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变厚度圆薄板非对称非线性弯曲问题*

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(我刊编委叶开沅来稿)

摘要: 首先将直角坐标系中的横向变厚度薄板的大挠度方程,转化到极坐标系中的变厚度圆薄板的非对称大挠度方程· 此方程和极坐标系中径向、切向两个平衡方程联立求解· 将物理方程和中面应变非线性变形方程,代入3个平衡方程,可得用3个变形位移表示的3个非对称非线性方程· 用Fourier级数表示的解代入基本方程,获得相应的基本方程· 在周边夹紧边界条件下,用修正迭代法求解· 作为算例,研究了余弦形式载荷作用下的问题,还给出了载荷与挠度的特征曲线,曲线依据变厚度参数变化而变化,其结果和物理概念完全吻合·

关 键 词: 变厚度; 非对称; 修正迭代法; 挠度
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引 言

在各种工程领域中,广泛应用着许多具有非均匀变厚度的结构元件(杆、板、壳)· 由于它们具有较好的结构特征优化的形状使得工程师设计时节约材料,但是对它们的力学性能的分析就增加了难度· 以往的大部分工作解决了轴对称的问题[1~4],但是对于变厚度圆薄板的非轴对称非线性变形问题尚未有人问津,只是对于等厚度非对称大变形问题作了一部分工作·

本文在文献[4~9]工作的基础上展开研究工作· 本文方法可推广到变厚度圆底壳的非轴对称大变形问题中应用·

1 基 本 方 程

在直角坐标系下关于薄板横向的大挠度方程为:

$$D\left(\frac{\partial^4 W}{\partial x^4}+\frac{\partial^4 W}{\partial y^4}+2\frac{\partial^4 W}{\partial x^2\partial y^2}\right)=q+N_x\frac{\partial^2 W}{\partial x^2}+N_y\frac{\partial^2 W}{\partial y^2}+2N_{xy}\frac{\partial^2 W}{\partial x\partial y}-$$
$$\frac{\partial^2 D}{\partial x^2}\left(\frac{\partial^2 W}{\partial x^2}+\mu\frac{\partial^2 W}{\partial y^2}\right)-\frac{\partial^2 D}{\partial y^2}\left(\frac{\partial^2 W}{\partial y^2}+\mu\frac{\partial^2 W}{\partial x^2}\right)-2\frac{\partial D}{\partial x}\left(\frac{\partial^3 W}{\partial x^3}+\frac{\partial^3 W}{\partial x\partial y^2}\right)-$$

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$$2\frac{\partial D}{\partial y}\left(\frac{\partial^3 W}{\partial y^3}+\frac{\partial^3 W}{\partial x^2\partial y}\right)-2(1-\mu)\frac{\partial^2 D}{\partial x\partial y}\frac{\partial^2 W}{\partial x\partial y}.$$

在极坐标系中径向、切向、横向平衡方程为:

$$\frac{\partial\sigma_r}{\partial r}+\frac{\sigma_r-\sigma_\theta}{r}+\frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta}=0, \tag{1}$$

$$\frac{\partial\tau_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta}+2\frac{\tau_{r\theta}}{r}=0, \tag{2}$$

$$\begin{aligned}D\Delta^2(W)={}&q+N_r\frac{\partial^2 W}{\partial r^2}+N_\theta\left(\frac{\partial^2 W}{\partial\theta^2}\frac{1}{r^2}+\frac{\partial W}{\partial r}\frac{1}{r}\right)+2N_{r\theta}\left(\frac{\partial^2 W}{\partial\theta\partial r}\frac{1}{r}-\frac{\partial W}{\partial\theta}\frac{1}{r^2}\right)-\\&\frac{\partial^2 D}{\partial r^2}\left(\frac{\partial^2 W}{\partial r^2}+\frac{\partial^2 W}{\partial\theta^2}\frac{\mu}{r^2}+\frac{\partial W}{\partial r}\frac{\mu}{r}\right)-\left(\frac{\partial^2 D}{\partial\theta^2}\frac{1}{r^2}+\frac{\partial D}{\partial r}\frac{1}{r}\right)\left(\frac{\partial^2 W}{\partial\theta^2}\frac{1}{r^2}+\frac{\partial W}{\partial r}\frac{1}{r}+\right.\\&\left.\mu\frac{\partial^2 W}{\partial r^2}\right)-2\frac{\partial D}{\partial r}\left(\frac{\partial^3 W}{\partial r^3}+\frac{\partial^3 W}{\partial\theta^2\partial r}\frac{1}{r^2}+\frac{\partial^2 W}{\partial r^2}\frac{1}{r}-\frac{\partial^2 W}{\partial\theta^2}\frac{2}{r^3}-\frac{\partial W}{\partial r}\frac{1}{r^2}\right)-\\&2\frac{\partial D}{\partial\theta}\frac{1}{r}\left(\frac{\partial^3 W}{\partial\theta^3}\frac{1}{r^3}+\frac{\partial^3 W}{\partial\theta\partial r^2}\frac{1}{r}+\frac{\partial^2 W}{\partial\theta\partial r}\frac{1}{r^2}\right)+\\&2(1-\mu)\left(\frac{\partial^2 D}{\partial\theta\partial r}\frac{1}{r}-\frac{\partial D}{\partial\theta}\frac{1}{r^2}\right)\left(\frac{\partial^2 W}{\partial\theta\partial r}\frac{1}{r}-\frac{\partial W}{\partial\theta}\frac{1}{r^2}\right).\end{aligned} \tag{3}$$

中面变形方程:

$$\varepsilon_r=\frac{\partial U}{\partial r}+\frac{1}{2}\left(\frac{\partial W}{\partial r}\right)^2,\quad \varepsilon_\theta=\frac{1}{r}\frac{\partial V}{\partial\theta}+\frac{U}{r}+\frac{1}{2}\left(\frac{1}{r}\frac{\partial W}{\partial\theta}\right)^2,$$

$$\varepsilon_{r\theta}=\frac{1}{r}\frac{\partial U}{\partial\theta}-\frac{V}{r}+\frac{\partial V}{\partial r}+\frac{1}{r}\frac{\partial W}{\partial\theta}\frac{\partial W}{\partial r};$$

物理方程:

$$N_r=\frac{hE}{1-\mu^2}(\varepsilon_r+\mu\varepsilon_\theta)=\sigma_r h,\quad N_\theta=\frac{hE}{1-\mu^2}(\varepsilon_\theta+\mu\varepsilon_r)=\sigma_\theta h,$$

$$N_{r\theta}=\frac{hE}{2(1+\mu)}\varepsilon_{r\theta}=\tau_{r\theta}h;$$

其中 E 为板的弹性模量, μ 为Poisson比. U、V、W 为板的径向、切向和横向位移,

$$D=\frac{Eh^3}{12(1-\mu)},\quad \Delta=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2},$$

N_r、N_θ 为中面薄膜张力.

将中面变形方程代入物理方程再代入式(1)、(2)、(3)可得:

$$\begin{aligned}&\frac{r}{1-\mu}\left\{\frac{\partial^2 U}{\partial r^2}+\frac{\partial W}{\partial r}\frac{\partial^2 W}{\partial r^2}+\mu\frac{\partial}{\partial r}\left[\frac{U}{r}+\frac{1}{r}\frac{\partial V}{\partial\theta}+\frac{1}{2}\left(\frac{1}{r}\frac{\partial W}{\partial\theta}\right)^2\right]\right\}+\\&\quad\left\{\frac{\partial U}{\partial r}+\frac{1}{2}\left(\frac{\partial W}{\partial r}\right)^2-\left[\frac{U}{r}+\frac{1}{r}\frac{\partial V}{\partial\theta}+\frac{1}{2}\left(\frac{1}{r}\frac{\partial W}{\partial\theta}\right)^2\right]\right\}+\\&\quad\frac{1}{2}\frac{\partial}{\partial\theta}\left[\frac{1}{r}\frac{\partial U}{\partial\theta}+r\frac{\partial}{\partial r}\left(\frac{V}{r}\right)+\frac{1}{r}\frac{\partial W}{\partial\theta}\frac{\partial W}{\partial r}\right]=0,\end{aligned} \tag{4}$$

$$\begin{aligned}&\frac{r}{2}\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial U}{\partial\theta}+r\frac{\partial}{\partial r}\left(\frac{V}{r}\right)+\frac{1}{r}\frac{\partial W}{\partial\theta}\frac{\partial W}{\partial r}\right]+\frac{1}{1-\mu}\frac{\partial}{\partial\theta}\left\{\frac{U}{r}+\frac{1}{r}\frac{\partial V}{\partial\theta}+\frac{1}{2}\left(\frac{1}{r}\frac{\partial W}{\partial\theta}\right)^2+\right.\\&\quad\left.\mu\left[\frac{\partial U}{\partial r}+\frac{1}{2}\left(\frac{\partial W}{\partial r}\right)^2\right]\right\}+\left[\frac{1}{r}\frac{\partial U}{\partial\theta}+r\frac{\partial}{\partial r}\left(\frac{V}{r}\right)+\frac{1}{r}\frac{\partial W}{\partial\theta}\frac{\partial W}{\partial r}\right]=0,\end{aligned} \tag{5}$$

$$\begin{aligned}D\Delta^2(W)={}&q+\frac{Eh}{1-\mu^2}\left\{\frac{\partial^2 W}{\partial r^2}\left[\frac{\partial U}{\partial r}+\frac{1}{2}\left(\frac{\partial W}{\partial r}\right)^2+\mu\left[\frac{U}{r}+\frac{1}{r}\frac{\partial V}{\partial\theta}+\frac{1}{2}\left(\frac{1}{r}\frac{\partial W}{\partial\theta}\right)^2\right]\right]+\right.\\&(1-\mu)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial W}{\partial\theta}\right)\left[\frac{1}{r}\frac{\partial U}{\partial\theta}+\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{V}{r}\right)+\frac{1}{r}\frac{\partial W}{\partial\theta}\frac{\partial W}{\partial r}\right]+\end{aligned}$$

$$\left(\frac{1}{r}\frac{\partial W}{\partial r}+\frac{1}{r^2}\frac{\partial^2 W}{\partial\theta^2}\right)\left[\frac{U}{r}+\frac{1}{r}\frac{\partial V}{\partial\theta}+\frac{1}{2}\left(\frac{1}{r}\frac{\partial W}{\partial\theta}\right)^2+\mu\left(\frac{\partial U}{\partial r}+\frac{1}{2}\left(\frac{\partial W}{\partial r}\right)^2\right)\right]\Bigg\}-$$
$$\frac{\partial^2 D}{\partial r^2}\left(\frac{\partial^2 W}{\partial r^2}+\frac{\partial^2 W}{\partial\theta^2}\frac{\mu}{r^2}+\frac{\partial W}{\partial r}\frac{\mu}{r}\right)-$$
$$\left(\frac{\partial^2 D}{\partial\theta^2}\frac{1}{r^2}+\frac{\partial D}{\partial r}\frac{1}{r}\right)\left(\frac{\partial^2 W}{\partial\theta^2}\frac{1}{r^2}+\frac{\partial W}{\partial r}\frac{1}{r}+\mu\frac{\partial^2 W}{\partial r^2}\right)-$$
$$2\frac{\partial D}{\partial r}\left(\frac{\partial^3 W}{\partial r^3}+\frac{\partial^3 W}{\partial\theta^2\partial r}\frac{1}{r^2}+\frac{\partial^2 W}{\partial r^2}\frac{1}{r}-\frac{\partial^2 W}{\partial\theta^2}\frac{2}{r^3}-\frac{\partial W}{\partial r}\frac{1}{r^2}\right)-$$
$$2\frac{\partial D}{\partial\theta}\frac{1}{r}\left(\frac{\partial^3 W}{\partial\theta^3}\frac{1}{r^3}+\frac{\partial^3 W}{\partial\theta\partial r^2}\frac{1}{r}+\frac{\partial^2 W}{\partial\theta\partial r}\frac{1}{r^2}\right)+$$
$$2(1-\mu)\left(\frac{\partial^2 D}{\partial\theta\partial r}\frac{1}{r}-\frac{\partial D}{\partial\theta}\frac{1}{r^2}\right)\left(\frac{\partial^2 W}{\partial\theta\partial r}-\frac{\partial W}{\partial\theta}\frac{1}{r^2}\right). \tag{6}$$

周边固定夹紧:

$$r=a\text{ 时},\ W=\frac{\partial W}{\partial r}=U=V=0; \tag{7}$$

$$r=0\text{ 时},\ W,\frac{\partial W}{\partial r},U,V\text{ 有限}. \tag{8}$$

引入无量纲量:

$$x=\frac{r}{a},\ u=\frac{aU}{h_0^2},\ v=\frac{aV}{h_0^2},\ w=\frac{W}{h_0},\ \frac{h}{h_0}=f,\ \frac{D}{D_0}=f_1,\ Q=\frac{a^4q}{h_0D_0}.$$

假设圆板厚度变化规律为:

$$h(r)=h_0f(r)=h_0\left(1+\varepsilon\frac{r}{a}+\varepsilon^2\frac{r^2}{a^2}+\varepsilon^3\frac{r^3}{a^3}+\cdots\right),$$

其中 h_0 为板中心厚度, ε 为变厚度小参数. 为了求解方便本文取 $f=1+\varepsilon x$, $f_1=(1+\varepsilon x)^3$, 则方程(4)、(5)、(6) 可化为:

$$\frac{x}{1-\mu}\left\{\frac{\partial^2 u}{\partial x^2}+\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}+\mu\frac{\partial}{\partial x}\left[\frac{u}{x}+\frac{1}{x}\frac{\partial v}{\partial\theta}+\frac{1}{2}\left(\frac{1}{x}\frac{\partial w}{\partial\theta}\right)^2\right]\right\}+$$
$$\left\{\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2-\left[\frac{u}{x}+\frac{1}{x}\frac{\partial v}{\partial\theta}+\frac{1}{2}\left(\frac{1}{x}\frac{\partial w}{\partial\theta}\right)^2\right]\right\}+$$
$$\frac{1}{2}\frac{\partial}{\partial\theta}\left[\frac{1}{x}\frac{\partial u}{\partial\theta}+x\frac{\partial}{\partial x}\left(\frac{v}{x}\right)+\frac{1}{x}\frac{\partial w}{\partial\theta}\frac{\partial w}{\partial x}\right]=0, \tag{9}$$

$$\frac{x}{2}\frac{\partial}{\partial x}\left[\frac{1}{x}\frac{\partial u}{\partial\theta}+x\frac{\partial}{\partial x}\left(\frac{v}{x}\right)+\frac{1}{x}\frac{\partial w}{\partial\theta}\frac{\partial w}{\partial x}\right]+\frac{1}{1-\mu}\frac{\partial}{\partial\theta}\left\{\frac{u}{x}+\frac{1}{x}\frac{\partial v}{\partial\theta}+\frac{1}{2}\left(\frac{1}{x}\frac{\partial v}{\partial\theta}\right)^2+\right.$$
$$\left.\mu\left[\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]\right\}+\left[\frac{1}{x}\frac{\partial u}{\partial\theta}+x\frac{\partial}{\partial x}\left(\frac{v}{x}\right)+\frac{1}{x}\frac{\partial w}{\partial x}\right]=0, \tag{10}$$

$$\nabla^2(w)=Q+12(1+\varepsilon x)\left\{\frac{\partial^2 w}{\partial x^2}\left\{\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2+\mu\left[\frac{u}{x}+\frac{1}{x}\frac{\partial v}{\partial\theta}+\frac{1}{2}\left(\frac{1}{x}\frac{\partial w}{\partial\theta}\right)^2\right]\right\}+\right.$$
$$(1-\mu)\frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial w}{\partial\theta}\right)\left[\frac{1}{x}\frac{\partial u}{\partial\theta}+x\frac{\partial}{\partial x}\left(\frac{v}{x}\right)+\frac{1}{x}\frac{\partial w}{\partial\theta}\frac{\partial w}{\partial x}\right]+$$
$$\left.\left(\frac{1}{x}\frac{\partial w}{\partial x}+\frac{1}{x^2}\frac{\partial^2 w}{\partial\theta^2}\right)\left\{\frac{u}{x}+\frac{1}{x}\frac{\partial v}{\partial\theta}+\frac{1}{2}\left(\frac{1}{x}\frac{\partial w}{\partial\theta}\right)^2+\mu\left[\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]\right\}\right\}-$$
$$6\varepsilon^2(1+\varepsilon x)\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial\theta^2}\frac{\mu}{x^2}+\frac{\partial w}{\partial x}\frac{\mu}{x}\right)-$$
$$\frac{3\varepsilon(1+\varepsilon x)^2}{x}\left(\frac{\partial^2 w}{\partial\theta^2}\frac{1}{x^2}+\frac{\partial w}{\partial x}\frac{1}{x}+\mu\frac{\partial^2 w}{\partial x^2}\right)-$$

$$6\varepsilon(1+\varepsilon x)^2\left(\frac{\partial^3 w}{\partial x^3}+\frac{\partial^3 w}{\partial\theta^2\partial x}\frac{1}{x^2}+\frac{\partial^2 w}{\partial\theta^2}\frac{2}{x^3}-\frac{\partial w}{\partial x}\frac{1}{x^2}\right)-$$
$$(\varepsilon^3x^3+3\varepsilon^2x^2+3\varepsilon x)\nabla^2 w, \tag{11}$$

其中

$$\nabla^2=\frac{\partial}{\partial x^2}+\frac{1}{x}\frac{\partial}{\partial x}+\frac{1}{x^2}\frac{\partial^2}{\partial\theta^2}.$$

相应的无量纲量下的边界条件为:

$$x=1\text{ 时},\ w=\partial w/\partial x=u=v=0; \tag{12}$$

$$x=0\text{ 时},\ w,\partial w/\partial x,u,v\text{ 有限}. \tag{13}$$

2　问题的求解

为了便于说明求解方法,本文只讨论在周边固定夹紧边界条件下的求解. 将 u、v、w、Q 展开成为 Fourier 级数,有

$$u(x,\theta)=\sum_{k=-\infty}^{+\infty}[u_{rk}(x)+\mathrm{i}u_{ik}(x)]\mathrm{e}^{\mathrm{i}k\theta}, \tag{14}$$

$$v(x,\theta)=\sum_{k=-\infty}^{+\infty}[v_{rk}(x)+\mathrm{i}v_{ik}(x)]\mathrm{e}^{\mathrm{i}k\theta}, \tag{15}$$

$$w(x,\theta)=\sum_{k=-\infty}^{+\infty}[w_{rk}(x)+\mathrm{i}w_{ik}(x)]\mathrm{e}^{\mathrm{i}k\theta}, \tag{16}$$

$$Q(x,\theta)=\sum_{k=-\infty}^{+\infty}[Q_{rk}(x)+\mathrm{i}Q_{ik}(x)]\mathrm{e}^{\mathrm{i}k\theta}, \tag{17}$$

其中, $u_{rk}(x)$, $u_{ik}(x)$, $v_{rk}(x)$, $v_{ik}(x)$, $w_{rk}(x)$, $w_{ik}(x)$, $Q_{rk}(x)$, $Q_{ik}(x)$ 为 x 的实函数. 将(14)、(15)、(16)、(17) 式代入(9)、(10)、(11)、(12)、(13), 由 $\mathrm{e}^{\mathrm{i}\theta}$ 的同次幂将实、虚部分开, 可得近似边值问题基本方程:

$$xu''_{rk}+u'_{rk}-\frac{2+k^2(1-\mu)}{2x}u_{rk}-\frac{k}{2}\left[(1+\mu)v'_{ik}-(3-\mu)\frac{v_{ik}}{x}\right]=$$
$$-\sum_{m=-\infty}^{+\infty}\left[x(w''_{rm}w'_{r,k-m}-w''_{im}w'_{i,k-m})+\frac{1-\mu}{2}(w'_{rm}w'_{r,k-m}-w'_{im}w'_{i,k-m})-\right.$$
$$\frac{(k-m)(k-k\mu+2m\mu)}{2x}(w'_{rm}w_{r,k-m}-w'_{im}w_{i,k-m})+$$
$$\left.\frac{m(k-m)(1+\mu)}{2x^2}(w_{rm}w_{r,k-m}-w_{im}w_{i,k-m})\right]=L_{1rk}(w), \tag{18}$$

$$xv''_{ik}+v'_{ik}-\frac{1-\mu+2k^2}{(1-\mu)x}v_{ik}+\frac{k}{1-\mu}\left[(1+\mu)u'_{rk}+(3-\mu)\frac{u_{rk}}{x}\right]=$$
$$-\sum_{m=-\infty}^{+\infty}\left[(k-m)(w''_{rm}w_{r,k-m}-w''_{im}w_{i,k-m})+\right.$$
$$\frac{m(1+\mu)}{1-\mu}(w'_{rm}w'_{r,k-m}-w'_{im}w'_{i,k-m})+\frac{(k-m)}{x}(w'_{rm}w_{r,k-m}-w'_{im}w_{i,k-m})-$$
$$\left.\frac{2m^2(k-m)}{(1-\mu)x^2}(w_{rm}w_{r,k-m}-w_{im}w_{i,k-m})\right]=L_{2ik}(w), \tag{19}$$

$$xu''_{ik}+u'_{ik}-\frac{2+k^2(1-\mu)}{2x}u_{ik}+\frac{k}{2}\left[(1+\mu)v'_{rk}-\frac{3-\mu}{x}v_{rk}\right]=$$
$$-\sum_{m=-\infty}^{+\infty}\left[x(w''_{im}w'_{r,k-m}+w''_{rm}w'_{i,k-m})+(1-\mu)(w'_{rm}w'_{i,k-m}+w'_{rm}w'_{r,k-m})/2-\right.$$

$$(k-m)(k-k\mu+2m\mu)(w'_{\mathrm{r}m}w_{\mathrm{i}k-m}+w'_{\mathrm{i}m}w_{\mathrm{r}k-m})/2x+$$
$$m(k-m)(1+\mu)(w_{\mathrm{r}m}w_{\mathrm{i}k-m}+w_{\mathrm{i}m}w_{\mathrm{r}k-m})/2x^2\Big]=L_{1\mathrm{i}k}(w), \tag{20}$$

$$xv''_{\mathrm{r}k}+v'_{\mathrm{r}k}-\frac{1-\mu+2k^2}{(1-\mu)x}v_{\mathrm{r}k}-\frac{k}{1-\mu}\left[(1+\mu)u'_{\mathrm{i}k}+\frac{3-\mu}{x}u_{\mathrm{i}k}\right]=$$
$$\sum_{m=-\infty}^{+\infty}\Big[(k-m)(w''_{\mathrm{r}m}w_{\mathrm{i}k-m}+w''_{\mathrm{i}m}w_{\mathrm{r}k-m})+\frac{m(1+\mu)}{1-\mu}(w'_{\mathrm{r}m}w'_{\mathrm{i}k-m}+w'_{\mathrm{i}m}w'_{\mathrm{r}k-m})+$$
$$\frac{k-m}{x}(w'_{\mathrm{r}m}w_{\mathrm{i}k-m}+w'_{\mathrm{i}m}w_{\mathrm{r}k-m})-$$
$$\frac{2m(k-m)}{(1-\mu)x^2}(w_{\mathrm{r}m}w_{\mathrm{i}k-m}+w_{\mathrm{i}m}w_{\mathrm{r}k-m})\Big]=L_{2\mathrm{r}k}(w), \tag{21}$$

$$H_k(w_{\mathrm{r}k})=Q_{\mathrm{r}k}+12(1+\varepsilon x)\sum_{m=-\infty}^{+\infty}\Big\{w''_{\mathrm{r}m}\Big[u'_{\mathrm{r}k-m}+\frac{1}{2}\sum_{l=-\infty}^{+\infty}(w'_{\mathrm{r}l}w'_{\mathrm{r}k-m-l}-w'_{\mathrm{i}l}w'_{\mathrm{i}k-m-l})+$$
$$\frac{\mu}{x}u_{\mathrm{r}k-m}-\frac{\mu}{x}(k-m)v_{\mathrm{i}k-m}-\frac{\mu}{2x^2}\sum_{l=-\infty}^{+\infty}l(k-m-l)(w_{\mathrm{r}l}w_{\mathrm{r}k-m-l}-w_{\mathrm{i}l}w_{\mathrm{i}k-m-l})\Big]-$$
$$w''_{\mathrm{i}m}\Big[u'_{\mathrm{i}k-m}+\frac{1}{2}\sum_{l=-\infty}^{+\infty}(w'_{\mathrm{r}l}w'_{\mathrm{i}k-m-l}+w'_{\mathrm{i}l}w'_{\mathrm{r}k-m-l})+\frac{\mu}{x}u_{\mathrm{i}k-m}+\frac{\mu}{x}(k-m)v_{\mathrm{r}k-m}-$$
$$\frac{\mu}{2x^2}\sum_{l=-\infty}^{+\infty}l(k-m-l)(w_{\mathrm{r}l}w_{\mathrm{i}k-m-l}+w_{\mathrm{i}l}w_{\mathrm{r}k-m-l})\Big]-$$
$$m(1-\mu)\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{w_{\mathrm{i}m}}{x}\right)\Big[-\frac{k-m}{x}u_{\mathrm{r}k-m}+x\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{v_{\mathrm{r}k-m}}{x}\right)-$$
$$\sum_{l=-\infty}^{+\infty}\frac{k-m-l}{x}(w'_{\mathrm{r}l}w_{\mathrm{i}k-m-l}+w'_{\mathrm{i}l}w_{\mathrm{r}k-m-l})\Big]-$$
$$m(1-\mu)\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{w_{\mathrm{i}m}}{x}\right)\Big[\frac{k-m}{x}u_{\mathrm{i}k-m}+x\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{v_{\mathrm{i}k-m}}{x}\right)-$$
$$\sum_{l=-\infty}^{+\infty}\frac{k-m-l}{x}(w'_{\mathrm{r}l}w_{\mathrm{r}k-m-l}-w'_{\mathrm{i}l}w_{\mathrm{i}k-m-l})\Big]+$$
$$\left(\frac{1}{x}w'_{\mathrm{r}m}-\frac{m^2}{x^2}w_{\mathrm{r}m}\right)\Big[\frac{1}{x}u_{\mathrm{r}k-m}-\frac{k-m}{x}v_{\mathrm{i}k-m}-\frac{1}{2x^2}\sum_{l=-\infty}^{+\infty}l(k-m-l)(w_{\mathrm{r}l}w_{\mathrm{r}k-m-l}-$$
$$w_{\mathrm{i}l}w_{\mathrm{i}k-m-l})+\mu u'_{\mathrm{r}k-m}+\frac{\mu}{2}\sum_{l=-\infty}^{+\infty}(w'_{\mathrm{r}l}w'_{\mathrm{r}k-m-l}+w'_{\mathrm{i}l}w'_{\mathrm{i}k-m-l})\Big]-$$
$$\left(\frac{1}{x}w'_{\mathrm{i}m}-\frac{m^2}{x^2}w_{\mathrm{i}m}\right)\Big[\frac{1}{x}u_{\mathrm{i}k-m}-\frac{k-m}{x}v_{\mathrm{r}k-m}-\frac{1}{2x^2}\sum_{l=-\infty}^{+\infty}l(k-m-l)(w_{\mathrm{r}l}w_{\mathrm{i}k-m-l}+$$
$$w_{\mathrm{i}l}w_{\mathrm{r}k-m-l})+\mu u'_{\mathrm{i}k-m}+\frac{\mu}{2}\sum_{l=-\infty}^{+\infty}(w'_{\mathrm{r}l}w'_{\mathrm{i}k-m-l}+w'_{\mathrm{i}l}w'_{\mathrm{r}k-m-l})\Big]\Big\}-$$
$$\sum_{l=-\infty}^{+\infty}\Big\{6\varepsilon^2(1+\varepsilon x)\Big[w''_{\mathrm{r}k}+\frac{\mu k^2}{x^2}(-w_{\mathrm{r}k})+\frac{\mu}{x}w'_{\mathrm{r}k}\Big]+$$
$$\frac{3\varepsilon(1+\varepsilon x)^2}{x}\Big[\frac{k^2}{x^2}(-w_{\mathrm{r}k})+\frac{1}{x}w'_{\mathrm{r}k}+\mu w''_{\mathrm{r}k}\Big]+$$
$$6\varepsilon(1+\varepsilon x)^2\Big[w'''_{\mathrm{r}k}+\frac{k^2}{x^2}(-w'_{\mathrm{r}k})+\frac{1}{x}w''_{\mathrm{r}k}+\frac{2k^2}{x^3}w_{\mathrm{r}k}-\frac{1}{x^2}w'_{\mathrm{r}k}\Big]+$$
$$(\varepsilon^3x^3+3\varepsilon^2x^2+3\varepsilon x)H_k(w_{\mathrm{r}k})\Big\}=$$
$$Q_{\mathrm{r}k}+L_{3\mathrm{r}k}(u,v,w), \tag{22}$$

$$
\begin{aligned}
H_k(w_{ik}) = {} & Q_{ik} + 12(1+\varepsilon x)\sum_{m=-\infty}^{+\infty}\Bigg\{ w''_{rm}\Bigg[u'_{ik-m} + \frac{1}{2}\sum_{l=-\infty}^{+\infty}(w'_{rl}w'_{ik-m-l} + w'_{il}w'_{rk-m-l}) + \\
& \frac{\mu}{x}u_{ik-m} + \frac{\mu}{x}(k-m)v_{rk-m} - \frac{\mu}{2x^2}\sum_{l=-\infty}^{+\infty} l(k-m-l)(w_{rl}w_{ik-m-l} + w_{il}w_{rk-m-l})\Bigg] + \\
& w''_{im}\Bigg[u'_{rk-m} + \frac{1}{2}\sum_{l=-\infty}^{+\infty}(w'_{rl}w'_{rk-m-l} - w'_{il}w'_{ik-m-l}) + \frac{\mu}{x}u_{rk-m} - \frac{\mu}{x}(k-m)v_{ik-m} - \\
& \frac{\mu}{2x^2}\sum_{m=-\infty}^{+\infty} l(k-m-l)(w_{rl}w_{rk-m-l} - w_{il}w_{ik-m-l})\Bigg] - \\
& m(1-\mu)\frac{d}{dx}\left(\frac{w_{im}}{x}\right)\Bigg[\frac{k-m}{x}u_{rk-m} + x\frac{d}{dx}\left(\frac{v_{im}}{x}\right) + \\
& \frac{1}{x}\sum_{l=-\infty}^{+\infty}(k-m-l)(w'_{rl}w_{rk-m-l} - w'_{il}w_{ik-m-l})\Bigg] + \\
& m(1-\mu)\frac{d}{dx}\left(\frac{w_{rm}}{x}\right)\Bigg[-\frac{k-m}{x}u_{ik-m} + x\frac{d}{dx}\left(\frac{v_{rk-m}}{x}\right) - \\
& \frac{1}{x}\sum_{l=-\infty}^{+\infty}(k-m-l)(w'_{rl}w_{ik-m-l} + w'_{il}w_{rk-m-l})\Bigg] + \\
& \left(\frac{1}{x}w'_{rm} - \frac{m^2}{x^2}w_{rm}\right)\Bigg[\frac{1}{x}u_{ik-m} + \frac{k-m}{x}v_{rk-m} - \frac{1}{2x^2}\sum_{l=-\infty}^{+\infty} l(k-m-l)(w_{rl}w_{ik-m-l} + \\
& w_{il}w_{rk-m-l}) + \mu u'_{ik-m} + \frac{\mu}{2}\sum_{l=-\infty}^{+\infty}(w'_{rl}w'_{ik-m-l} + w'_{il}w'_{rk-m-l})\Bigg] + \\
& \left(\frac{1}{x}w'_{im} - \frac{m^2}{x^2}w_{im}\right)\Bigg[\frac{1}{x}u_{rk-m} - \frac{k-m}{x}v_{ik-m} - \frac{1}{2x^2}\sum_{l=-\infty}^{+\infty} l(k-m-l)(w_{rl}w_{rk-m-l} - \\
& w_{il}w_{ik-m-l}) + \mu u'_{rk-m} + \frac{\mu}{2}\sum_{l=-\infty}^{+\infty}(w'_{rl}w'_{rk-m-l} - w'_{il}w'_{ik-m-l})\Bigg]\Bigg\} - \\
& \sum_{k=-\infty}^{+\infty}\Bigg\{6\varepsilon^2(1+\varepsilon x)\left[w''_{ik} + \frac{\mu k^2}{x^2}(-w_{irk}) + \frac{\mu}{x}w'_{ik}\right] + \\
& \frac{3\varepsilon(1+\varepsilon x)^2}{x}\left[\frac{k^2}{x^2}(-w_{irk}) + \frac{1}{x}w'_{irk} + \mu w''_{ik}\right] + \\
& 6\varepsilon(1+\varepsilon x)^2\left[w'''_{ik} + \frac{k^2}{x^2}(-w'_{ik}) + \frac{1}{x}w''_{ik} + \frac{2k^2}{x^3}w_{ik} - \frac{1}{x^2}w'_{ik}\right] + \\
& (\varepsilon^3x^3 + 3\varepsilon^2x^2 + 3\varepsilon x)H_k(w_{ik})\Bigg\} = \\
& Q_{ik} + L_{3ik}(u, v, w). \qquad (23)
\end{aligned}
$$

边界条件:

$$x = 1\ 时,\ u_{rk} = u_{ik} = v_{rk} = v_{ik} = 0; \qquad (24)$$

$$w_{rk} = w_{ik} = w'_{rk} = w'_{ik} = 0. \qquad (25)$$

$$x = 0\ 时,\ u_{rk},\ u_{ik},\ v_{rk},\ v_{ik},\ w'_{rk},\ w'_{ik}\ 有限. \qquad (26)$$

其中

$$(\ )' = \frac{d}{dx},\ (\ )'' = \frac{d}{dx^2},\ H_k = \left(\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} - \frac{k^2}{x^2}\right)^2 \qquad (k = 0,\ \pm 1,\ \pm 2,\ \dots).$$

从式(18)到(26)构成了问题的基本方程.

3 算 例

设有一个半径为 a, 厚度为 h 的圆薄板, 周边固定夹紧, 承受横向载荷 $q(r,\theta)=2q_0\cos n\theta$, 其中 $n\geqslant 1$, $n\neq 2, 4$, 材料的泊松比 $\mu=0.3$• 此时

$$Q_{rk}(x)=\begin{cases}Q, & k=\pm n,\\ 0, & \text{其它};\end{cases}\qquad Q_{ik}=0\qquad (k=0,\ \pm 1,\ \pm 2,\ \cdots),$$

用修正迭代法[6,7,9]求解, 可得挠度的二次近似解析解•

取 $n=1$, $\theta_0=0$, $x=1/2$, 则可得:

$$Q=\left(\frac{18}{5}+\frac{104\,652}{4\,375}\varepsilon+\frac{16\,497}{3\,500}\varepsilon^2+\frac{57\,663}{112\,000}\varepsilon^3\right)y_0+\left(-\frac{20\,625\,700\,588\,320\,625\,078\,399}{4\,670\,845\,203\,843\,000\,000\,000}\varepsilon+\frac{658\,520\,634\,927\,565\,745\,122\,057}{102\,758\,594\,484\,546\,000\,000\,000}\right)y_0^3• \tag{27}$$

由(27)式绘出相应载荷与点 (x_0,θ_0) 的挠度特征曲线如图 1、图 2:

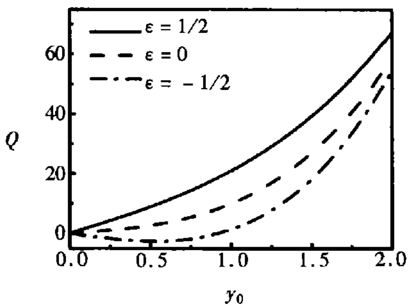

图 1 特征曲线图

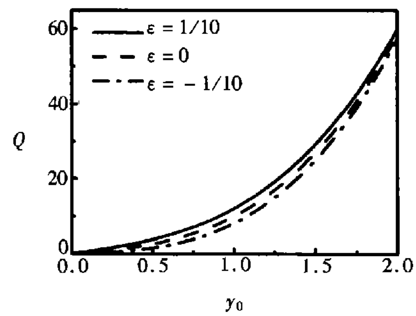

图 2 特征曲线图

4 讨 论

采用修正迭代法, 求出了挠度的二次近似解• 由以上的讨论可以看出, 板在沿 θ 方向按余弦规律变化的载荷作用下, 挠度 w 的展开式中一定只含余弦项, 不含正弦项• 将本文所选坐标旋转 90° 即可推出板在沿 θ 方向按正弦变化的载荷作用下, 挠度 w 的展开式中一定只含正弦项, 不含余弦项• 在余弦规律变化的载荷作用下, 取定 n、θ、x, 得到挠度和载荷的关系式, 含有待定量 ε, 根据 ε 可得到不同的曲线, 由图 1、图 2 可知, ε 的值越小, 曲线越趋于重合, 当 ε 的值趋向于零, 则曲线趋向于等厚度时的特征曲线•

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Unsymmetrical Nonlinear Bending Problem of Circular Thin Plate With Variable Thickness

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Abstract: Firstly, the cross large deflection equation of circular thin plate with variable thickness in rectangular coordinates system was transformed into unsymmetrical large deflection equation of circular thin plate with variable thickness in polar coordinates system. This cross equation in polar coordinates system is united with radical and tangential equations in polar coordinates system, and then three equilibrium equations were obtained. Physical equations and nonlinear deformation equations of strain at central plane are substituted into superior three equilibrium equations, and then three unsymmetrical nonlinear equations with three deformation displacements were obtained. Solution with expression of Fourier series is substituted into fundamental equations; correspondingly fundamental equations with expression of Fourier series were obtained. The problem was solved by modified iteration method under the boundary conditions of clamped edges. As an example, the problem of circular thin plate with variable thickness subjected to loads with cosin form was studied. Characteristic curves of the load varying with the deflection were plotted. The curves vary with the variation of the parameter of variable thickness. Its solution is accordant with physical conception.

Key words: variable thickness; unsymmetrical bending; modified iteration method; deflection