\\
&R=\frac{\bar h}{a^2}R_x,\quad r=\frac{R_x}{R_y},\quad \bar h=\frac{h_c}{2},\quad r_1=\frac{R_x}{R_{xy}}=0
\end{aligned}\right\}\tag{2.18}$$

此时，平衡方程为

$$\left.\begin{aligned}
&\frac{\partial^2\Psi_\xi}{\partial\xi^2}+\frac{1-\nu}{2}\lambda^2\frac{\partial^2\Psi_\xi}{\partial\eta^2}+\frac{1+\nu}{2}\lambda\frac{\partial^2\Psi_\eta}{\partial\xi\partial\eta}-\frac{1}{\varepsilon}\left(\Psi_\xi+\frac{\partial W}{\partial\xi}\right)=0\\
&\frac{1+\nu}{2}\lambda\frac{\partial^2\Psi_\xi}{\partial\xi\partial\eta}+\frac{1-\nu}{2}\frac{\partial^2\Psi_\eta}{\partial\xi^2}+\lambda^2\frac{\partial^2\Psi_\eta}{\partial\eta^2}-\frac{1}{\varepsilon}\left(\Psi_\eta+\lambda\frac{\partial W}{\partial\eta}\right)=0\\
&\frac{\partial\Psi_\xi}{\partial\xi}+\lambda\frac{\partial\Psi_\eta}{\partial\eta}+L_1W-\varepsilon_fL_1^2W-\varepsilon P\left(\frac{\partial^2W}{\partial\xi^2}+\rho\lambda^2\frac{\partial^2W}{\partial\eta^2}\right.\\
&\quad\left.-\frac{1}{R}-\rho\frac{r}{R}\right)=-\varepsilon(Q+F)\\
&L_1^2\Phi=\varepsilon_0(1-\nu^2)\lambda^2A\\
&\frac{\partial^2U}{\partial\xi^2}+\frac{1-\nu}{2}\lambda^2\frac{\partial^2U}{\partial\eta^2}+\frac{1+\nu}{2}\lambda\frac{\partial^2V}{\partial\xi\partial\eta}=-B\\
&\frac{1+\nu}{2}\lambda\frac{\partial^2U}{\partial\xi\partial\eta}+\frac{1-\nu}{2}\frac{\partial^2V}{\partial\xi^2}+\lambda^2\frac{\partial^2V}{\partial\eta^2}=-C
\end{aligned}\right\}\tag{2.19a～f}$$

$$L_1=\frac{\partial^2}{\partial\xi^2}+\lambda^2\frac{\partial^2}{\partial\eta^2}\tag{2.20}$$

求解平面内可移边界问题，(2.19a～d)构成了基本方程组

$$\left.\begin{aligned}F&=\lambda^2\left[\frac{\partial^2\Phi}{\partial\eta^2}\left(\frac{\partial^2W}{\partial\zeta^2}-\frac{1}{R}\right)+\frac{\partial^2\Phi}{\partial\zeta^2}\left(\frac{\partial^2W}{\partial\eta^2}-\frac{r}{\lambda^2R}\right)-2\frac{\partial^2\Phi}{\partial\zeta\partial\eta}\frac{\partial^2W}{\partial\zeta\partial\eta}\right]\\A&=\left(\frac{\partial^2W}{\partial\zeta\partial\eta}\right)^2-\frac{\partial^2W}{\partial\zeta^2}\frac{\partial^2W}{\partial\eta^2}+\frac{1}{R}\frac{\partial^2W}{\partial\eta^2}+\frac{r}{\lambda^2R}\frac{\partial^2W}{\partial\zeta^2}\end{aligned}\right\}\tag{2.21}$$

求解平面内不可移边界问题，(2.19a～c, e, f)构成基本方程组

$$\left.\begin{aligned}F=\varepsilon_0\Big\{&\left[\frac{\partial U}{\partial\zeta}+\frac{1}{2}\left(\frac{\partial W}{\partial\zeta}\right)^2+\frac{W}{R}\right]\left[\frac{\partial^2W}{\partial\zeta^2}+\nu\lambda^2\frac{\partial^2W}{\partial\eta^2}-\left(\frac{1}{R}+\frac{\nu r}{R}\right)\right]\\&+\lambda\left[\frac{\partial V}{\partial\eta}+\frac{\lambda}{2}\left(\frac{\partial W}{\partial\eta}\right)^2+\frac{rW}{\lambda R}\right]\left[\lambda^2\frac{\partial^2W}{\partial\eta^2}+\nu\frac{\partial^2W}{\partial\zeta^2}-\left(\frac{r}{R}+\frac{\nu}{R}\right)\right]\\&+(1-\nu)\lambda\left(\lambda\frac{\partial U}{\partial\eta}+\frac{\partial V}{\partial\zeta}+\lambda\frac{\partial W}{\partial\zeta}\frac{\partial W}{\partial\eta}\right)\frac{\partial^2W}{\partial\zeta\partial\eta}\Big\}\\B=&\frac{\partial W}{\partial\zeta}\left(\frac{\partial^2W}{\partial\zeta^2}+\frac{1-\nu}{2}\lambda^2\frac{\partial^2W}{\partial\eta^2}\right)+\frac{1+\nu}{2}\lambda^2\frac{\partial W}{\partial\eta}\frac{\partial^2W}{\partial\zeta\partial\eta}+\left(\frac{1}{R}+\frac{\nu r}{R}\right)\frac{\partial W}{\partial\zeta}\\C=&\lambda\frac{\partial W}{\partial\eta}\left(\frac{1-\nu}{2}\frac{\partial^2W}{\partial\zeta^2}+\lambda^2\frac{\partial^2W}{\partial\eta^2}\right)+\frac{1+\nu}{2}\lambda\frac{\partial W}{\partial\zeta}\frac{\partial^2W}{\partial\zeta\partial\eta}+\left(\frac{r}{R}+\frac{\nu}{R}\right)\lambda\frac{\partial W}{\partial\eta}\end{aligned}\right\}\tag{2.22}$$

无量纲化后的边界条件，以简支边界为例，有

$$\left.\begin{aligned}&\zeta=\frac{a_0}{a},\ 1\text{时},\quad\frac{\partial^2\Phi}{\partial\eta^2}=0,\ \frac{\partial^2\Phi}{\partial\zeta\partial\eta}=0,\ W=0,\ \frac{\partial\Psi_\zeta}{\partial\zeta}=0,\ \Psi_\eta=0,\ \frac{\partial^2W}{\partial\zeta^2}=0\\&\eta=\frac{b_0}{b},\ 1\text{时},\quad\frac{\partial^2\Phi}{\partial\zeta\partial\eta}=0,\ \frac{\partial^2\Phi}{\partial\zeta^2}=0,\ W=0,\ \Psi_\zeta=0,\ \frac{\partial\Psi_\eta}{\partial\eta}=0,\ \frac{\partial^2W}{\partial\eta^2}=0\end{aligned}\right\}\tag{2.23}$$

对于工程中常见的扁壳（如扁球壳、扁柱壳、平板以及中面为平移曲面的扁壳），都有 $1/R_{xy}=0$，因此(2.18)式中令 $r_1=R_x/R_{xy}=0$。这里，若 $r=1$，为扁球壳；$r=0$，为扁柱壳；$1/R=0$，扁壳则退化为平板。

这里获得的平衡方程，若有 $1/R=0$，则与文[3]中所用的矩形夹层板的基本方程相一致。同时，根据钱伟长[4]对壳体理论的分类可知，本文涉及的是Kármán型大挠度问题。本文的下一部分，具体地求解多夹层扁壳的弯曲与稳定。

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Nonlinear Theory of Multilayer Sandwich Shells and Its Application (Ⅱ) —Fundamental Equations for Orthotropic Shallow Shells

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Abstract

This paper applied the simplified theory for multilayer sandwich shells undergoing moderate/small rotations in Ref. [1] to shallow shells. The equilibrium equations and boundary conditions of large deflection of orthotropic and the special case, isotropic shells, are presented.

Key words multilayer sandwich shallow shells, orthotropic fundamental equation