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横观各向同性饱和地基与中厚圆板的非轴对称动力相互作用*

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(我刊原编委黄义来稿)

摘要: 研究横观各向同性饱和土地基上中厚弹性圆板的非轴对称振动问题, 即首先利用 Fourier 展开和 Hankel 变换技术, 求解了简谐激励下横观各向同性饱和土地基的非轴对称 Biot 波动方程, 然后按混合边值问题建立地基与弹性中厚圆板非轴对称动力相互作用的对偶积分方程, 并将对偶积分方程化为易于数值计算的第二类 Fredholm 积分方程• 文末给出了算例• 数值结果表明, 在一定频率范围内, 地基表面的位移幅值随激振频率增加而增大, 随距离的增大以振荡形式衰减变化•

关 键 词: Biot 波动方程; 横观各向同性饱和土; 中厚弹性圆板; 振动; Fredholm 积分方程

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引 言

多孔饱和地基上板的振动是一个复杂的动力接触问题, 同时也是地震工程和岩土工程中的重要课题, 所以近年来已有许多学者对此问题进行了研究[1~3]• 但值得注意的是, 已有文献大多只考虑多孔饱和均质半无限地基上刚性或弹性圆薄板的轴对称动力响应, 而对非均质的横观各向同性饱和多孔介质与中厚弹性圆板的非轴对称动力相互作用, 由于涉及复杂的混合边值问题, 在数学求解上存在较大困难, 至今尚未见到相关报道• 本文利用横观各向同性饱和多孔介质的波动方程, 在文献[4]至文献[7]的基础上, 求解简谐激励下横观各向同性饱和土地基上弹性中厚圆板的非轴对称振动问题, 并给出了算例•

1 Biot 波动方程及非轴对称稳态解

在柱坐标系下横观各向同性饱和土地基的波动方程为[4],[7]

$$B_1\left[\nabla^2 u_r - \frac{1}{r}\left(2\frac{\partial u_\theta}{r\partial\theta} + \frac{u_r}{r}\right)\right] + B_5\frac{\partial^2 u_r}{\partial z^2} + \rho_1^*\ \omega^2 u_r + \left(B_1 + B_2 - \frac{B_6^2}{B_8}\right)\frac{\partial e_h}{\partial r} +$$

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$$\left(B_3+B_5-\frac{B_6B_7}{B_8}\right)\frac{\partial}{\partial r}\frac{\partial u_z}{\partial r}+\left(\frac{B_6}{B_8}+\rho_l\omega^2\beta_1\right)\frac{\partial f}{\partial r}=0, \tag{1a}$$

$$B_1\left[\nabla^2u_\theta-\frac{1}{r}\left(-2\frac{\partial u_r}{r\partial\theta}+\frac{u_\theta}{r}\right)\right]+B_5\frac{\partial^2u_\theta}{\partial z^2}+\rho_1^*\omega^2u_\theta+\left(B_1+B_2-\frac{B_6^2}{B_8}\right)\frac{\partial e_h}{r\partial\theta}+\left(B_3+B_5-\frac{B_6B_7}{B_8}\right)\frac{\partial}{r\partial\theta}\frac{\partial u_z}{\partial z}+\left(\frac{B_6}{B_8}+\rho_l\omega^2\beta_1\right)\frac{\partial f}{r\partial\theta}=0, \tag{1b}$$

$$B_5\nabla^2u_z+\rho_3^*\omega^2u_z+\left(B_3+B_5-\frac{B_6B_7}{B_8}\right)\frac{\partial e_h}{\partial z}+\left(B_4-\frac{B_7^2}{B_8}\right)\frac{\partial^2u_z}{\partial z^2}+\left(\frac{B_7}{B_8}+\rho_l\omega^2\beta_3\right)\frac{\partial f}{\partial z}=0, \tag{1c}$$

$$\left(\frac{B_6}{B_8}+\rho_l\omega^2\beta_1\right)e_h+\left(\frac{B_7}{B_8}+\rho_l\omega^2\beta_3\right)\frac{\partial u_z}{\partial z}-\beta_1\nabla^2f-\beta_3\frac{\partial^2f}{\partial z^2}-\frac{f}{B_8}=0, \tag{1d}$$

式中

$$\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{\partial}{r\partial r}+\frac{\partial^2}{r^2\partial\theta^2},\quad e_h=\frac{\partial u_r}{\partial r}+\frac{u_r}{r}+\frac{\partial u_\theta}{r\partial\theta},$$

$$\rho_j^*=\rho-\beta_j\rho_l^2\omega^2,\quad \beta_j=\frac{1}{m_j\omega^2-\mathrm{i}\omega r_j}\qquad(j=1,2,3),$$

u_r、u_θ 和 u_z 分别为介质奄径向、周向和轴向的位移分量, f 是孔隙水压力, ω 是简谐激励的圆频率; i 是虚数单位; ρ 是饱和多孔介质密度 $\rho=(1-\phi)\rho_s+\phi\rho_l$, ρ_l 孔隙流体密度, ρ_s 是骨架固体材料密度; $m_j(j=1,2,3)$ 是反映孔隙流体宏观流动速度和微观流动速度之间的量, η 是孔隙流体动力粘滞系数, r_j 是与孔隙形状有关的量, $r_j=\eta/K_j$, $B_j(j=1,2,\cdots,8)$ 为弹性常数, 可用横观各向同性介质弹性常数 c_{ij}、骨架固体体积模量 K_s、孔隙流体体积模量 K_l 和孔隙率 ϕ 表示为[7]

$$\begin{cases} B_1=c_{66},\ B_2=c_{12}+\dfrac{B_6^2}{B_8},\ B_3=c_{33}+\dfrac{B_6B_7}{B_8},\ B_4=c_{33}+\dfrac{B_7^2}{B_8},\ B_5=c_{44}, \\ B_6=-\left(1-\dfrac{c_{11}+c_{12}+c_{13}}{3K_s}\right)B_8,\ B_7=-\left[1-2\left(1-\dfrac{c_{33}+c_{13}}{3K_s}\right)\right]B_8, \\ B_8=\left[\dfrac{1-\phi}{K_s}+\dfrac{\phi}{K_l}-\dfrac{2c_{11}+2c_{12}+4c_{13}+c_{33}}{9K_s^2}\right]^{-1}. \end{cases} \tag{1e}$$

将 u_r、u_z、e_h、f、u_θ 和 σ_z、σ_r、σ_θ、τ_{rz}、$\tau_{\theta z}$、$\tau_{r\theta}$ 沿 θ 方向作 Fourier 展开, 取如下形式

$$[u_r(r,\theta,z)\quad u_z(r,\theta,z)\quad e_h(r,\theta,z)\quad f(r,\theta,z)]^{\mathrm{T}}=\sum_{n=0}^{\infty}[u_{rn}(r,z)\quad u_{zn}(r,z)\quad e_{hn}(r,z)\quad f_n(r,z)]^{\mathrm{T}}\cos n\theta, \tag{2a}$$

$$[\sigma_z(r,\theta,z)\quad \sigma_r(r,\theta,z)\quad \sigma_\theta(r,\theta,z)\quad \tau_{rz}(r,\theta,z)]^{\mathrm{T}}=\sum_{n=0}^{\infty}[\sigma_{zn}(r,z)\quad \sigma_{rn}(r,z)\quad \sigma_{\theta n}(r,z)\quad \tau_{rzn}(r,z)]^{\mathrm{T}}\cos n\theta, \tag{2b}$$

$$[u_\theta(r,\theta,z)\quad \tau_{\theta z}(r,\theta,z)\quad \tau_{r\theta}(r,\theta,z)]^{\mathrm{T}}=\sum_{n=0}^{\infty}[u_{\theta n}(r,z)\quad \tau_{\theta zn}(r,z)\quad \tau_{r\theta n}(r,z)]^{\mathrm{T}}\sin n\theta, \tag{2c}$$

将式(2)代入式(1), 再经 Hankel 变换, 并求解所得方程(详细推导见文献[4]和文献[5]), 得到简谐激励下横观各向同性饱和土地基的位移和水压力响应幅值

$$u_{rn}=n\int_0^\infty[D_{4n}\exp(-\lambda_0 z)]\,\frac{\mathrm{J}_n(kr)}{r}\mathrm{d}k-\sum_{s=1}^{3}\int_0^\infty\left\{d_s[D_{sn}\exp(-\lambda_s z)]\right\}\mathrm{J}_n'(kr)\,k\,\mathrm{d}k,\tag{3a}$$

$$u_{\theta n}=-\int_0^\infty[D_{4n}\exp(-\lambda_0 z)]\,\mathrm{J}_n'(kr)\,k\,\mathrm{d}k+n\sum_{s=1}^{3}\int_0^\infty\left\{d_s[D_{sn}\exp(-\lambda_s z)]\right\}\frac{\mathrm{J}_n(kr)}{r}\mathrm{d}k,\tag{3b}$$

$$u_{zn}=-\sum_{s=1}^{3}\int_0^\infty\left\{\gamma_s[D_{sn}\exp(-\lambda_s z)]\right\}\mathrm{J}_n(kr)\,k\,\mathrm{d}k,\tag{3c}$$

$$f_n=\sum_{s=1}^{3}\int_0^\infty[D_{sn}\exp(-\lambda_s z)]\,\mathrm{J}_n(kr)\,k\,\mathrm{d}k,\tag{3d}$$

以及变换域上的应力通解

$$\sigma_{zn}=\sum_{s=1}^{3}\xi_s D_{sn}\exp(-\lambda_s z),\tag{4a}$$

$$\tau_{zn}=\tau_{rzn}+\tau_{\theta zn}=-B_5\lambda_0 D_{4n}\exp(-\lambda_0 z)-B_5\sum_{s=1}^{3}\zeta_s D_{sn}\exp(-\lambda_s z),\tag{4b}$$

$$\tau_{dn}=\tau_{rzn}-\tau_{\theta zn}=B_5\lambda_0 D_{4n}\exp(-\lambda_0 z)-B_5\sum_{s=1}^{3}\zeta_s D_{sn}\exp(-\lambda_s z),\tag{4c}$$

式中 D_{4n} 和 $D_{sn}(s=1,2,3)$ 为与 z 无关的待定系数, 其余各变量和参数的定义为 $\lambda_s(s=1,2,3)$ 是特征方程 $a_1\lambda^6+a_2\lambda^4+a_3\lambda^2+a_4=0$ 的根$(\mathrm{Re}\,\lambda_s\geqslant 0)$, 该代数方程的系数为

$$a_1=B_5 b_4\beta_3,\quad a_2=B_5(b_5\beta_3+b_4 b_8+b_7^2)+b_1 b_4\beta_3+k^2 b_3^2\beta_3,$$
$$a_3=B_5 b_5 b_8+b_1(\beta_3 b_5+b_4 b_8+b_7^2)+(b_3^2 b_8+2b_3 b_6 b_7-b_6^2 b_4)k^2,$$
$$a_4=b_1 b_5 b_8-b_6^2 b_5 k^2,$$
$$b_1=\omega^2\rho_1^*-\left(2B_1+B_2-\frac{B_6^2}{B_8}\right)k^2,\quad b_2=B_1+B_2-\frac{B_6^2}{B_8},\quad b_3=B_3+B_5-\frac{B_6B_7}{B_8},$$
$$b_4=B_4-\frac{B_7^2}{B_8},\quad b_5=\omega^2\rho_3^*-B_5k^2,\quad b_6=\frac{B_6}{B_8}+\omega^2\rho_1\beta_1,\quad b_7=\frac{B_7}{B_8}+\omega^2\rho_1\beta_3,\quad b_8=\frac{1}{B_8}-\beta_1k^2,$$

另外

$$\alpha_s=[(b_4b_4-b_3b_7)\lambda_s^2+b_5b_6]\,\frac{k^2}{\Delta_s},\quad \gamma_s=-[B_5b_7\lambda_s^2+b_1b_7+b_3b_7k^2]\,\frac{\lambda_s}{\Delta_s},$$
$$\Delta_s=B_5b_4\lambda_s^4+(B_5b_4+b_1b_4+b_3^2k^2)\lambda_s^2+b_1b_5,$$
$$\lambda_0=\left[\frac{B_1k^2-\omega^2\rho_1^*}{B_5}\right]^{1/2},\quad d_s=\frac{k}{B_5}\left(\frac{b_2\alpha_s+b_3\gamma_s\lambda_s+b_6}{\lambda_s^2-\lambda_0^2}\right),$$
$$\xi_s=\left(B_3-\frac{B_6B_7}{B_8}\right)\alpha_s+\left(B_4-\frac{B_7^2}{B_8}\right)\gamma_s\lambda_s+\frac{B_7}{B_8},\quad \zeta_s=d_s\lambda_s-k\gamma_s\qquad(s=1,2,3)\cdot$$

2 混合边值问题

半径为 a 的弹性中厚度圆板, 支承在横观各向同性饱和地基的表面上, 圆板的 Poisson 比为 ν, 挠曲刚度为 D, 板密度为 ρ, 板受任意非轴对称动荷载作用大小为 $p(r,\theta,t)\cdot$ 地基反力为 $q(r,\theta,t)\cdot$ 文献[8] 给出了直角坐标下中厚度矩形板的动力方程, 经坐标变换后得到极坐标下的中厚度圆板的动力方程

$$\nabla^2\nabla^2 w-\left(\frac{k\tau\rho}{G}+\frac{\rho I}{D}\right)\frac{\partial^2}{\partial t^2}\nabla^2 w+\frac{k\tau\rho}{G}\frac{\rho I}{D}\frac{\partial^4 w}{\partial t^4}+\frac{\rho I}{D}\frac{\partial^2 w}{\partial t^2}=$$
$$\frac{p-q}{D}+\frac{k\tau}{Gh}\left[\frac{\rho I}{D}\frac{\partial^2}{\partial t^2}-\nabla^2\right](p-q)+\frac{k\tau}{Ch}\nabla^2(p-q),\tag{5a}$$

$$
\begin{aligned}
&Q_r - \frac{1-\nu}{2}\frac{k_\tau D}{Gh}\left[\nabla^2 Q_r + 2\frac{\partial Q_r}{r\partial r} + \frac{Q_r}{r^2}\right] + \frac{k_\tau \rho J}{Gh}\frac{\partial^2}{\partial t^2}Q_r = \\
&\quad -D\frac{\partial}{\partial r}\nabla^2 w + \left(\rho J + \frac{1+\nu}{2}\frac{k_\tau D\rho}{G}\right)\frac{\partial}{\partial r}\left(\frac{\partial^2 w}{\partial t^2}\right) - \frac{(1-\nu)k_\tau D\rho}{G}\frac{1}{r}\frac{\partial^2 w}{\partial t^2} + \frac{\rho J}{D}\frac{\partial^2 w}{\partial t^2} + \\
&\quad \left(\frac{Dk_\sigma}{Ch} - \frac{1+\nu}{2}\frac{Dk_\tau}{Gh}\right)\frac{\partial}{\partial r}(p-q) + \frac{(1-\nu)Dk_\tau}{Gh}\frac{1}{r}(p-q),
\end{aligned}
\tag{5b}
$$

$$
\begin{aligned}
&Q_\theta - \frac{1-\nu}{2}\frac{k_\tau D}{Gh}\left[\nabla^2 Q_\theta + 2\frac{\partial Q_r}{r^2\partial\theta} - \frac{Q_\theta}{r^2}\right] + \frac{k_\tau \rho J}{Gh}\frac{\partial^2 Q_\theta}{\partial t^2} = \\
&\quad -D\frac{\partial}{\partial\theta}\nabla^2 w + \left(\rho J + \frac{1+\nu}{2}\frac{k_\tau D\rho}{G}\right)\frac{\partial}{r\partial\theta}\left(\frac{\partial^2 w}{\partial t^2}\right) + \\
&\quad \left(\frac{k_\sigma D}{Ch} - \frac{1+\nu}{2}\frac{k_\tau D}{Gh}\right)\frac{\partial}{r\partial\theta}(p-q),
\end{aligned}
\tag{5c}
$$

板的内力表示为

$$
M_r = -D\left[\frac{\partial^2 w}{\partial r^2} + \nu\left(\frac{\partial w}{r\partial r} + \frac{\partial^2 w}{r^2\partial\theta^2}\right)\right] + \frac{Dk_\tau}{Gh}\left[\frac{\partial Q_r}{\partial r^2} + \nu\left(\frac{Q_r}{r} + \frac{\partial Q_\theta}{r\partial\theta}\right)\right] + \frac{Dk_\sigma}{Gh}(p-q),
\tag{6a}
$$

$$
M_\theta = -D\left[\frac{\partial w}{r\partial r} + \frac{\partial^2 w}{r^2\partial\theta^2} + \nu\frac{\partial^2 w}{\partial r^2}\right] + \frac{Dk_\tau}{Gh}\left[\frac{Q_r}{r} + \frac{\partial Q_\theta}{r\partial\theta} + \nu\frac{\partial Q_r}{\partial r}\right] + \frac{Dk_\sigma}{Gh}(p-q),
\tag{6b}
$$

$$
M_{r\theta} = -D(1-\nu)\frac{\partial}{\partial r}\left(\frac{\partial w}{r\partial\theta}\right) + \frac{1-\nu}{2}\left(\frac{\partial Q_r}{r\partial\theta} - \frac{Q_\theta}{r} + \frac{\partial Q_\theta}{\partial r}\right),
\tag{6c}
$$

式中 $C = E/((1+\nu)\nu)$ 为板的挤压刚度, $J = h^3/12$ 为板单位宽度截面惯性矩, 当 $k_\tau = k_\sigma = 1.2$ 为 Reissner 板; $k_\tau = 1$, $k_\sigma = 0$ 为 Hencky 板; $k_\tau = 1.2$, $k_\sigma = 0$ 为 ½±ÃÀ³ 板; $k_\tau = 12/\pi^2$, $k_\sigma = 0$ 为 Mindlin 板•

在非轴对称简谐激励 $p(r, \theta, t) = p(r, \theta)e^{i\omega t}$ 下(为简化符号, 激励及各响应的幅值仍采用原先的记号), 将激励及响应幅值沿 θ 方向展为 Fourier 级数形式, 令

$$
\begin{aligned}
&[p(r,\theta) \quad q(r,\theta) \quad w(r,\theta) \quad M_r(r,\theta) \quad M_\theta(r,\theta) \quad Q_r(r,\theta)]^{\mathrm T} = \\
&\quad \sum_{n=0}^{\infty}[p_n(r) \quad q_n(r) \quad w_n(r) \quad M_{rn}(r) \quad M_{\theta n}(r) \quad Q_{rn}(r)]^{\mathrm T}\cos n\theta,
\end{aligned}
\tag{7a}
$$

$$
[Q_\theta(r,\theta) \quad M_{r\theta}(r,\theta)]^{\mathrm T} = \sum_{n=0}^{\infty}[Q_{\theta n}(r) \quad M_{r\theta n}(r)]^{\mathrm T}\sin n\theta,
\tag{7b}
$$

将以上各式代入方程(5), 得到

$$
\begin{aligned}
&\nabla^2_{(n)}\nabla^2_{(n)} w_n + \left(\frac{k_\tau\rho}{G} + \frac{\rho J}{D}\right)\omega^2\nabla^2_{(n)} w_n + \left(\frac{k_\tau\rho}{G}\frac{\rho J}{D}\omega^4 - \frac{\rho J}{D}\omega^2\right) w_n = \\
&\quad \left[\frac{1}{D} - \frac{k_\tau}{Gh}\left(\frac{\rho J\omega^2}{D} - \nabla^2_{(n)}\right) + \frac{k_\tau}{Ch}\nabla^2_{(n)}\right](p_n - q_n),
\end{aligned}
\tag{8a}
$$

$$
\begin{aligned}
&Q_{rn} - \frac{1-\nu}{2}\frac{k_\tau D}{Gh}\left[\nabla^2_{(n)} Q_{rn} + 2\frac{\partial Q_{rn}}{r\partial r} + \frac{Q_{rn}}{r^2}\right] - \frac{k_\tau\rho J\omega^2 Q_{rn}}{Gh} = \\
&\quad -D\frac{\partial}{\partial r}\nabla^2_{(n)} w_n + \omega^2\left(\rho J + \frac{1+\nu}{2}\frac{k_\tau D\rho}{G}\right)\frac{\partial w_n}{\partial r} + \frac{(1-\nu)k_\tau D\rho\omega^2}{G}\frac{w_n}{r} + \\
&\quad \left(\frac{Dk_\sigma}{Ch} - \frac{1+\nu}{2}\frac{Dk_\tau}{Gh}\right)\frac{\partial}{\partial r}(p_n - q_n) + \frac{(1-\nu)Dk_\tau}{Gh}\frac{1}{r}(p_n - q_n),
\end{aligned}
\tag{8b}
$$

$$
\begin{aligned}
&Q_{\theta n} - \frac{1-\nu}{2}\frac{k_\tau D}{Gh}\left[\nabla^2_{(n)} Q_{\theta n} - \frac{2Q_{rn}}{r^2} - \frac{Q_{\theta n}}{r^2}\right] - \frac{k_\tau\rho J\omega^2 Q_{\theta n}}{Gh} = \\
&\quad D\frac{n}{r}\nabla^2_{(n)} w_n + \omega^2\left(\rho J + \frac{1+\nu}{2}\frac{k_\tau D\rho}{G}\right)\frac{n w_n}{r} - \left(\frac{k_\sigma D}{Ch} - \frac{1+\nu}{2}\frac{k_\tau D}{Gh}\right)\frac{n}{r}(p_n - q_n),
\end{aligned}
\tag{8c}
$$

式中 $\nabla^2_{(n)} = \dfrac{\mathrm{d}^2}{\mathrm{d}r^2} + \dfrac{\mathrm{d}}{r\,\mathrm{d}r} + \dfrac{n^2}{r^2}$,

令

$$\beta^4 = \frac{\rho h\omega^2}{D},\quad S = \frac{k\tau D}{Gh},\quad R = \frac{\rho J}{\rho h},$$

$$\delta_1^2 = \frac{1}{2}\left[R + S + \sqrt{(R - S)^2 + 4\beta}\right],\quad \delta_2^2 = \frac{1}{2}\left[R + S - \sqrt{(R - S)^2 + 4\beta}\right],$$

$$\delta_3^2 = \frac{2(RS\beta - 1)}{S(1 - \nu)},\quad R_0 = \frac{k_\sigma D}{Ch},\quad R_1 = \frac{1}{D}\left(1 - \frac{k\tau\rho J\omega^2}{Gh}\right),\quad R_3 = \frac{2D}{S(1 - \nu)},$$

$$R_4 = \frac{2DR\beta^4}{S(1 - \nu)} + \frac{(1 + \nu)D\beta^4}{(1 - \nu)S},\quad R_5 = 2D\beta^4,\quad R_4 = \frac{2Gk_\sigma}{Ck\tau(1 - \nu)} - \frac{1 + \nu}{1 - \nu}$$

方程(8)可表示为

$$(\nabla^2_{(n)} + \delta_1^2)(\nabla^2_{(n)} + \delta_2^2)w_n = (R_1 + R_2\nabla^2_{(n)})(p_n - q_n), \tag{9a}$$

$$(\nabla^2_{(n)} + \delta_3^2)(rQ_{rn}) = R_3 r\frac{\mathrm{d}}{\mathrm{d}r}\nabla^2_{(n)}w_n + R_4 r\frac{\mathrm{d}w_n}{\mathrm{d}r} - R_5 w_n - \left(R_6 r\frac{\mathrm{d}}{\mathrm{d}r} + 2\right)(p_n - q_n), \tag{9b}$$

同时,用平衡方程

$$\frac{1}{r}\frac{\partial Q_\theta}{\partial\theta} + \frac{\partial Q_r}{\partial r} + \frac{Q_r}{r} + p - q - \rho h\frac{\partial^2 w}{\partial t^2} = 0,$$

代替(8c),并注意到式(6a)和(6b),化简后得

$$\frac{n}{r}Q_{\theta n} = -\beta^4 Drw_n - \frac{\mathrm{d}Q_{rn}}{\mathrm{d}r} - \frac{Q_{rn}}{r} - (p_n - q_n)\cdot \tag{9c}$$

假设圆板与地基之间为光滑接触,将地基应力在表面沿 θ 方向作 Fourier 展开,对圆板的挠度、激励的幅值和地基反力的幅值 $q(r, \theta)$ 也作相应展开,边界条件可以写为

$$\sigma_{zn}(r) = \begin{cases} -q_n(r) & (0 \leqslant r \leqslant a), \\ 0 & (a \leqslant r \leqslant \infty), \end{cases} \tag{10a}$$

$$\tau_{\theta zn}(r) = 0 \qquad (0 \leqslant r \leqslant \infty), \tag{10b}$$

$$\tau_{rzn}(r) = 0 \qquad (0 \leqslant r \leqslant \infty), \tag{10c}$$

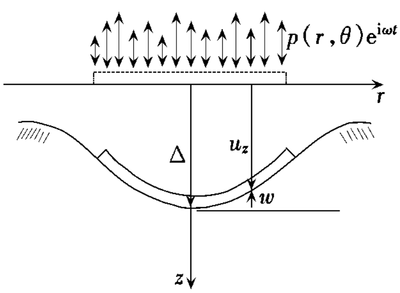

图 1 圆板振动的力学模型

地基表面完全排水时

$$f_n(r) = 0 \qquad (0 \leqslant r \leqslant \infty), \tag{10d}$$

完全不排水时

$$\frac{\partial f_n(r)}{\partial z} = 0 \qquad (0 \leqslant r \leqslant \infty), \tag{10e}$$

位移连续条件

$$u_{zn} = \Delta_n - w_n(r) \qquad (0 \leqslant r \leqslant a), \tag{10f}$$

式中 Δ_n 为圆板中心位移的 Fourier 展开项,$w_n(r)$ 为圆板相对于板中心挠度的 Fourier 展开项,须满足方程(9)·

引入以下变换

$$w_n(r) = \frac{1}{D}\int_0^\infty k[p_n(k) - q_n(k)]w(r, k)\,\mathrm{d}k, \tag{11}$$

$$H(r) = rQ_{rn} = \frac{1}{D}\int_0^\infty k[p_n(k) - q_n(k)]H(r, k)\,\mathrm{d}k, \tag{12}$$

其中

$$[p_n(k) - q_n(k)] = \int_0^\infty r[p_n(k) - q_n(k)]\mathrm{J}_n(kr)\,\mathrm{d}r, \tag{13}$$

将式(11)代入式(9),则 $w(r,k)$ 应满足以下方程

$$(\nabla^2_{(n)}+\delta_1^2)(\nabla^2_{(n)}+\delta_2^2)w(r,k)=(R_1+R_2k^2)\mathrm{J}_n(kr) \tag{14}$$

解出

$$\begin{aligned}w(r,k)={}&A_1\mathrm{J}_n(\delta_1 r)+A_2\mathrm{J}_n(\delta_2 r)+B_1Y_n(\delta_1 r)+B_2Y_n(\delta_2 r)+\\&\frac{R_1-R_2k^2}{(\delta_1^2-k^2)(\delta_2^2-k^2)}\mathrm{J}_n(kr),\end{aligned} \tag{15}$$

上式中,系数 A_1、A_2、B_1、B_2 应由板的边界条件确定· 考虑到实心圆板, 当 $r=0$ 时 w_n 及 $\mathrm{d}w_n/\mathrm{d}r$ 均为有限值, 故取 $B_1=B_2=0$, 这样,经 Hankel 逆变换后可以得出

$$\begin{aligned}w_n(r)={}&\frac{1}{D}\int_0^\infty k[p_n(k)-\\&q_n(k)]\left[A_1\mathrm{J}_n(\delta_1 r)+A_2\mathrm{J}_n(\delta_2 r)+\frac{R_1-R_2k^2}{(\delta_1^2-k^2)(\delta_2^2-k^2)}\mathrm{J}_n(kr)\right]\mathrm{d}k,\end{aligned} \tag{16}$$

将式(12)和(16)代入(9b),并利用 Bessel 函数的性质,化简后得到

$$\begin{aligned}(\nabla^2_{(n)}+\delta_1^2)H(r,k)={}&A_1[(-R_3\delta_1^3+R_4\delta_1)r\mathrm{J}_n'(\delta_1 r)-R_5\mathrm{J}_n(\delta_1 r)]+\\&A_2[(-R_3\delta_2^3+R_4\delta_2)r\mathrm{J}_n'(\delta_2 r)-R_5\mathrm{J}_n(\delta_2 r)]+\\&(-R_3k^3\lambda+R_4k\lambda-R_6k)r\mathrm{J}_n'(kr)-(R_5-2)\mathrm{J}_n(kr),\end{aligned} \tag{17}$$

其中 $\lambda=(R_1-R_2k^2)/((\delta_1^2-k^2)(\delta_2^2-k^2))$

进一步解出 $H(r,k)$, 并结合式(12),得

$$\begin{aligned}Q_{rn}(r)={}&\int_0^\infty k[p_n(k)-q_n(k)]\left[A_1C_1\mathrm{J}_n'(\delta_1 r)+\frac{A_1D_1\mathrm{J}_n(\delta_1 r)}{r}+A_2C_2\mathrm{J}_n'(\delta_2 r)+\right.\\&\left.\frac{A_2D_2\mathrm{J}_n(\delta_2 r)}{r}+\frac{A_3\mathrm{J}_n(\delta_3 r)}{r}+C_3\mathrm{J}_n'(kr)+\frac{D_3\mathrm{J}_n(kr)}{r}\right]\mathrm{d}k,\end{aligned} \tag{18}$$

将式(16)、(18)和(9c)代入式(6),化简后得到

$$\begin{aligned}M_{rn}(r)={}&\int_0^\infty k[p_n(k)-q_n(k)]\times\\&\left\{A_1\mathrm{J}_n'(\delta_1 r)\left[\frac{(1-\nu)SD_1\delta_1-D\nu\delta_1}{r}+D\delta_1-(1-\nu)SC_1\right]+\right.\\&A_1\mathrm{J}_n(\delta_1 r)\left[\frac{-(1-\nu)SD_1+D\nu n^2}{r^2}-S\nu D\beta^4+\frac{1}{\delta_1 r}(D\delta_1+(1-\nu)SC_1)(-\delta_1^2r^2+n^2)\right]+\\&A_2\mathrm{J}_n'(\delta_2 r)\left[\frac{(1-\nu)SD_2\delta_2+D\nu\delta_2}{r}+D\delta_2-(1-\nu)SC_2\right]+\\&A_2\mathrm{J}_n(\delta_2 r)\left[\frac{-(1-\nu)SD_2+D\nu n^2}{r^2}-S\nu D\beta^4+\frac{1}{\delta_2 r}(D\delta_2+(1-\nu)SC_2)(-\delta_2^2r^2+n^2)\right]+\\&A_3\mathrm{J}_n'(\delta_3 r)[(1-\nu)S\delta_3]+A_3\mathrm{J}_n(\delta_3 r)\left(-\frac{1-\nu}{r}S\right)+\\&\mathrm{J}_n'(kr)\left[\frac{(1-\nu)SD_3k-D\nu\lambda k}{r}+D\lambda k-(1-\nu)SC_3\right]+\\&\mathrm{J}_n(kr)\left[\frac{-(1-\nu)SD_3+D\nu n^2\lambda}{r^2}-S\nu D\beta^4\lambda+R_0-\nu S+\right.\\&\left.\left.\frac{1}{kr}(Dk\lambda+(1-\nu)SC_3)(-k^2r^2+n^2)\right]\right\}\mathrm{d}k,\end{aligned} \tag{19}$$

$$M_{r\theta n}(r)=\int_0^\infty k[p_n(k)-q_n(k)]\times$$

$$
\begin{aligned}
&\Bigg\{A_1 J_n'(\delta_1 r)\Bigg[\Bigg(-\frac{1-\nu}{2}\frac{S}{n}\Bigg)\Bigg(\frac{C_1 n^2 - C_1 - 2D_1\delta_1}{r} + C_1 n^2 - C_1 r^2\delta_1^2\Bigg) + \\
&(1-\nu)D\Bigg(\frac{n\delta_1}{r} - \frac{S\beta\delta_1 r}{2n}\Bigg)\Bigg] + A_1 J_n(\delta_1 r)\Bigg[\Bigg(-\frac{1-\nu}{2}\frac{S}{n}\Bigg)\Bigg(\frac{-C_1 n^2 r + 2D_1\delta_1 n^2}{\delta_1 r^2} - \\
&C_1\delta_1 r - D_1\delta_1^2\Bigg) - \frac{n(1-\nu)D}{r^2}\Bigg] + A_2 J_n'(\delta_2 r)\Bigg[\Bigg(-\frac{1-\nu}{2}\frac{S}{n}\Bigg)\Bigg(\frac{C_2 n^2 - C_2 - 2D_2\delta_2}{r} + \\
&C_2 n^2 - C_2 r^2\delta_2^2\Bigg) + (1-\nu)D\Bigg(\frac{n\delta_2}{r} - \frac{S\beta\delta_2 r}{2n}\Bigg)\Bigg] + \\
&A_2 J_n(\delta_2 r)\Bigg[\Bigg(-\frac{1-\nu}{2}\frac{S}{n}\Bigg)\Bigg(\frac{-C_2 n^2 r + 2D_2\delta_2 n^2}{\delta_2 r^2} - C_2\delta_2 r - D_2\delta_2^2\Bigg) - \frac{n(1-\nu)D}{r^2}\Bigg] + \\
&A_3 J_n'(\delta_3 r)\Bigg[\Bigg(\frac{1-\nu}{2}\frac{S}{n}\Bigg)\Bigg(\frac{\delta_3 + \delta_3 r}{r}\Bigg)\Bigg] + A_3 J_n(\delta_3 r)\Bigg[\Bigg(\frac{1-\nu}{2}\frac{S}{n}\Bigg)\Bigg(\frac{\delta_3^2 r^2 - 2(n^2-1)}{r^2}\Bigg)\Bigg] + \\
&J_n'(kr)\Bigg[\Bigg(-\frac{1-\nu}{2}\frac{S}{n}\Bigg)\Bigg(\frac{C_3(n^2-1) + kr^2 - 2D_3 k}{r} + C_3 n^2 - C_3 r^2 k^2\Bigg) + \\
&(1-\nu)D\Bigg(\frac{n\lambda k}{r} - \frac{S\beta\lambda kr}{2n}\Bigg)\Bigg] + J_n(kr)\Bigg[\Bigg(-\frac{1-\nu}{2}\frac{S}{n}\Bigg)\Bigg(\frac{-C_3 n^2 r + 2D_3 k n^2}{kr^2} - \\
&C_3 kr - D_3 k^2\Bigg) - \frac{n(1-\nu)\lambda D}{r^2}\Bigg]\Bigg\}\mathrm{d}k,
\end{aligned}
\tag{20}
$$

以上各式中常数 C_1、C_2、C_3、D_1、D_2 及 D_3 定义为

$$
C_1 = \frac{R_4\delta_1 - R_3\delta_1^3}{\delta_3^2 - \delta_1^2},\quad C_2 = \frac{R_4\delta_2 - R_3\delta_2^3}{\delta_3^2 - \delta_2^2},\quad C_3 = \frac{R_4 k\lambda - R_6 k - R_3 k^3\lambda}{\delta_3^2 - k^2},
$$

$$
D_1 = \frac{(2R_4 + R_5)\delta_1^2 - 2R_3\delta_1^4 - R_5\delta_3^2}{(\delta_3^2 - \delta_1^2)^2},\quad D_2 = \frac{(2R_4 + R_5)\delta_2^2 - 2R_3\delta_2^4 - R_5\delta_3^2}{(\delta_3^2 - \delta_2^2)^2},
$$

$$
D_3 = \frac{(2R_4 + R_5)k^2\lambda - 2R_3 k^4\lambda - R_5\delta_3^2\lambda - 2R_6 k^2 - 2\delta_3^2 + 2k^2}{(\delta_3^2 - k^2)^2}.
$$

对周边自由的中厚板，注意到式(6a)、(6c)，边界条件为

$$
r = a:\ M_r = 0,\ M_{r\theta} = 0,\ Q_r = 0,
$$

沿 θ 方向作 Fourier 展开后，有

$$
r = a:\ M_{rn} = 0,\ M_{r\theta n} = 0,\ Q_{rn} = 0,
\tag{21}
$$

将式(18)、(19)和(20)代入式(21)，解出 A_1、A_2 和 A_3. 当给定板的相关参数和激励后，A_1、A_2 和 A_3 都是形如 $\int_0^\infty g_i(k)q(k,0)\mathrm{d}k$ 的表达式，借助符号运算软件(如Matlab)可计算出 $g_i(k)$，进而求出 A_1、A_2 和 A_3，再代入(16)式，记

$$
W(k,r) = A_1 J_n(\delta_1 r) + A_2 J_n(\delta_2 r) + \lambda J_n(kr),
\tag{22}
$$

则
$$
w_n(r) = \frac{1}{D}\int_0^\infty [p_n(k) - q_n(k)]W(r,k)\mathrm{d}k.
$$

综合(3)、(4)式，并结合边界条件(10)，可以得出如下积分方程

$$
\int_0^\infty G(k)q_n(k)J_n(kr)k\mathrm{d}k = \Delta_n - \int_0^\infty [p_n(k) - q_n(k)]W(r,k)\mathrm{d}k \quad (0 \leqslant r \leqslant a),
\tag{23a}
$$

$$
\int_0^\infty kq_n(k)J_n(kr)\mathrm{d}k = 0 \qquad (a < r \leqslant \infty),
\tag{23b}
$$

在式(23a)中，当地基表面完全排水时 $G(k) = G_d(k)$；当表面完全不排水时 $G(k) = G_{ud}(k)$，

具体形式如下

$$G_d(k)=[\gamma_1(\zeta_2-\zeta_3)+\gamma_2(\zeta_3-\zeta_1)+\gamma_3(\zeta_1-\zeta_2)]/\Delta_d,$$

$$G_{ud}(k)=[\gamma_1(\lambda_3\zeta_2-\lambda_2\zeta_3)+\gamma_2(\lambda_1\zeta_3-\lambda_3\zeta_1)+\gamma_3(\lambda_2\zeta_1-\lambda_1\zeta_2)]/\Delta_{ud},$$

其中 $$\Delta_d=\det\begin{bmatrix}\xi_1&\xi_2&\xi_3\\\zeta_1&\zeta_2&\zeta_3\\1&1&1\end{bmatrix},\quad\Delta_{ud}=\det\begin{bmatrix}\xi_1&\xi_2&\xi_3\\\zeta_1&\zeta_2&\zeta_3\\\lambda_1&\lambda_2&\lambda_3\end{bmatrix}.$$

将(23a)和(23b)两式合并，得

$$\int_0^\infty\left\{[G(k)-1]\operatorname{curl}\left(\frac{l}{a}\right)+1\right\}q_n(k)\mathrm{J}_n(kr)k\mathrm{d}k=$$
$$\Delta_n\operatorname{curl}\left(\frac{l}{a}\right)-\left\{\int_0^\infty[p_n(k)-q_n(k)]W(r,k)\mathrm{d}k\right\}\operatorname{curl}\left(\frac{l}{a}\right),\tag{24}$$

其中圆函数 curl(x) 定义为

$$\operatorname{curl}(x)=\begin{cases}0, & x>1,\\1, & 0\leqslant x\leqslant 1.\end{cases}$$

对式(24)两边关于 r 作 n 阶 Hankel 变换，对具体的激励荷载，经过适当运算，可将对偶边界积分方程(23)化为如下关于 $q_n(k,0)$ 的第二类 Fredholm 积分方程

$$q_n(\xi,0)+\int_0^\infty K(\xi,k)q_n(k,0)k\mathrm{d}k=F(\xi),\tag{25}$$

式中

$$K(\xi,k)=[G(k)-1]\int_0^a r\mathrm{J}_n(\xi r)\mathrm{J}_n(kr)\mathrm{d}r-\int_0^a r\mathrm{J}_n(\xi r)W(k,r)\mathrm{d}r,$$

$$F(\xi)=\Delta_n\int_0^a r\mathrm{J}_n(\xi r)\mathrm{d}r-\int_0^\infty p_n(k,0)\left[\int_0^a r\mathrm{J}_n(\xi r)W(k,r)\mathrm{d}r\right]k\mathrm{d}k.$$

采用数值方法，由方程(25)解出 $q_n(k,0)$ 后，就可求出地基和圆板的各项响应．

3　数值算例

横观各向同性饱和地基的弹性常数 $B_1\sim B_8$ 具体取值分别为：$4\times10^6\ \mathrm{N/m^2}$，$7.803\times10^6\ \mathrm{N/m^2}$，$13.313\times10^6\ \mathrm{N/m^2}$，$12.864\times10^6\ \mathrm{N/m^2}$，$3.2\times10^6\ \mathrm{N/m^2}$，$-6.91\times10^6\ \mathrm{N/m^2}$，$-6.326\times10^6\ \mathrm{N/m^2}$，$8.228\times10^6\ \mathrm{N/m^2}$；板 $a=10$ m，$\nu=0.167$，$E=8\times10^7\ \mathrm{N/m^2}$，$\Delta_0=1$ m，$p_0=1\ \mathrm{kN/m^2}$．地基表面完全排水．本文借助 Matlab 软件编制相应的程序，并将第二类 Fredholm 积分方程(25)化为线性代数方程组求解．图 2 给出 $q(k,0)$ 的数值结果(图中以 $T(k,0)$ 表示)．结果显示 $q(k,0)$ 衰减很快，k 值只在小范围内 $q(k,0)$ 才有较大变化，当 $k>100$ 时 $q(k,0)$ 的值基本趋于零．因此对方程(25)求解时，可以把 k 从零到无穷大的变化区间缩小为零到有限值的变化区间，相应的无穷积分就变为有限域积分．

图 3 是地基与板的相互作用力 $q(r,0)$ 的响应，图中 $q(r,0)$ 出现负值(即地基产生了拉应力)，主要由于计算模型中假设了地基和板之间的法向位移完全协调而无脱离现象发生引起的，如何合理地解决这一问题值得进一步深入研究．图 4 描述了不同厚度($h/a=0.1$、$h/a=0.2$、$h/a=0.3$)板的挠度曲线，当 $h/a=0.3$ 时，板的挠度基本上趋于 0，可按刚性板计算．图 5、图 6 和图 7 给出板在不同频率激励($f=50$ Hz，$f=200$ Hz，$f=800$ Hz)下，半空间饱和地基表面的竖向位移响应幅值 $u_z(r,0)$．从结果可以看出，激励的频率从 50 Hz 变化到 200 Hz 时，地基表面的位移幅值随激振频率的增大而增大，当频率在增加时，地基表面的位移幅值增

加不明显• 同时可见, 位移幅值随距离的增大以减幅振荡形式变化•

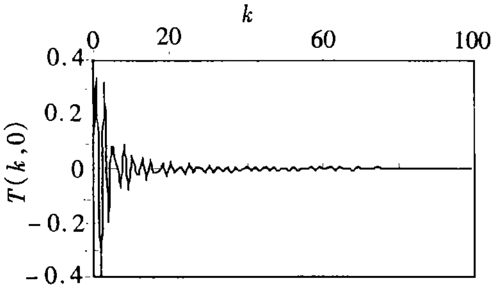

图 2 $T(k,0)$ 的数值结果

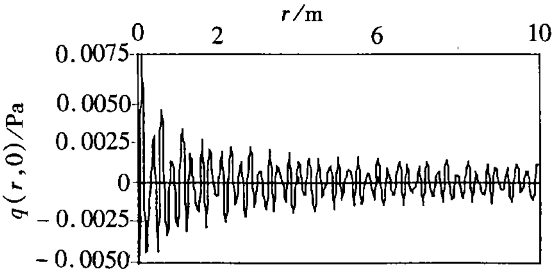

图 3 $q(r,0)$ 的数值结果

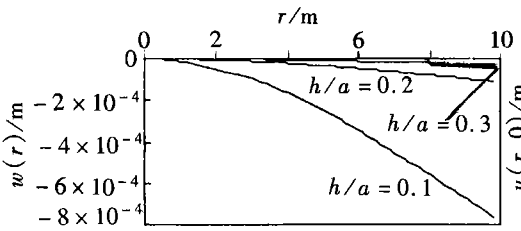

图 4 不同刚度板的挠度比较

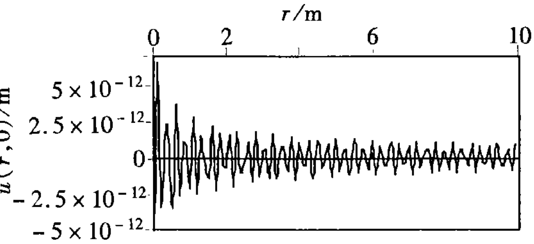

图 5 f = 50 Hz 时地基表位移幅值

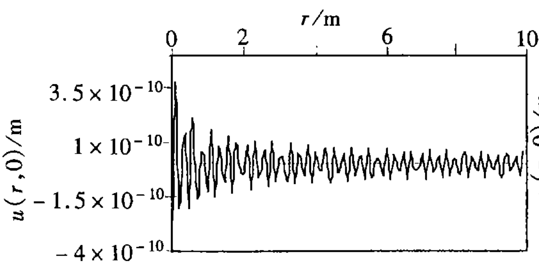

图 6 f = 200 Hz 时地基表位移幅值

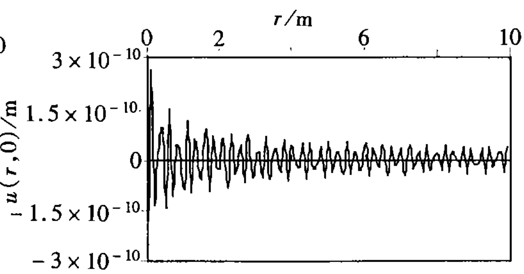

图 7 f = 800 Hz 时地基表位移幅值

4 结 语

本文讨论了横观各向同性饱和土地基与弹性中厚度圆板的非轴对称动力相互作用问题, 给出了一般的求解方法, 文中所提出的求解方法具有一般性, 对同类问题的求解具有一定的指导意义• 作为本理论的应用, 讨论了一个实例• 数值结果分析表明, 在一定频率范围内半空间饱和地基表面的位移幅值随激振频率的增大而增大, 地基表面位移幅值随距离的增大以减幅振荡形式变化•

[参 考 文 献]

[1] 陈龙珠, 陈胜利. 饱和地基上弹性圆板的动力响应[J]. 力学学报, 2001, **33**(6): 821—827.

[2] 陈胜利, 陈龙珠. 饱和地基上含刚性圆核弹性圆板的竖向振动分析[J]. 力学学报, 2002, **34**(1): 77—84.

[3] 金波. 具有混合边界透水条件的多孔饱和半空间上刚性圆板的垂直振动[J]. 固体力学学报, 1999, **20**(3): 267—271.

[4] 张引科, 黄义. 横观各向同性饱和弹性多孔介质的非轴对称动力响应[J]. 应用数学和力学, 2001, **22**(1): 56—70.

[5] 张引科, 黄义. 弹性饱和多孔介质在非轴对称荷载下的稳态的动力响应[J]. 土木工程学报, 2002, **35**(3): 41－45.
[6] Kazi_Aoual, Guy Bonnet, Paul Jouanna. Green' s founction in an infinite transversely isotropic saturated poroelastic medium[J]. J Acoust Soc Am, 1988, **84**(5): 1883－1889.
[7] 黄义, 张玉红. 饱和土三维非轴对称 Lamb 问题[J]. 中国科学(E 辑), 2000, **30**(4): 375－384.
[8] 曹志远, 扬 田. 厚板动力学及其应用[M]. 北京: 科学出版社, 1983, 36－38.

Dynamic Interaction between Elastic Thick Circular Plate and Transversely Isotropic Saturated Soil Ground

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Abstract: A study of the dynamic interaction between foundation and the underlying soil has been presented in a recent paper based on the assumption of saturated ground and elastic circular plate excited by the axisymmetical harmonic source. However, the assumption may not always by valid. The work is extended to the case of a circular plate resting on transversely isotropic saturated soil and subjected to a non_axisymmetical harmonic force. The analysis is based on the theory of elastic wave in transversely isotropic saturated poroelastic media established. By the techuique of Fourier expansion and Hankel transform, the governing different equation for transversely isotropic saturated soil are easily solved and the cooresponding Hankel transformed stress and displacement solutions are obtained. Then, under the contact conditions, the problem leads to a pair of dual integral equations which describes the mixed boundary_value problem. Furthermore, the dual integral equations can be reduced to the Fredholm integral equations of the second kind and solved by numerical procedure. At the end, a numerical result is presented which indicates that on a certain frequency range, the displacement amplitude of the surface of the foundation is increased with the increase of the frequency of the exciting force, and decreased in vibration form with the increase of the distance.

Key words: Biot' s wave equation; transversely isotropic saturated soil; elastic circular plate; non_axisymmetical harmonic response; Fredholm integral equation of the second kind