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对称迭层矩形板的平面应力分析*

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(钟万勰推荐)

摘要: 常用的对称迭层板为各向异性板• 根据平面应力问题的基本方程精确地用应力函数解法求得了各向异性板的一般解析解• 推导出平面内应力和位移的一般公式, 其中积分常数由边界条件来决定• 一般解包括三角函数和双曲函数组成的解, 它能满足 4 个边为任意边界条件的问题• 还有代数多项式解, 它能满足 4 个角的边界条件• 因此一般解可用以求解任意边界条件下的平面应力问题• 以 4 边承受均匀法向和切向载荷以及非均匀法向载荷的对称迭层方板为例, 进行了计算和分析•

关 键 词: 对称迭层法; 各向异性; 应力函数解法; 应变; 位移

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引 言

复合材料以其重量轻、比强度和比刚度高, 已广泛应用于航空、航天和船舶结构等各种不同工程中, 常用的复合材料板多为各向异性板• Kalmanok[1]用迭加法, 黄炎[2]用一般解析法求解了各向同性矩形板的平面应力问题• 张承宗[3]用位移解法求解了各向异性矩形板的平面应力问题• 姚伟岸、苏滨、钟万勰[4]基于相似性原理并引入 Hamilton 体系求解了正交各向异性板弯曲问题• 平面应力问题的各种解析解均为三角函数和双曲函数混合解以及代数多项式解, 文献[3]的代数多项式解是不完整的, 而且有误, 因而未能全面精确地分析各种不同的边界问题• 本文采用应力函数解法精确地求得了各向异性矩形板平面应力问题的一般解, 从而可用以完整的求解各种问题•

1 基本方程的解

各向异性矩形板平面应力问题的基本方程(图 1) 为[5]

$$\frac{\partial N_x}{\partial x}+\frac{\partial N_{xy}}{\partial y}=0,\quad \frac{\partial N_{xy}}{\partial x}+\frac{\partial N_y}{\partial y}=0, \tag{1}$$

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$$\varepsilon_x = \frac{\partial u}{\partial x},\quad \varepsilon_y = \frac{\partial v}{\partial y},\quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x},\tag{2}$$

$$\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{16}\\ a_{12} & a_{22} & a_{26}\\ a_{16} & a_{26} & a_{66}\end{bmatrix}\begin{Bmatrix}N_x\\ N_y\\ N_{xy}\end{Bmatrix},\tag{3}$$

式中 a_{11}、a_{12}、a_{16}、a_{22}、a_{26}、a_{66} 为板的柔度系数, N_x、N_y、N_{xy} 为应力, ε_x、ε_y、γ_{xy} 为应变, u、v 为位移, 设应力函数 F 为

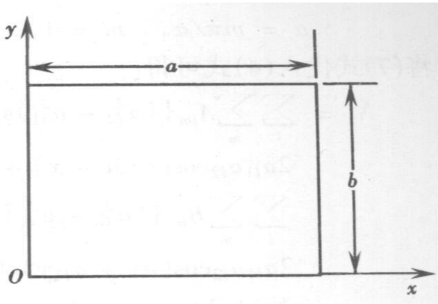

图 1　板的坐标系

$$N_x = \frac{\partial^2 F}{\partial y^2},\quad N_y = \frac{\partial^2 F}{\partial x^2},\quad N_{xy} = -\frac{\partial^2 F}{\partial x\partial y}.\tag{4}$$

上式满足平衡方程(1), 将(2)式进行微分可得应变的协调方程

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x\partial y}.$$

将(4)式代入(3)式然后代入上式可得

$$a_{11}\frac{\partial^4 F}{\partial y^2} - 2a_{16}\frac{\partial^4 F}{\partial x\partial y^3} + (2a_{12}+a_{66})\frac{\partial^4 F}{\partial x^2\partial y^2} - 2a_{26}\frac{\partial^4 F}{\partial x^3\partial y} + a_{22}\frac{\partial^4 F}{\partial x^4} = 0.\tag{5}$$

采用复数形式的解[6], 令 $F = e^{\pm i(\alpha x + \alpha' y)}$ 代入上式可得

$$a_{11}\alpha'^4 - 2a_{16}\alpha\alpha'^3 + (2a_{12}+a_{66})\alpha^2\alpha'^2 - 2a_{26}\alpha^3\alpha' + a_{22}\alpha^4 = 0.$$

上式的解为 $\alpha' = \alpha_{11} \pm i\alpha_{12}$ 和 $\alpha_{21} \pm i\alpha_{22}$, 故有

$$F = e^{\pm i(\alpha x + \alpha_{11}y) + \alpha_{12}y},\quad F = e^{\pm i(\alpha x + \alpha_{21}y) + \alpha_{22}y}.\tag{6}$$

上式也可表成三角函数和双曲函数

$$F = [A_1\sin(\alpha x + \alpha_{11}y) + B_1\cos(\alpha x + \alpha_{11}y)](C_1\mathrm{sh}(\alpha_{12}y) + D_1\mathrm{ch}(\alpha_{12}y)) + [A_2\sin(\alpha x + \alpha_{21}y) + B_2\cos(\alpha x + \alpha_{21}y)](C_2\mathrm{sh}(\alpha_{22}y) + D_2\mathrm{ch}(\alpha_{22}y)).$$

如令 $F = e^{\pm i(\beta y + \beta' x)}$ 代入(5)式还可求得另一类相似的解, 此外还有代数多项式解[6]

$$F = t_{11}\zeta\eta + t_{20}\zeta^2 + t_{02}\eta^2 + t_{21}\zeta^2\eta + t_{12}\zeta\eta^2 + t_{30}\zeta^3 + t_{03}\eta^3 + t_{31}(\zeta^3\eta + t_{26}\zeta^2\eta^2) + t_{13}(\zeta\eta^3 + t_{16}\zeta^2\eta^2),$$

式中 $\zeta = x/a$, $\eta = y/b$, $t_{26} = 3a_{26}b/(a(2a_{12}+a_{66}))$, $t_{16} = 3a_{16}a/(b(2a_{12}+a_{66}))$.

各向异性矩形板具有中心对称性[7]. 板的边界条件有时也具有中心对称性, 为便于求解可设一般解为[1~3],[6~8]

$$\begin{aligned} F = {}& \sum_l\Big\{\sum_m A_{lm}\sin[\alpha(a-x) + \alpha_{l1}(b-y)]\mathrm{sh}(\alpha_{l2}y) + \\ & B_{lm}\sin(\alpha x + \alpha_{l1}y)\mathrm{sh}(\alpha_{l2}(b-y))\Big\}\Big/\mathrm{sh}(\alpha_{l2}b) + \\ & \sum_l\Big\{\sum_n C_{ln}\sin[\beta(b-y) + \beta_{l1}(a-x)]\mathrm{sh}(\beta_{l2}x) + \\ & D_{ln}\sin(\beta y + \beta_{l1}x)\mathrm{sh}(\beta_{l2}(a-x))\Big\}\Big/\mathrm{sh}(\beta_{l2}a) + \\ & [t_1\eta + t_4(1-\eta)](\zeta - \zeta^3) + [t_2(1-\eta) + t_3\eta](2\zeta - 3\zeta^2 + \zeta^3) + \\ & [t_5\zeta + t_8(1-\zeta)](\eta - \eta^3) + [t_6(1-\zeta) + t_7\zeta](2\eta - 3\eta^2 + \eta^3) + \\ & t_9\zeta\eta - t_{12}(\zeta - \zeta^2)(\eta - \eta^2), \end{aligned}\tag{7}$$

式中

$$t_{12} = (t_1 + t_2 - t_3 - t_4)t_{26} + (t_5 + t_6 - t_7 - t_8)t_{16};$$

$$\alpha = m\pi/a,\quad m = 1, 2, 3, \cdots;\ \beta = n\pi/b,\quad n = 1, 2, 3, \cdots;\ l = 1, 2.$$

将(7)式代入(4)式可得

$$\begin{aligned}
N_x = &\sum_l\sum_m A_{lm}\Big\{(\alpha_{l2}^2-\alpha_{l1}^2)\sin[\alpha(a-x)+\alpha_{l2}(b-y)]\,\mathrm{sh}(\alpha_{l2}y)-\\
&2\alpha_{l1}\alpha_{l2}\cos[\alpha(a-x)+\alpha_{l1}(b-y)]\,\mathrm{ch}(\alpha_{l2}y)\Big\}/\mathrm{sh}(\alpha_{l2}b)+\\
&\sum_l\sum_m B_{lm}\Big\{(\alpha_{l2}^2-\alpha_{l1}^2)\sin(\alpha x+\alpha_{l1}y)\,\mathrm{sh}(\alpha_{l2}(b-y))-\\
&2\alpha_{l1}\alpha_{l2}\cos(\alpha x+\alpha_{l1}y)\,\mathrm{ch}(\alpha_{l2}(b-y))\Big\}/\mathrm{sh}(\alpha_{l2}b)-\\
&\sum_l\sum_n\Big\{C_{ln}\beta^2\sin[\beta(b-y)+\beta_{l1}(a-x)]\,\mathrm{sh}(\beta_{l2}x)+\\
&D_{ln}\beta^2\sin(\beta y+\beta_{l1}x)\,\mathrm{sh}(\beta_{l2}(a-x))\Big\}/\mathrm{sh}(\beta_{l2}a)-\\
&6[t_5\zeta\eta+t_6(1-\zeta)(1-\eta)+t_7\zeta(1-\eta)+t_8\eta(1-\zeta)]/b^2+\\
&2t_{12}(\zeta-\zeta^2)/b^2,
\end{aligned}\tag{8a}$$

$$\begin{aligned}
N_y = -&\sum_l\sum_m A_{lm}\Big\{\alpha^2\sin[\alpha(a-x)+\alpha_{l1}(b-y)]\,\mathrm{sh}(\alpha_{l2}y)+\\
&B_{lm}\alpha^2\sin(\alpha x+\alpha_{l1}y)\,\mathrm{sh}(\alpha_{l2}(b-y))\Big\}/\mathrm{sh}(\alpha_{l2}b)+\\
&\sum_l\sum_n C_{ln}\Big\{(\beta_{l2}^2-\beta_{l1}^2)\sin[\beta(b-y)+\beta_{l1}(a-x)]\,\mathrm{sh}(\beta_{l2}x)-\\
&2\beta_{l1}\beta_{l2}\cos[\beta(b-y)+\beta_{l1}(a-x)]\,\mathrm{ch}(\beta_{l2}x)\Big\}/\mathrm{sh}(\beta_{l2}a)+\\
&\sum_l\sum_n D_{ln}\Big\{(\beta_{l2}^2-\beta_{l1}^2)\sin(\beta y+\beta_{l1}x)\,\mathrm{sh}(\beta_{l2}(a-x))-\\
&2\beta_{l1}\beta_{l2}\cos(\beta y+\beta_{l1}x)\,\mathrm{ch}(\beta_{l2}(a-x))\Big\}/\mathrm{sh}(\beta_{l2}a)-\\
&6[t_1\zeta\eta+t_2(1-\zeta)(1-\eta)+t_3(1-\zeta)\eta+t_4\zeta(1-\eta)]/a^2+\\
&2t_{12}(\eta-\eta^2)/a^2,
\end{aligned}\tag{8b}$$

$$\begin{aligned}
N_{xy} = &\sum_l\sum_m A_{lm}\Big\{\alpha\alpha_{l1}\sin[\alpha(a-x)+\alpha_{l1}(b-y)]\,\mathrm{sh}(\alpha_{l2}y)+\\
&\alpha\alpha_{l2}\cos[\alpha(a-x)+\alpha_{l1}(b-y)]\,\mathrm{ch}(\alpha_{l2}y)\Big\}/\mathrm{sh}(\alpha_{l2}b)+\\
&\sum_l\sum_m B_{lm}\Big\{\alpha\alpha_{l1}\sin(\alpha x+\alpha_{l1}y)\,\mathrm{sh}(\alpha_{l2}(b-y))+\\
&\alpha\alpha_{l2}\cos(\alpha x+\alpha_{l1}y)\,\mathrm{ch}(\alpha_{l2}(b-y))\Big\}/\mathrm{sh}(\alpha_{l2}b)+\\
&\sum_l\sum_n C_{ln}\Big\{\beta\beta_{l1}\sin[\beta(b-y)+\beta_{l1}(a-x)]\,\mathrm{sh}(\beta_{l2}x)+\\
&\beta\beta_{l2}\cos[\beta(b-y)+\beta_{l1}(a-x)]\,\mathrm{ch}(\beta_{l2}x)\Big\}/\mathrm{sh}(\beta_{l2}a)+\\
&\sum_l\sum_n D_{ln}\Big\{\beta\beta_{l1}\sin(\beta y+\beta_{l1}x)\,\mathrm{sh}(\beta_{l2}(a-x))+\\
&\beta\beta_{l2}\cos(\beta y+\beta_{l1}x)\,\mathrm{ch}(\beta_{l2}(a-x))\Big\}/\mathrm{sh}(\beta_{l2}a)+\\
&[(t_4-t_1)(1-3\zeta^2)+(t_2-t_3)(2-6\zeta+3\zeta^2)+(t_8-t_5)(1-3\eta^2)+\\
&(t_6+t_7)(2-6\eta+3\eta^2)-t_9+t_{12}(1-2\zeta)(1-2\eta)]/ab.
\end{aligned}\tag{8c}$$

将(2)式的第1、2式和(8)式代入(3)式的第1、2式积分后可得

$$\begin{aligned}
u = &\sum_l\sum_m A_{lm}\Big\{(2a_{11}\alpha_{l1}/\alpha-a_{16})\alpha_{l2}\sin[\alpha(a-x)+\alpha_{l1}(b-y)]\,\mathrm{ch}(\alpha_{l2}y)+\\
&[a_{11}(\alpha_{l2}^2-\alpha_{l1}^2)/\alpha-a_{12}\alpha+a_{16}\alpha_{l1}]\cos[\alpha(a-x)+\\
&\alpha_{l1}(b-y)]\,\mathrm{sh}(\alpha_{l2}y)\Big\}/\mathrm{sh}(\alpha_{l2}b)-\sum_l\sum_m B_{lm}\Big\{(2a_{11}\alpha_{l1}/\alpha-a_{16})\alpha_{l2}\sin(\alpha x+\\
&\alpha_{l1}y)\,\mathrm{ch}(\alpha_{l2}(b-y))+[a_{11}(\alpha_{l2}^2-\alpha_{l1}^2)/\alpha-a_{12}\alpha+a_{16}\alpha_{l1}]\cos(\alpha x+
\end{aligned}$$

$$
\begin{aligned}
&\alpha_{l1}y)\,\mathrm{sh}(\alpha_{l2}(b-y))\Big\}\Big/\mathrm{sh}(\alpha_{l2}b)-\sum_l\sum_n C_{ln}\Big\{[a_{11}\beta^2/(\beta_{l1}^2+\beta_{l2}^2)-\\
&a_{12}]\beta_{l2}\sin[\beta(b-y)+\beta_{l1}(a-x)]\,\mathrm{ch}(\beta_{l2}x)+[a_{11}\beta^2\beta_{l1}/(\beta_{l1}^2+\beta_{l2}^2)+\\
&a_{12}\beta_{l1}-a_{16}\beta]\cos[\beta(b-y)+\beta_{l1}(a-x)]\,\mathrm{sh}(\beta_{l2}x)\Big\}\Big/\mathrm{sh}(\beta_{l2}a)+\\
&\sum_l\sum_n D_{ln}\Big\{[a_{11}\beta^2/(\beta_{l1}^2+\beta_{l2}^2)-a_{12}]\beta_{l2}\sin(\beta y+\beta_{l1}x)\,\mathrm{ch}(\beta_{l2}(a-x))+\\
&[a_{11}\beta^2\beta_{l1}/(\beta_{l1}^2+\beta_{l2}^2)+a_{12}\beta_{l1}-a_{16}\beta]\cos(\beta y+\beta_{l1}x)\,\mathrm{sh}(\beta_{l2}(a-x))\Big\}\Big/\mathrm{sh}(\beta_{l2}a)-\\
&3[t_5\zeta^2\eta+t_6(2\zeta-\zeta^2)(1-\eta)+t_7\zeta^2(1-\eta)+t_8(2\zeta-\zeta^2)\eta]a_{11}a/b^2-\\
&3[t_1\zeta^2\eta+t_2(2\zeta-\zeta^2)(1-\eta)+t_3(2\zeta-\zeta^2)\eta+t_4\zeta^2(1-\eta)]a_{12}/a+\\
&[(t_4-t_1)(\zeta-\zeta^3)+(t_2-t_3)(2\zeta-3\zeta^2+\zeta^3)+(t_8-t_5)\zeta(1-3\eta^2)+\\
&(t_6-t_7)\zeta(2-6\eta+3\eta^2)-t_9\zeta]a_{16}/b+t_{12}[(\zeta^2-2\zeta^3/3)a_{11}a/b^2+\\
&2\zeta(\eta-\eta^2)a_{12}/a+(\zeta-\zeta^2)(1-2\eta)a_{16}/b]+f_1(y),
\end{aligned}
\tag{9a}
$$

$$
\begin{aligned}
v=-&\sum_l\sum_m A_{lm}\Big\{[a_{22}\alpha^2/(\alpha_{l1}^2+\alpha_{l2}^2)-a_{12}]\alpha_{l2}\sin[\alpha(a-x)+\alpha_{l1}(b-y)]\,\mathrm{ch}(\alpha_{l2}y)+\\
&[a_{22}\alpha^2\alpha_{l1}/(\alpha_{l1}^2+\alpha_{l2}^2)+a_{12}\alpha_{l1}-a_{26}\alpha]\cos[\alpha(a-x)+\\
&\alpha_{l1}(b-y)]\,\mathrm{sh}(\alpha_{l2}y)\Big\}\Big/\mathrm{sh}(\alpha_{l2}b)+\sum_l\sum_m B_{lm}\Big\{[a_{22}\alpha^2/(\alpha_{l1}^2+\alpha_{l2}^2)-a_{12}]\alpha_{l2}\sin(\alpha x+\\
&\alpha_{l1}y)\,\mathrm{ch}(\alpha_{l2}(b-y))+[a_{22}\alpha^2\alpha_{l1}/(\alpha_{l1}^2+\alpha_{l2}^2)+a_{12}\alpha_{l1}-a_{26}\alpha]\cos(\alpha x+\\
&\alpha_{l1}y)\,\mathrm{sh}(\alpha_{l2}(b-y))\Big\}\Big/\mathrm{sh}(\alpha_{l2}b)+\sum_l\sum_n C_{ln}\Big\{(2a_{22}\beta_{l1}/\beta-\\
&a_{26})\beta_{l2}\sin[\beta(b-y)+\beta_{l1}(a-x)]\,\mathrm{ch}(\beta_{l2}x)+[a_{22}(\beta_{l2}^2-\beta_{l1}^2)/\beta-\\
&a_{12}\beta+a_{26}\beta_{l2}]\cos[\beta(b-y)+\beta_{l1}(a-x)]\,\mathrm{sh}(\beta_{l2}x)\Big\}\Big/\mathrm{sh}(\beta_{l2}a)-\\
&\sum_l\sum_n D_{ln}\Big\{(2a_{22}\beta_{l2}/\beta-a_{26})\beta_{l2}\sin(\beta y+\beta_{l1}x)\,\mathrm{ch}(\beta_{l2}(a-x))+\\
&[a_{22}(\beta_{l2}^2-\beta_{l1}^2)/\beta-a_{12}\beta+a_{26}\beta_{l2}]\cos(\beta y+\beta_{l1}x)\,\mathrm{sh}(\beta_{l2}(a-x))\Big\}\Big/\mathrm{sh}(\beta_{l2}a)-\\
&3[t_5\zeta\eta^2+t_6(1-\zeta)(2\eta-\eta^2)+t_7\zeta(2\eta-\eta^2)+t_8(1-\zeta)\eta^2]a_{12}/b-\\
&3[t_1\zeta\eta^2+t_2(1-\zeta)(2\eta-\eta^2)+t_3\eta^2(1-\zeta)+t_4\zeta(2\eta-\eta^2)]a_{22}b/a^2+\\
&[(t_4-t_1)(1-3\zeta^2)\eta+(t_2-t_3)(2-6\zeta+3\zeta^2)\eta+(t_8-t_5)(\eta-\eta^3)+\\
&(t_6-t_7)(2\eta-3\eta^2+\eta^3)-t_9\eta]a_{26}/a+t_{12}[2(\zeta-\zeta^2)\eta a_{12}/b+\\
&(\eta^2-2\eta^3/3)a_{22}b/a^2+(1-2\zeta)(\eta-\eta^2)a_{26}/a]+f_2(x),
\end{aligned}
\tag{9b}
$$

式中$f_1(y)$为y的函数,$f_2(x)$为x的函数• 将(9)式代入(2)式的第3式然后和(8)式一齐代入(3)式的第3式,注意将(8)式和(9)式中的三角函数和双曲函数项仍用(6)式来表示(全部消除),最后将x和y的两个代数多项式分别排列在等式的两边,则应等于一常数,设该常数为θ可得

$$
\begin{aligned}
f_1(y)=\ &u_0-[6t_6a_{16}/b+(t_1-2t_2+2t_3-t_4+t_9/2-t_{12}/2)a_{66}/a+\theta b]\eta+\\
&[(t_1-t_3)\eta^3+(t_4-t_2)(3\eta^2-\eta^3)]a_{22}b^2/a^3+\\
&[(t_5+t_8)\eta^3+(t_7-t_6)(3\eta^2-\eta^3)-t_{12}\eta^2](a_{12}+a_{66})/a+\\
&3(t_6-t_8)\eta^2a_{16}/b+[6(t_2-t_3)\eta^2+2t_{12}(\eta^2-2\eta^3/3)]a_{26}b/a^2,
\end{aligned}
\tag{10a}
$$

$$
\begin{aligned}
f_2(x)=\ &v_0-[6t_2a_{26}/a+(t_5-2t_6+2t_7-t_8+t_9/2-t_{12}/2)a_{66}/b-\theta a]\zeta+\\
&[(t_5-t_7)\zeta^3+(t_8-t_6)(3\zeta^2-\zeta^3)]a_{11}a^2/b^3+\\
&[(t_1-t_4)\zeta^3+(t_3-t_2)(3\zeta^2-\zeta^3)-t_2\zeta^2](a_{12}+a_{66})/b+\\
&\quad 3(t_2-t_4)\zeta^2a_{26}/a+[6(t_6-t_7)\zeta^2+2t_{12}(\zeta^2-2\zeta^3/3)]a_{16}a/b^2•
\end{aligned}
\tag{10b}
$$

以上各式共有 $2\cdot 2(m+n)+12$ 个积分常数• 一般来说, 每个边有 2 个边界条件: 即边界法向载荷或位移、切向载荷或位移应分别等于边界的已知值• 将法向边界条件方程式中的非正弦函数展成正弦级数, 切向边界条件的非余弦函数展成余弦级数, 根据正交性可得 $2(2m+2n)+4$ 个方程式• 每个角有两个法向载荷或位移, 均应等于角点的已知值• 对于全部边界为载荷的问题, 4 个余弦级数的常数项方程是相同的, 则还应确定一个刚性转角以及 x 和 y 方向的刚性位移, 共得 12 个方程式来求解全部积分常数, 有关非三角函数展成三角级数的公式可参看文献[9]•

2 例题和分析

2.1 均匀载荷的例

设板的 4 边承受均匀法向载荷和切向载荷, 板的中点固定, 这个问题可简单地由(8)式的代数多项式部分来求解, 此时角点条件和中心固定条件为

$$N_{x(0,0)}=N_{x(0,b)}=N_{x(a,0)}=N_{x(a,b)}=N_x, \tag{11a}$$

$$N_{y(0,0)}=N_{y(0,b)}=N_{y(a,0)}=N_{y(a,b)}=N_y, \tag{11b}$$

$$N_{xy(0,0)}=N_{xy},\quad \theta=0,\quad u_{(a/2,b/2)}=v_{(a/2,b/2)}=0\bullet \tag{11c}$$

将(8)式代入(11a, b)式和(11c)的第 1 式, 并令 $A_{lm}=B_{lm}=C_{ln}=D_{ln}=0$, 可得

$$t_1=t_2=t_3=t_4=-N_y a^2/6,\quad t_5=t_6=t_7=t_8=-N_x b^2/6,\quad t_9=N_{xy}ab,$$

上式满足所有边界条件, 且板内各点的应力亦为均匀分布• 将上式代入(9)式和(10)式, 并应用到(11c)的后 3 式可得板内各点的位移为

$$u=(a_{11}N_x+a_{12}N_y+a_{16}N_{xy})(x-a/2)+(a_{16}N_x+a_{26}N_y+a_{66}N_{xy})(y-b/2)/2,$$

$$v=(a_{12}N_x+a_{22}N_y+a_{26}N_{xy})(x-a/2)+(a_{16}N_x+a_{26}N_y+a_{66}N_{xy})(x-a/2)/2\bullet$$

设板为正方形 $a=b$, 且 $a_{11}=a_{22}$, $a_{16}=a_{26}$, $N_x=N_y$• 由上式容易看出, 此正方形均匀变大且为菱形, 包括 $N_{xy}=0$ 即仅为均匀拉力作用时• 这和各向异性方板弯曲的等高线图形是相似的[10]• 当 a_{16} 为正时, 对角线 $x=y$ 增长而对角线 $x+y=a$ 减短• 当 a_{16} 为负时, 则相反• 反之, 如欲均匀变大则应有 $N_{xy}=-2N_x a_{16}/a_{66}$•

2.2 非均匀载荷的例

对称角铺设 45°/-45°/45° 玻璃/环氧复合材料迭层正方形板, 弹性模数 $E_1/E_2=3$, $G_{12}/E_2=0.5$, Poisson 比 $\nu_{12}=0.25$• 容易求得[11]

$$a_{11}=a_{22}=0.7234/E_2h,\quad a_{12}=-0.2756/E_2h,\quad a_{16}=a_{26}=-0.0852/E_2h,$$

$$a_{66}=1.1489/E_2h,\quad \lambda=-2a_{16}/a_{11},\quad \mu=(2a_{12}+a_{66})/a_{11},$$

h 为板厚, 且

$$\alpha_{l1}=\left(-\lambda\pm\sqrt{\lambda^2-4\mu+8}\right)\alpha/4,$$

$$\alpha_{l2}=\sqrt{4\mu+8-2\lambda^2\pm2\lambda\sqrt{\lambda^2-4\mu+8}}\,\alpha/4,$$

式中 $l=1$ 时取正号, $l=2$ 时取负号• 设 4 边的边界条件为

$$(N_x)_{x=0}=(N_x)_{x=a}=N\eta(1-\eta);\ (N_y)_{y=0}=(N_y)_{y=b}=N\zeta(1-\zeta), \tag{12a}$$

$$(N_{xy})_{x=0}=(N_{xy})_{x=a}=(N_{xy})_{y=0}=(N_{xy})_{y=b}=0\bullet \tag{12b}$$

将(8)式代入以上各式• 由于边界是中心对称的, 故应有 $A_{lm}=B_{lm}$, $C_{ln}=D_{ln}$, $t_1=t_2$, $t_3=t_4$, $t_5=t_6$, $t_7=t_8$, 又由于 $a=b$, $a_{11}=a_{22}$, $a_{16}=a_{26}$, 故变形与对角线 $x=y$ 亦为对称, 因而又有 $C_{ln}=A_{ln}$, $t_2=t_6$, $t_4=t_8$• 首先由(12a)和(12b)的第 1 式可以求得

$$\sum_l\sum_m A_{lm}\frac{4}{b}\frac{\beta^3\alpha_{l1}\alpha_{l2}(\cos m\pi-\cos n\pi)}{(\beta^2-\alpha_{l1}^2+\alpha_{l2}^2)^2+4\alpha_{l1}^2\alpha_{l2}^2}\left(\cos n\pi\,\mathrm{cth}\,\alpha_{l2}b-\frac{\cos\alpha_{l1}b}{\mathrm{sh}\,\alpha_{l2}b}\right)-$$

$$\sum_l A_{ln}\beta^2-t_6\frac{12}{n\pi b^2}+t_8\frac{12\cos n\pi}{n\pi b^2}=N\frac{4(1-\cos n\pi)}{(n\pi)^3},$$

$$\sum_l\sum_m A_{lm}\frac{2\alpha}{b}\frac{\cos m\pi+\cos n\pi}{(\beta^2-\alpha_{l1}^2+\alpha_{l2}^2)+4\alpha_{l1}^2\alpha_{l2}^2}\Big\{[\alpha_{l1}^2(\alpha_{l1}^2+\alpha_{l2}^2-\beta^2)+$$

$$\alpha_{l2}^2(\alpha_{l1}^2+\alpha_{l2}^2+\beta^2)]\cos n\pi-2\alpha_{l1}\alpha_{l2}(\alpha_{l1}^2+\alpha_{l2}^2)\frac{\sin(\alpha_{l1}b)}{\mathrm{sh}(\alpha_{l2}b)}\Big\}+$$

$$\sum_l A_{ln}\beta\beta_{l2}\left(\cos n\pi\frac{\cos(\beta_{l1}a)}{\mathrm{sh}(\beta_{l2}a)}+\mathrm{cth}(\beta_{l2}a)\right)+$$

$$\sum_l\sum_k A_{lk}\gamma\left(\gamma_{l1}+\gamma_{l2}\cos k\pi\frac{\sin(\gamma_{l1}a)}{\mathrm{sh}(\gamma_{l2}b)}\right)K_n+$$

$$(t_6-t_8)\left[1+\cos n\pi-\frac{2\lambda}{\mu}(1-\cos n\pi)\right]\frac{12}{(n\pi)^2}=0,$$

$$\sum_l\sum_m A_{lm}\frac{\alpha}{b}(1+\cos m\pi)\left(1-\frac{2\alpha_{l1}\alpha_{l2}}{\alpha_{l1}^2+\alpha_{l2}^2}\frac{\sin\alpha_{l1}b}{\mathrm{sh}\,\alpha_2 b}\right)+$$

$$\sum_l\sum_k A_{lk}\gamma\left(\gamma_{l1}+\gamma_{l2}\cos k\pi\frac{\sin\gamma_{l1}a}{\mathrm{sh}\,\gamma_{l2}a}\right)\frac{1-\cos k\pi}{k\pi}+t_6-t_8-t_9=0,$$

式中 $K_n=4k/(\pi k^2-\pi n^2)$ 当 $n\pm k$ 为奇数, 否则 $K_n=0$, k 和 γ、γ_{l1}、γ_{l2} 分别与 n 和 β、β_{l1}、β_{l2} 相似• 由角点 $N_{x(0,0)}=0$ 和 $N_{x(0,b)}=0$ 二个条件可得

$$\sum_l\sum_m A_{lm}\alpha_1\alpha_{l2}(\cos m\pi\cos(\alpha_1 b)/\mathrm{sh}(\alpha_{l2}b)+\mathrm{cth}(\alpha_2 b))+3t_6/b^2=0,$$

$$\sum_l\sum_m A_{lm}\alpha_1\alpha_{l2}(\cos m\pi\,\mathrm{cth}(\alpha_{l2}b)/\cos(\alpha_{l1}b)/\mathrm{sh}(\alpha_{l2}b))+3t_8/b^2=0•$$

此时其他边界条件均自动满足, 由以上 5 式即可求得 A_{lm}、t_6、t_8 和 t_9, 板内各点的正应力和剪应力见表 1 和表 2, N_y 的值和 N_x 的值是对称的, 故未列入• 可以看出两种内力对 $x=a/2$ 和 $y=b/2$ 都是不对称的, 这和上例是不相同的• 因为上例载荷是均匀的, 仅用到代数多项式解, 内力的计算未涉及到材料特性• 但对 $x=y$ 和 $x+y=a$ 两个对角线则是对称的• 这是因为本例题的材料是以对角线为主轴的正交异性方板的缘故•

表 1　板内各点的正应力 (N_x/N)

η	ζ				
	0.1	0.3	0.5	0.7	0.9
0.1	0.1657	0.1860	0.2177	0.1711	0.1421
0.3	0.1856	0.1989	0.1926	0.1927	0.1635
0.5	0.2361	0.2017	0.1945	0.2017	0.2361
0.7	0.1635	0.1927	0.1926	0.1989	0.1856
0.9	0.1421	0.1711	0.2177	0.1860	0.1657

表 2　板内各点的正应力 (N_{xy}/N)

η	ζ				
	0.1	0.3	0.5	0.7	0.9
0.1	0.0263	0.0277	0.0026	-0.0305	-0.0248
0.3	0.0277	0.0235	-0.0011	-0.0200	-0.0305
0.5	0.0026	-0.0011	-0.0010	-0.0011	0.0026
0.7	-0.0305	-0.0200	-0.0011	0.0235	0.0277
0.9	-0.0248	-0.0305	0.0026	0.0277	0.0263

3　结　　论

根据各向异性矩形板平面应力问题的全部基本方程, 采用应力函数解法, 精确完整地求得了一般解析解• 可用以求解各种边界条件问题, 也可以求解组合板的问题, 但其中相连的板边则改用边界连续性条件来代替• 另外对一些简单的边界问题可仅由代数多项式解来求解•

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Analysis of Symmetric Laminated Rectangular Plates in Plane Stress

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Abstract: Symmetric laminated plates usually used are anisotropic plates. Based on fundamental equation for anisotropic rectangular plates in plane stress problem, a general analytical solution was established by method of stress function accurately. Therefore it gives the general formula of stress and displacement in plane. The integral constants in general formula can be determined by boundary conditions. This general solution composes the composite solution made by trigonometric function and hyperbolic function which can satisfy the problem of arbitrary boundary conditions along four edges. The algebraic polynomial solutions which can satisfy the problem of boundary conditions at four corners. Consequently this general solution can be used to solve the plane stress problem with arbitrary boundary conditions. For example, a symmetric laminated square plate acted with uniform normal load and tangential load and non_uniform normal load on four edges has been calculated and analyzed.

Key words: symmetric laminated plate; anisotropic; stress function methodology; strain; displacement