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大范围运动刚体上矩形薄板力学行为分析*

肖世富[1], 陈 滨[2]

(1. 中国工程物理研究院 结构力学研究所, 四川 绵阳 621900;
2. 北京大学 力学与工程科学系, 北京 100871)

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摘要: 采用 Hamilton 变分原理建立了大范围运动平板的动力学模型. 从理论上证明了不同大范围运动状态下平板中既可存在动力刚化效应, 也可存在动力软化效应, 且动力软化效应还可使板的平衡状态发生分岔而失稳. 采用假设模态法验证了理论分析结果并得到了分岔临界值和近似后屈曲解.

关 键 词: 柔性多体系统动力学; 动力刚化; 动力软化; 稳定性; 分岔; 后屈曲
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引 言

自从 Kane[1] 首次提出动力刚化概念以来, 柔性多体系统中的动力刚化研究引起了广泛的兴趣[2~9]. 柔性多体系统中的刚柔耦合机理非常复杂, 根据具体构形的不同, 刚体运动既可引起柔性体的刚度强化[1~9], 也可引起其刚度软化, 甚至使其平衡位置发生分岔而失稳[10~13]. 另外, 刚体和柔性附件还可能引起耦合模态特征[13, 14]. 甚至在同一系统中, 不同的刚体运动可能引起不同的动力刚化效应或动力软化效应.

这些工作的研究对象主要是柔性梁构成的简单刚柔耦合系统. 而对于大范围运动下的板, 研究成果还比较少, Banerjee 和 Kane(1989)[15] 建立了给定大范围刚体运动的简支板的离散程序, 并分析了其一阶频率; Chang 和 Shabana(1990)[16] 建立了空间任意运动板的非线性有限元公式; Boutaghou, Erdman 和 Stolarski(1992)[17] 采用广义 Hamilton 原理建立了任意大运动梁、板的连续偏微分方程形式的动力学方程; Yoo 和 Pierre(2003)[18] 建立了转动悬臂板的线性离散振动方程, 数值分析了转动板的固有频率. 上述工作, 对大范围运动板的动力刚化现象进行了数值研究. 而 Bloch[2] 对转动强化悬臂梁存在的动力刚化效应进行了理论证明; Musat 和 Epureanu[19] 提出了一种新的方法, 研究了转动参考系中小幅振动的多刚体系统的动力学和稳定性.

本文对于由中心空心刚性柱_平板所构成的简单刚柔耦合系统, 采用 Hamilton 原理建立了系统的非线性刚柔耦合动力学模型; 从理论上分析证明了系统中既可存在动力刚化效应, 也可存在动力软化效应, 并采用假设模态法对理论结果进行了验证, 分析了动力软化时平板的近似

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作者简介: 肖世富(1970—), 男, 四川人, 副研究员, 博士(联系人. Tel: + 86_816_2485465; Fax: + 86_816_2281485; E_mail: sfxiao@pku.org.cn).

临界分岔值和后屈曲解．研究内容如下，第 1 节采用 Hamilton 原理建立系统的非线性刚柔耦合动力学模型，第 2 节分析平动状态下板的力学行为，第 3 节分析转动状态下板的力学行为，第 4 节给出简要结论．

1 动力学建模

考虑如图 1 所示的中心空心刚性柱_平板所构成的简单刚柔耦合系统，其中 $o^*\_x^*y^*z^*$ 为惯性坐标系（单位矢量 $\boldsymbol{e}_{r1}$, $\boldsymbol{e}_{r2}$, $\boldsymbol{e}_{r3}$），而 O_XYZ 为描述板变形运动的浮动坐标系（单位矢量 $\boldsymbol{e}_{f1}$, $\boldsymbol{e}_{f2}$, $\boldsymbol{e}_{f3}$）．设刚性柱在 $o^*\_x^*y^*$ 平面内运动，其中心在惯性坐标系 $o^*\_x^*y^*z^*$ 中的坐标为 (x_c^*, y_c^*)；而浮动坐标系相对惯性坐标系的转角为 $\alpha(t)$．刚性柱半径为 $|R|$，质量为 M_r，相对中轴线的中心转动惯量为 I_r；设其中心控制合力（主动和被动）为 $\boldsymbol{F} = F_1\boldsymbol{e}_{r1} + F_2\boldsymbol{e}_{r2}$，合力矩（主动和被动）为 $\boldsymbol{m}(t) = m(t)\boldsymbol{e}_{r3}$ 作用；其上固结一 $a \times b \times h$ 均匀等厚矩形板，R 的取值可正可负，分别对应外悬臂板和内悬臂板．板的密度为 ρ，弹性模量为 E，Possion 比为 ν，弯曲刚度为 D．

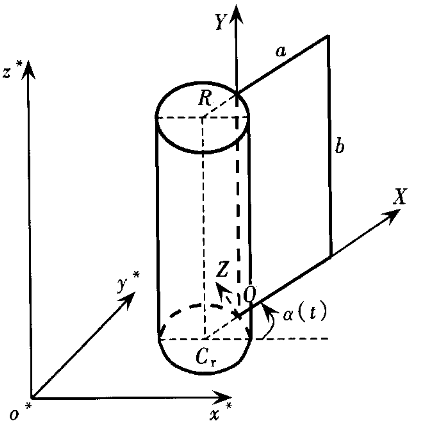

图 1 大范围运动刚体上平板的构形

下面采用 Hamilton 原理建立系统的动力学方程．

在惯性坐标系中，刚性柱中心的位移场为

$$\boldsymbol{r}_{C_r} = x_c^*\boldsymbol{e}_{r1} + y_c^*\boldsymbol{e}_{r2}. \tag{1}$$

在浮动坐标系 O_XYZ 中，设板中面位移分量为 $\{u(x,y,t), v(x,y,t), w(x,y,t)\}$，根据 Love_Kirchhoff 假设，板的位移分量 $\{u_1(x,y,z,t), u_2(x,y,z,t), u_3(x,y,z,t)\}$ 为

$$u_1 = u - z\frac{\partial w}{\partial x},\quad u_2 = v - z\frac{\partial w}{\partial y},\quad u_3 = w. \tag{2}$$

从而板在惯性坐标系 $o^*\_x^*y^*z^*$ 中的位移场为

$$\boldsymbol{r} = x_c^*\boldsymbol{e}_{r1} + y_c^*\boldsymbol{e}_{r2} + (R + x + u_1)\boldsymbol{e}_{f1} + (y + u_2)\boldsymbol{e}_{f2} + u_3\boldsymbol{e}_{f3}. \tag{3}$$

则系统的总动能为

$$T = T_1 + T_2, \tag{4}$$

式中

$$T_1 = \frac{1}{2}M[(\dot{x}_c^*)^2 + (\dot{y}_c^*)^2] + \frac{1}{2}I\dot{\alpha}^2 + I_{rf}(-\dot{x}_c^*\sin\alpha + \dot{y}_c^*\cos\alpha)\dot{\alpha}, \tag{4a}$$

$$\begin{aligned} T_2 = {} & \frac{1}{2}\int_0^a\int_0^b \rho h\left\{(\dot{u} - w\dot{\alpha})^2 + (\dot{w} + u\dot{\alpha})^2 + \dot{v}^2\right\}\mathrm{d}y\,\mathrm{d}x + \\ & \frac{1}{24}\int_0^a\int_0^b \rho h^3\left\{\left(\frac{\partial^2 w}{\partial x\partial t}\right)^2 + \left(\frac{\partial^2 w}{\partial y\partial t}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\dot{\alpha}^2\right\}\mathrm{d}y\,\mathrm{d}x + \\ & \int_0^a\int_0^b \rho h(\dot{x}_c^*\cos\alpha + \dot{y}_c^*\sin\alpha)(\dot{u} - w\dot{\alpha})\mathrm{d}y\,\mathrm{d}x + \\ & \int_0^a\int_0^b \rho h[(-\dot{x}_c^*\sin\alpha + \dot{y}_c^*\cos\alpha) + (R + x)\dot{\alpha}](\dot{w} + u\dot{\alpha})\mathrm{d}y\,\mathrm{d}x, \end{aligned} \tag{4b}$$

而 $M = M_r + \rho abh$；$I = I_r + (\rho bh/3)[(R + a)^3 - R^3]$；$I_{rf} = \rho Rabh + \rho a^2bh/2$.

下面分析板的应变能．非线性几何关系为

$$\begin{cases} \varepsilon_x = \dfrac{\partial u}{\partial x} + \dfrac{1}{2}\left(\dfrac{\partial w}{\partial x}\right)^2 - z\dfrac{\partial^2 w}{\partial x^2}, \\ \varepsilon_y = \dfrac{\partial v}{\partial y} + \dfrac{1}{2}\left(\dfrac{\partial w}{\partial y}\right)^2 - z\dfrac{\partial^2 w}{\partial y^2}, \\ \gamma_{xy} = \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} + \dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} - 2z\dfrac{\partial^2 w}{\partial x \partial y}, \end{cases} \tag{5}$$

其余应变分量忽略．取线弹性本构关系

$$\sigma_x = \frac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y),\quad \sigma_y = \frac{E}{1-\nu^2}(\varepsilon_y + \nu\varepsilon_x),\quad \tau_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy}. \tag{6}$$

设板没有弯曲挠度时的面内内力 $\{N_x, N_{xy}, N_y\}$ 为

$$N_x = \frac{Eh}{1-\nu^2}\left(\frac{\partial u}{\partial x} + \nu\frac{\partial v}{\partial y}\right),\quad N_y = \frac{Eh}{1-\nu^2}\left(\frac{\partial v}{\partial y} + \nu\frac{\partial u}{\partial x}\right),\quad N_{xy} = \frac{Eh}{2(1+\nu)}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right). \tag{7}$$

于是板的应变能为

$$U = \frac{1}{2}\int_0^a\int_0^b\int_{-h/2}^{h/2}(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \tau_{xy}\gamma_{xy})\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x = U_1 + U_2 + U_3, \tag{8}$$

式中

$$U_1 = \frac{1}{2}\int_0^a\int_0^b\left[N_x\frac{\partial u}{\partial x} + N_y\frac{\partial v}{\partial y} + N_{xy}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right]\mathrm{d}y\,\mathrm{d}x,$$

$$U_2 = \frac{1}{2}\int_0^a\int_0^b\left\{\left[N_x\left(\frac{\partial w}{\partial x}\right)^2 + N_y\left(\frac{\partial w}{\partial y}\right)^2 + 2N_{xy}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right] + \frac{Eh}{4(1-\nu^2)}\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right]^2\right\}\mathrm{d}y\,\mathrm{d}x,$$

$$U_3 = \frac{1}{2}D\int_0^a\int_0^b\left\{\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)^2 - 2(1-\nu)\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x\partial y}\right)^2\right]\right\}\mathrm{d}y\,\mathrm{d}x.$$

控制力和力矩作的虚功为

$$\delta W = F_1(t)\,\delta x_c^* + F_2(t)\,\delta y_c^* + m(t)\,\delta\alpha. \tag{9}$$

Hamilton 最小作用量原理要求

$$\int_{t_1}^{t_2}[\delta(T-U) + \delta W]\,\mathrm{d}t = 0. \tag{10}$$

对上式变分, 即可得到系统的非线性动力学耦合方程组:

$$M\ddot{x}_c^* - I_{\mathrm{rf}}\ddot{\alpha}\sin\alpha - I_{\mathrm{rf}}\dot{\alpha}^2\cos\alpha + \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_0^a\int_0^b \rho h[(\dot{u} - w\dot{\alpha})\cos\alpha - (\dot{w} + u\dot{\alpha})\sin\alpha]\,\mathrm{d}y\,\mathrm{d}x\right] = F_1, \tag{11a}$$

$$M\ddot{y}_c^* + I_{\mathrm{rf}}\ddot{\alpha}\cos\alpha - I_{\mathrm{rf}}\dot{\alpha}^2\sin\alpha + \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_0^a\int_0^b \rho h[(\dot{u} - w\dot{\alpha})\sin\alpha + (\dot{w} + u\dot{\alpha})\cos\alpha]\,\mathrm{d}y\,\mathrm{d}x\right] = F_2, \tag{11b}$$

$$\begin{aligned} &I_{\mathrm{r}}\ddot{\alpha} + (-\ddot{x}_c^*\sin\alpha + \ddot{y}_c^*\cos\alpha)\int_0^a\int_0^b\rho h(R + x + u)\,\mathrm{d}y\,\mathrm{d}x - \\ &\quad(\ddot{x}_c^*\cos\alpha + \ddot{y}_c^*\sin\alpha)\int_0^a\int_0^b\rho h w\,\mathrm{d}y\,\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_0^a\int_0^b\rho h\left\{-w\dot{u} + (R + x + u)\dot{w} + \right.\right. \\ &\quad\left.\left.\left[w^2 + (R + x + u)^2 + \frac{1}{12}h^2\left(\frac{\partial w}{\partial x}\right)^2\right]\dot{\alpha}\right\}\mathrm{d}y\,\mathrm{d}x\right] = m(t), \end{aligned} \tag{11c}$$

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = \rho h\left[\frac{\partial^2 u}{\partial t^2} - w\ddot{\alpha} - 2\dot{w}\dot{\alpha} - (R + x + u)\dot{\alpha}^2 + (\ddot{x}_c^*\cos\alpha + \ddot{y}_c^*\sin\alpha)\right], \tag{11d}$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = \rho h \frac{\partial^2 v}{\partial t^2}, \tag{11e}$$

$$\begin{aligned} &D\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) + \rho h \frac{\partial^2 w}{\partial t^2} - \frac{1}{12}\rho h^3\left(\frac{\partial^4 w}{\partial x^2 \partial t^2} + \frac{\partial^4 w}{\partial y^2 \partial t^2}\right) = \\ &\quad \frac{\partial}{\partial x}\left(N_x \frac{\partial w}{\partial x} + N_{xy}\frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial y}\left(N_{xy}\frac{\partial w}{\partial x} + N_y \frac{\partial w}{\partial y}\right) + \\ &\quad \frac{Eh}{2(1-\nu^2)}\frac{\partial}{\partial x}\left\{\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right]\frac{\partial w}{\partial x}\right\} + \frac{Eh}{2(1-\nu^2)}\frac{\partial}{\partial y}\left\{\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right]\frac{\partial w}{\partial y}\right\} - \\ &\quad \rho h\left[(R + x + u)\ddot{\alpha} + 2\dot{u}\dot{\alpha} + \frac{1}{12}h^2\frac{\partial^2 w}{\partial x^2}\dot{\alpha}^2 - w\dot{\alpha}^2 - (\ddot{x}_c^* \sin\alpha - \ddot{y}_c^* \cos\alpha)\right]. \end{aligned} \tag{11f}$$

边界条件为

$$\begin{cases} x = 0:\ u = 0,\ v = 0;\ w = 0,\ \dfrac{\partial w}{\partial x} = 0, \\ x = a:\ N_x = 0,\ N_{xy} = 0;\ \dfrac{\partial^2 w}{\partial x^2} + \nu\dfrac{\partial^2 w}{\partial y^2} = 0, \\ \qquad D\left[\dfrac{\partial^3 w}{\partial x^3} + (2-\nu)\dfrac{\partial^3 w}{\partial x \partial y^2}\right] - \dfrac{Eh}{2(1-\nu^2)}\left[\left(\dfrac{\partial w}{\partial x}\right)^2 + \left(\dfrac{\partial w}{\partial y}\right)^2\right]\dfrac{\partial w}{\partial x} + \\ \qquad \dfrac{\rho h^3}{12}\left(\dfrac{\partial w}{\partial x}\dot{\alpha}^2 - \dfrac{\partial^3 w}{\partial x \partial t^2}\right) = 0, \\ y = 0, b:\ N_y = 0,\ N_{xy} = 0;\ \dfrac{\partial^2 w}{\partial y^2} + \nu\dfrac{\partial^2 w}{\partial x^2} = 0, \\ \qquad D\left[\dfrac{\partial^3 w}{\partial y^3} + (2-\nu)\dfrac{\partial^3 w}{\partial x^2 \partial y}\right] - \dfrac{Eh}{2(1-\nu^2)}\left[\left(\dfrac{\partial w}{\partial x}\right)^2 + \left(\dfrac{\partial w}{\partial y}\right)^2\right]\dfrac{\partial w}{\partial y} - \\ \qquad \dfrac{\rho h^3}{12}\dfrac{\partial^3 w}{\partial y \partial t^2} = 0. \end{cases} \tag{12}$$

2 平动板力学行为分析

2.1 平动板动态特性分析

当刚性柱只在 $\boldsymbol{e}_{r1}$ 方向上平动时, $y_c^* \equiv 0$, $\alpha(t) \equiv 0$, 控制合力应满足

$$F_2 = \int_0^a \int_0^b \rho A \ddot{w}(x, t) \mathrm{d}y \mathrm{d}x, \tag{13}$$

控制合力矩应满足

$$m(t) = \int_0^a \int_0^b \rho h w[(R + x + u)\ddot{w} - w\ddot{u}] \mathrm{d}y \mathrm{d}x - \ddot{x}_c^* \int_0^a \int_0^b \rho h w \mathrm{d}y \mathrm{d}x. \tag{14}$$

此时,系统整体平动时的动力学控制方程组为

$$M\ddot{x}_c^* + \int_0^a \int_0^b \rho h \ddot{u} \mathrm{d}y \mathrm{d}x = F_1, \tag{15a}$$

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = \rho h\left(\frac{\partial^2 u}{\partial t^2} + \ddot{x}_c^*\right), \tag{15b}$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = \rho h \frac{\partial^2 v}{\partial t^2}, \tag{15c}$$

$$D\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) + \rho h \frac{\partial^2 w}{\partial t^2} - \frac{1}{12}\rho h^3\left(\frac{\partial^4 w}{\partial x^2 \partial t^2} + \frac{\partial^4 w}{\partial y^2 \partial t^2}\right) =$$

$$\frac{\partial}{\partial x}\left(N_x\frac{\partial w}{\partial x}+N_{xy}\frac{\partial w}{\partial y}\right)+\frac{\partial}{\partial y}\left(N_{xy}\frac{\partial w}{\partial x}+N_y\frac{\partial w}{\partial y}\right)+$$
$$\frac{Eh}{2(1-\nu^2)}\frac{\partial}{\partial x}\left\{\left[\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right]\frac{\partial w}{\partial x}\right\}+\frac{Eh}{2(1-\nu^2)}\frac{\partial}{\partial y}\left\{\left[\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right]\frac{\partial w}{\partial y}\right\} \tag{15d}$$

及边界条件(12)(边界条件中 $\dot{\theta}\equiv 0$).

在匀加速度即 $\ddot{x}_c^* \equiv A$ 时,系统(15)、(12)存在平凡解

$$N_{x0}=\rho hA(x-a),\quad N_{xy0}=N_{y0}=0,\quad w_0=0. \tag{16}$$

在平凡解(16)邻域线性化并忽略厚度方向上的小量,即可得到匀加速度平动状态下平板的线性自由振动方程及其边界条件

$$\begin{cases} D\left(\dfrac{\partial^4 w}{\partial x^4}+2\dfrac{\partial^4 w}{\partial x^2\partial y^2}+\dfrac{\partial^4 w}{\partial y^4}\right)+\rho h\dfrac{\partial^2 w}{\partial t^2}=\dfrac{\partial}{\partial x}\left\{\rho hA(x-a)\dfrac{\partial w}{\partial x}\right\}, \\ x=0:\ w=0,\ \dfrac{\partial w}{\partial x}=0;\quad x=a:\ \dfrac{\partial^2 w}{\partial x^2}+\nu\dfrac{\partial^2 w}{\partial y^2}=0,\ \dfrac{\partial^3 w}{\partial x^3}+(2-\nu)\dfrac{\partial^3 w}{\partial x\partial y^2}=0, \\ y=0,b:\ \dfrac{\partial^2 w}{\partial y^2}+\nu\dfrac{\partial^2 w}{\partial x^2}=0,\ \dfrac{\partial^3 w}{\partial y^3}+(2-\nu)\dfrac{\partial^3 w}{\partial x^2\partial y}=0. \end{cases} \tag{17}$$

定常边界条件下悬臂板的自由振动方程为

$$\begin{cases} D\left(\dfrac{\partial^4 w}{\partial x^4}+2\dfrac{\partial^4 w}{\partial x^2\partial y^2}+\dfrac{\partial^4 w}{\partial y^4}\right)+\rho h\dfrac{\partial^2 w}{\partial t^2}=0, \\ x=0:\ w=0,\ \dfrac{\partial w}{\partial x}=0;\quad x=a:\ \dfrac{\partial^2 w}{\partial x^2}+\nu\dfrac{\partial^2 w}{\partial y^2}=0,\ \dfrac{\partial^3 w}{\partial x^3}+(2-\nu)\dfrac{\partial^3 w}{\partial x\partial y^2}=0, \\ y=0,b:\ \dfrac{\partial^2 w}{\partial y^2}+\nu\dfrac{\partial^2 w}{\partial x^2}=0,\ \dfrac{\partial^3 w}{\partial y^3}+(2-\nu)\dfrac{\partial^3 w}{\partial x^2\partial y}=0. \end{cases} \tag{18}$$

比较系统(17)和(18),我们有下面的定理.

定理 1　设系统(18)的振动频率为 ω_i^*, $i=1,2,3,\cdots$,而在刚体匀加速度平动时系统(17)的振动频率为 ω_i, $i=1,2,3,\cdots$,则当 $A>0$ 时,系统(17)的振动频率降低,即 $\omega_i<\omega_i^*$, $i=1,2,3,\cdots$,且存在临界值 $\bar{A}$,当 $A=\bar{A}$ 时,系统基频降为零,平衡态(16)发生分岔而失稳;而 $A<0$ 时,系统(17)的振动频率升高,即 $\omega_i>\omega_i^*$, $i=1,2,3,\cdots$.

证明　设系统(17)的势能为 V_T,系统(18)的势能为 V^*. 由 Rayleigh 商,只需证明当 $A>0$ 时,$V_T<V^*$,则 $\omega_i<\omega_i^*$, $i=1,2,3,\cdots$;而当 $A<0$ 时,$V_T>V^*$,则 $\omega_i>\omega_i^*$, $i=1,2,3,\cdots$.

系统(17)的势能 V_T 为

$$V_T=\frac{1}{2}D\int_0^a\int_0^b\left\{\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)^2-2(1-\nu)\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2\right]\right\}\mathrm{d}y\,\mathrm{d}x+$$
$$\frac{1}{2}\int_0^a\int_0^b\rho hA(x-a)\left(\frac{\partial w}{\partial x}\right)^2\mathrm{d}y\,\mathrm{d}x, \tag{19}$$

系统(18)的势能 V^* 为

$$V^*=\frac{1}{2}D\int_0^a\int_0^b\left\{\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)^2-2(1-\nu)\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2\right]\right\}\mathrm{d}y\,\mathrm{d}x. \tag{20}$$

显然当 $A>0$ 时,$V_T<V^*$;而当 $A<0$ 时,$V_T>V^*$. 故由 Rayleigh 商有:当 $A>0$ 时,$\omega_i<\omega_i^*$, $i=1,2,3,\cdots$;而 $A<0$ 时,$\omega_i>\omega_i^*$, $i=1,2,3,\cdots$.

另外,在 $A>0$ 且 $w(x)\neq 0$ 的情况下,当 $A\mapsto 0^+$, $V_T\mapsto V^*>0$,而当 $A\mapsto+\infty$ 时,存

在 $w(x) \neq 0$, 使得 $V_{\mathrm{T}} < 0$, 由 $V_{\mathrm{T}}(w(x), A)$ 的连续性, 则系统存在临界值A, 当 $A = A$ 时, 系统基频降为零, 平凡解发生分岔而失稳. 证毕.

定理 1 表明当 $A > 0$ 时, 系统存在动力软化效应; 而当 $A < 0$ 时, 系统存在动力刚化效应. 因此, 刚柔耦合系统中, 不同整体运动状态下, 系统中柔性附件既可存在动力刚化效应, 也可存在动力软化效应.

2.2 平动板力学行为近似解析分析

在板作整体匀加速度平动时, 忽略板拉伸的影响和厚度方向上的小量, 则系统对应的 Lagrange 函数为

$$\Pi_{\mathrm{T}} = \frac{1}{2}\int_0^a\int_0^b \rho h\left(\frac{\partial w}{\partial t}\right)^2 \mathrm{d}y\,\mathrm{d}x - \frac{1}{2}\int_0^a\int_0^b \rho h A(x-a)\left(\frac{\partial w}{\partial x}\right)^2 \mathrm{d}y\,\mathrm{d}x - \frac{1}{2}D\int_0^a\int_0^b\left\{\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)^2 - 2(1-\nu)\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x\partial y}\right)^2\right]\right\}\mathrm{d}y\,\mathrm{d}x - \frac{Eh}{8(1-\nu^2)}\int_0^a\int_0^b\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right]^2 \mathrm{d}y\,\mathrm{d}x. \tag{21}$$

取悬臂板的一阶假设模态, 设

$$w(x, y, t) = [Z(t)/a^3](x^4 - 4ax^3 + 6a^2x^2), \tag{22}$$

式中 Z 为广义坐标. 将式(22)代入式(21)并应用 Hamilton 变分原理, 即可得到整体匀加速度平动时板的非线性自由振动方程

$$\beta^2\ddot{Z} + \left(\frac{27}{26}\delta^2 - \frac{81}{52}\beta^2 A\right)Z + \frac{34\,992}{1\,183}Z^3 = 0, \tag{23}$$

式中 $\beta = \sqrt{\rho a^2(1-\nu^2)/E}$, $\delta = h/a$ 和$A = A/a$. 由式(23)的线性化方程可得到系统的第 1 阶近似振动频率

$$\omega_1 = \sqrt{\frac{27}{26}\frac{\delta^2}{\beta^2} - \frac{81}{52}A}. \tag{24}$$

式(24)表明: 当 $A < 0$ 时, 振动频率升高, 系统出现动力刚化效应; 当 $A > 0$ 时, 振动频率下降, 且存在临界值 $A_c = (2/3)(\delta^2/\beta^2)$, 当 $A = A_c$ 时, $\omega_1 = 0$, 系统平衡状态发生分岔而失稳, 系统出现了动力软化效应. 下面再分析式(23)中广义坐标 Z 的平衡解, 此时存在 2 种情况:

a. 当 $A < A_c$ 时, 系统只存在稳定的零平衡解 $Z = 0$;

b. 当 $A \geqslant A_c$ 时, 零平衡解失稳, 叉式分岔出 1 对对称稳定的后屈曲解

$$Z = \pm\sqrt{\frac{91}{2\,592}\left(\frac{3}{2}\beta^2 A - \delta^2\right)}. \tag{25}$$

3 转动板力学行为分析

3.1 转动板动态特性分析

当刚性柱纯转动时, $x_c^* \equiv 0$, $y_c^* \equiv 0$, 控制合力应满足

$$\begin{cases} F_1 = -I_{\mathrm{rf}}\ddot{\alpha}\sin\alpha - I_{\mathrm{rf}}\dot{\alpha}^2\cos\alpha + \dfrac{\mathrm{d}}{\mathrm{d}t}\left[\displaystyle\int_0^a\int_0^b \rho h[(\dot{u} - w\dot{\alpha})\cos\alpha - (\dot{w} + u\dot{\alpha})\sin\alpha]\,\mathrm{d}y\,\mathrm{d}x\right], \\ F_2 = I_{\mathrm{rf}}\ddot{\alpha}\cos\alpha - I_{\mathrm{rf}}\dot{\alpha}^2\sin\alpha + \dfrac{\mathrm{d}}{\mathrm{d}t}\left[\displaystyle\int_0^a\int_0^b \rho h[(\dot{u} - w\dot{\alpha})\sin\alpha + (\dot{w} + u\dot{\alpha})\cos\alpha]\,\mathrm{d}y\,\mathrm{d}x\right]. \end{cases} \tag{26}$$

而整体转动时系统的动力学控制方程组为

$$
\begin{cases}
I_r\ddot{\alpha}+\dfrac{\mathrm{d}}{\mathrm{d}t}\Bigg[\displaystyle\int_0^a\int_0^b\rho h\Bigg\{-w\dot{u}+(R+x+u)\dot{w}+ \\
\qquad\left[w^2+(R+x+u)^2+\dfrac{1}{12}h^2\left(\dfrac{\partial w}{\partial x}\right)^2\right]\dot{\alpha}\Bigg\}\mathrm{d}y\,\mathrm{d}x\Bigg]=m(t), \\
\dfrac{\partial N_x}{\partial x}+\dfrac{\partial N_{xy}}{\partial y}=\rho h\left[\dfrac{\partial^2 u}{\partial t^2}-w\ddot{\alpha}-2\dot{w}\dot{\alpha}-(R+x+u)\dot{\alpha}^2\right], \\
\dfrac{\partial N_{xy}}{\partial x}+\dfrac{\partial N_y}{\partial y}=\rho h\dfrac{\partial^2 v}{\partial t^2}, \\
D\left(\dfrac{\partial^4 w}{\partial x^4}+2\dfrac{\partial^4 w}{\partial x^2\partial y^2}+\dfrac{\partial^4 w}{\partial y^4}\right)+\rho h\dfrac{\partial^2 w}{\partial t^2}-\dfrac{1}{12}\rho h^3\left(\dfrac{\partial^4 w}{\partial x^2\partial t^2}+\dfrac{\partial^4 w}{\partial y^2\partial t^2}\right)= \\
\qquad\dfrac{\partial}{\partial x}\left(N_x\dfrac{\partial w}{\partial x}+N_{xy}\dfrac{\partial w}{\partial y}\right)+\dfrac{\partial}{\partial y}\left(N_{xy}\dfrac{\partial w}{\partial x}+N_y\dfrac{\partial w}{\partial y}\right)+ \\
\qquad\dfrac{Eh}{2(1-\nu^2)}\dfrac{\partial}{\partial x}\left\{\left[\left(\dfrac{\partial w}{\partial x}\right)^2+\left(\dfrac{\partial w}{\partial y}\right)^2\right]\dfrac{\partial w}{\partial x}\right\}+\dfrac{Eh}{2(1-\nu^2)}\dfrac{\partial}{\partial y}\left\{\left[\left(\dfrac{\partial w}{\partial x}\right)^2+\left(\dfrac{\partial w}{\partial y}\right)^2\right]\dfrac{\partial w}{\partial y}\right\}- \\
\qquad\rho h\left[(R+x+u)\ddot{\alpha}+2\dot{u}\dot{\alpha}+\dfrac{1}{12}h^2\dfrac{\partial^2 w}{\partial x^2}\dot{\alpha}^2-w\dot{\alpha}^2\right].
\end{cases}
\tag{27}
$$

对应的边界条件为式(12).

当刚性柱匀速转动时,略去板拉伸的影响,可得到系统的近似平凡解

$$
\begin{cases}
m(t)=0,\ \dot{\alpha}_0=\Omega=\text{const};\ N_{xy0}=N_{y0}=0;\ w_0=0, \\
N_{x0}=\rho h\Omega^2\left[\dfrac{1}{2}(a^2-x^2)+R(a-x)\right].
\end{cases}
\tag{28}
$$

在平凡解(28)邻域线性化并忽略厚度方向上的小量,则可得到整体匀速转动状态下平板的线性振动方程及其边界条件

$$
\begin{cases}
D\left(\dfrac{\partial^4 w}{\partial x^4}+2\dfrac{\partial^4 w}{\partial x^2\partial y^2}+\dfrac{\partial^4 w}{\partial y^4}\right)+\rho h\dfrac{\partial^2 w}{\partial t^2}= \\
\qquad\dfrac{\partial}{\partial x}\left\{\rho h\Omega^2\left[\dfrac{1}{2}(a^2-x^2)+R(a-x)\right]\dfrac{\partial w}{\partial x}\right\}+\rho h\Omega^2 w, \\
x=0:\ w=0,\ \dfrac{\partial w}{\partial x}=0;\quad x=a:\ \dfrac{\partial^2 w}{\partial x^2}+\nu\dfrac{\partial^2 w}{\partial y^2}=0,\ \dfrac{\partial^3 w}{\partial x^3}+(2-\nu)\dfrac{\partial^3 w}{\partial x\partial y^2}=0, \\
y=0,b:\ \dfrac{\partial^2 w}{\partial y^2}+\nu\dfrac{\partial^2 w}{\partial x^2}=0,\ \dfrac{\partial^3 w}{\partial y^3}+(2-\nu)\dfrac{\partial^3 w}{\partial x^2\partial y}=0.
\end{cases}
\tag{29}
$$

比较系统(29)和系统(18),我们有下面的定理:

定理 2 设系统(18)的振动频率为 ω_i^*, $i=1,2,3,\cdots$, 而整体匀速转动系统(29)的振动频率为 ω_i, $i=1,2,3,\cdots$, 则当 $R>0$ 时, $\omega_i>\omega_i^*$, $i=1,2,3,\cdots$; 而当 $R<0$, 且 $a<|R|$ 时, $\omega_i<\omega_i^*$, $i=1,2,3,\cdots$, 且存在一个临界转速 Ω_c, 当 $\Omega\geqslant\Omega_c$ 时,系统基频 $\omega_1=0$, 平衡态(28)发生分岔而失稳.

为证明定理 2, 首先证明下面引理:

引理 1 设 $f(x)\in L^{(4)}[0,1]$, 且 $f(0)=f'(0)=0$; $f''(1)=f'''(1)=0$, 则

$$
\int_0^1\left[\frac{1}{2}(1-x^2)(f')^2-f^2\right]\mathrm{d}x\geqslant 0. \tag{30}
$$

证明 由边界条件及 Schwarz 不等式有

$$\int_0^1 f^2 \mathrm{d}x = \int_0^1 \left(\int_0^x f'(\xi)\, \mathrm{d}\xi \right)^2 \mathrm{d}x \leqslant \int_0^1 x \int_0^x (f')^2 \mathrm{d}\xi \mathrm{d}x. \tag{31}$$

令 $$g(x) = -\frac{1}{2}\int_0^x (f'(\xi))^2 \mathrm{d}\xi, \tag{32}$$

则由分步积分有

$$\int_0^1 \left[\frac{1}{2}(1-x^2)(f')^2 \right] \mathrm{d}x = \int_0^1 -(1-x^2) g' \,\mathrm{d}x =$$

$$-(1-x^2)g(x)\Big|_0^1 - \int_0^1 2xg\,\mathrm{d}x = \int_0^1 x \int_0^x (f')^2 \mathrm{d}\xi \mathrm{d}x. \tag{33}$$

考虑到式(31)和(33),则不等式(30)成立. 证毕.

下面证明定理2.

证明 设系统(29)的势能为 V_R,系统(18)的势能为 V^*. 同理由Rayleigh商,只需证明:当 $R>0$ 时,$V_R > V^*$;而当 $R<0$,且 $a<|R|$ 时,$V_R < V^*$.

系统(29)的势能 V_R 为

$$V_R = \frac{1}{2} D \int_0^a \int_0^b \left\{ \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)^2 - 2(1-\nu) \left[\frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right] \right\} \mathrm{d}y \mathrm{d}x +$$

$$\frac{1}{2} \int_0^a \int_0^b \rho h \Omega^2 \left\{ \left[\frac{1}{2}(a^2 - x^2) + R(a-x) \right] \left(\frac{\partial w}{\partial x} \right)^2 - w^2 \right\} \mathrm{d}y \mathrm{d}x. \tag{34}$$

由引理1,当 $R>0$ 时,$V_R > V^*$;而当 $R<0$,且 $a<|R|$ 时,$V_R<V^*$. 故由Rayleigh商有:当 $R>0$ 时,$\omega_i > \omega_i^*$,$i=1,2,3,\cdots$;而当 $R<0$,且 $a<|R|$ 时,$\omega_i<\omega_i^*$,$i=1,2,3,\cdots$.

另外,可同理由 $V_R(w, \Omega)$ 的连续性证明临界转速的存在性. 证毕.

定理2表明:当悬臂板在刚性空心柱内部($R<0$,且 $a<|R|$)时,系统中存在动力软化现象,且存在一个临界转速 Ω_c,当 $\Omega \geqslant \Omega_c$ 时,板平衡位置可能发生分岔而失稳;而当平板固结在刚性柱外部($R>0$)时,系统存在动力刚化现象,板的刚度增强,平凡解(28)是稳定的.

3.2 转动板力学行为近似解析分析

在板整体匀速转动时,忽略板拉伸的影响和厚度方向上的小量,则系统对应的Lagrange函数为

$$\Pi_R = \frac{1}{2} \int_0^a \int_0^b \rho h \left(\frac{\partial w}{\partial t} \right)^2 \mathrm{d}y \mathrm{d}x - \frac{1}{2} \int_0^a \int_0^b \rho h \Omega^2 \left[\frac{1}{2}(a^2 - x^2) + R(a-x) \right] \left(\frac{\partial w}{\partial x} \right)^2 \mathrm{d}y \mathrm{d}x -$$

$$\frac{1}{2} D \int_0^a \int_0^b \left\{ \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)^2 - 2(1-\nu) \left[\frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right] \right\} \mathrm{d}y \mathrm{d}x -$$

$$\frac{Eh}{8(1-\nu^2)} \int_0^a \int_0^b \left[\left(\frac{\partial w}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial y} \right)^2 \right]^2 \mathrm{d}y \mathrm{d}x + \frac{1}{2} \int_0^a \int_0^b \rho h \Omega^2 w^2 \mathrm{d}y \mathrm{d}x. \tag{35}$$

同样取悬臂板的1阶假设模态(22),且将式(22)代入(35)并应用Hamilton变分原理,即可得到整体匀速转动时板的非线性自由振动方程

$$\beta^2 \ddot{Z} + \left(\frac{27}{26}\delta^2 + \frac{9}{52}(1+9R)\beta^2 \Omega^2 \right) Z + \frac{34\,992}{1\,183} Z^3 = 0, \tag{36}$$

式中 $\beta = \sqrt{\rho a^2 (1-\nu^2)/E}$,$\delta = h/a$ 和 $R = R/a$. 由式(36)的线性化方程可得到系统的第1阶近似振动频率

$$\omega_1 = \sqrt{\frac{27}{26}\frac{\delta^2}{\beta^2} + \frac{9}{52}(1+9R)\Omega^2}. \tag{37}$$

式(37)表明: 当 $R > -1/9$ 时, 振动频率升高, 系统出现动力刚化效应; 当 $R < -1/9$ 时, 振动频率下降, 且存在临界值 $\Omega_c = \sqrt{-6/(1+9R)}\,\delta/\beta$, 当 $\Omega = \Omega_c$ 时, $\omega_1 = 0$, 系统平衡状态发生分岔而失稳, 系统出现了动力软化效应. 下面再分析式(36)中广义坐标 Z 的平衡解, 此时存在3种情况:

a. 当 $R > -1/9$ 时, 系统只存在稳定的零平衡解 $Z = 0$;

b. 当 $R < -1/9$ 且 $\Omega < \Omega_c$ 时, 系统也只存在稳定的零平衡解 $Z = 0$;

c. 当 $R < -1/9$ 且 $\Omega \geqslant \Omega_c$ 时, 零平衡解失稳叉式分岔出1对对称稳定的后屈曲解

$$Z = \pm\sqrt{-\frac{91}{2\,592}\left[\delta^2 + \frac{1}{6}(1+9R)\beta^2\Omega^2\right]}. \tag{38}$$

4 结　　论

本文对于由中心刚性空心柱带有柔性悬臂板组成的一类简单刚柔耦合系统, 采用Hamilton原理建立了系统的非线性刚柔耦合动力学模型, 分析了中心刚体在等加速度平动状态和匀速转动状态下平板的动态特性. 从理论上证明了: 柔性多体系统中, 刚体运动对柔性附件产生拉伸作用的预应力时, 柔性附件的固有频率升高, 存在动力刚化效应; 刚体运动对柔性附件产生压缩作用的预应力时, 柔性附件的固有频率下降, 甚至当压缩预应力达到临界值时, 柔性附件的平衡位置还将发生分岔而失稳, 即存在动力软化效应. 进一步采用假设模态法近似解析分析了上述两种状态下板的力学行为, 得到了板的第1阶近似振动频率、临界分岔值和后屈曲解, 验证了上述理论结论.

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Dynamic Behavior of a Thin Rectangular Plate Attached to a Moving Rigid

XIAO Shi_fu[1], CHEN Bin[2]

(1. Southwest Institute of Structural Mechanics, P.O. Box 919_401, Mianyang, Sichuan 621900, P.R. China;
2. Department of Mechanics and Engineering Science, Peking University, Beijing 100871, P.R. China)

Abstract: A nonlinear dynamic model of a thin rectangular plate attached to a moving rigid is established by employing the general Hamilton's variational principle. Based on the new model, both phenomena of dynamic stiffening and dynamic softening can occur in the plate was proved theoretically when the rigid undergoes different large overall motions including overall translational and rotary motions. It was also proved that dynamic softening effect even can make the trivial equilibrium of the plate lose its stability through bifurcation. Assumed modes method was employed to validate the theoretical result and analyze the approximately critical bifurcation value and the post_buckling equilibria.

Key words: flexible multi_body system; dynamic stiffening; dynamic softening; stability; bifurcation; post_buckling