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压电热弹性体的变分原理及正则方程和齐次方程*

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(钟万勰推荐)

摘要: 结合对偶变量理论, 为压电热弹性体混合层合板问题推导了齐次的控制方程和 Hamilton 等参元列式. 首先根据广义的 Hamilton 变分原理推导了压电热弹性体非齐次的 Hamilton 正则方程. 然后进一步考虑了热平衡方程与导热方程中变量的对偶关系, 通过增加正则方程的维数, 成功地将非齐次的正则方程转化为能独立求解压电热弹性体耦合问题的齐次控制方程. 为了推导四节点 Hamilton 等参元列式的方便, 可将温度梯度关系类比成本构关系并构建新的变分原理. 齐次方程大大简化了人们在分析压电热弹性体耦合问题时, 通常要求解非齐次方程和关于平衡方程和导热方程的二阶微分方程的繁琐方法, 同时也减少了数值计算工作量.

关 键 词: 压电热弹性体; Hamilton 原理; Hamilton 正则方程; 对偶变量; 齐次方程; 等参元齐次列式

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引 言

由于压电材料固有的机械磁场和电场的耦合效应, 它们在空间结构、航空器、汽车、摩托车的振动控制等方面得到了广泛应用. 近年来压电材料的应用也大大地促进了理论的发展.

常见的关于压电材料板壳或混合层合板壳的解析法可分为三类, 一是采用扩展的复合材料层合板理论[1]分析压电材料混合层板[2-4]; 第二类是利用广义的 Eshelby_Stroh 公式[5]; 第三类与 Eshelby_Stroh 公式方法类似, 即状态空间法[6-11]. 当然还有其它一些很有代表性的分析法, 例如, 文献[12] 就圆柱壳的对称问题提出了 Volterra 积分方程和对应的数值方法.

就上述的解析法而言, 压电热材料层合板壳的控制方程大多是从基本方程出发进行繁琐的变形或消元推导得到的, 求解过程也很复杂. 主要原因是最后的控制方程是一个非齐次方程, 在求解时还需结合热平衡方程和导热方程, 所以结合边界条件求解由热平衡方程和导热方程导出的另一个二阶微分方程是常见的处理方法[8-11].

钟万勰教授在著作中[13]详细地阐述了弹性力学的 Hamilton 正则方程. 弹性力学的正则

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方程与现代控制论的方法是相似的,它的表达形式有许多优点. 钟万勰[13]证明了 Hamilton 正则方程在弹性力学中的普遍性. 钟万勰在著作[14]中进一步论述了对偶变量理论及其广泛的应用领域. 在此基础上,本文推导了压电热材料层合板齐次控制微分方程和相应原Hamilton等参元列式方法. 齐次方程可大大简化通常要求解非齐次方程和关于平衡方程和导热方程的二阶微分方程的方法.

1 基 本 公 式

压电热弹性体的基本方程可分为3组,即本构关系、梯度关系和平衡方程

$$\sigma = \boldsymbol{c}\varepsilon - \boldsymbol{e}^{\mathrm{T}}\boldsymbol{E} - \lambda T,\quad \boldsymbol{d} = \boldsymbol{e}\varepsilon - \kappa\boldsymbol{E} - \boldsymbol{r}T,\quad \boldsymbol{s} = \lambda^{\mathrm{T}}\varepsilon + \boldsymbol{r}^{\mathrm{T}}\boldsymbol{E} + c_0 T, \tag{1}$$

其中 σ 为应力向量(分量是 $\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{23}, \sigma_{13}, \sigma_{12}$),$\varepsilon$ 为应变向量(分量 $\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, 2\varepsilon_{23}, 2\varepsilon_{13}, 2\varepsilon_{12}$),$\lambda$ 为应力_温度系数向量(分量 λ_i, $i = 1, 2, 3$),$\boldsymbol{c}$ 为刚度系数矩阵(分量 $c_{ij} = c_{ji}$, $i, j = 1, 2, 3, 4, 5, 6$); $\boldsymbol{e}$ 为 3×6 的压电系数矩阵(分量 e_{ij}, $i = 1, 2, 3, j = 1, 2, 3, 4, 5, 6$),$\boldsymbol{E}$ 为电场强度向量(分量 E_i, $i = 1, 2, 3$),$\boldsymbol{d}$ 为电位移向量(分量 d_i, $i = 1, 2, 3$),κ 为介电系数矩阵(分量 $\kappa_{ij} = \kappa_{ji}$, $i, j = 1, 2, 3$),$\boldsymbol{r}$ 为热电系数向量(分量 r_i, $i = 1, 2, 3$),T 为温度增量,c_0 为比热容,s 为单位体积熵,

$$\varepsilon_{ij} = (u_{i,j} + u_{j,i})/2,\quad E_i = -\phi_{,i},\quad p_i = -k_{ii}T_{,i}, \tag{2}$$

式中,ϕ 是标量电势,p_i 是热通量分量,k_{ii} 是热传导系数

$$\sigma_{ij,j} = 0,\quad d_{i,i} = 0,\quad p_{i,i} = 0. \tag{3}$$

2 热电弹性体修正后的H_R 变分原理和控制微分方程

在直角坐标系下考虑稳态问题. 将本构关系(1)前两个方程变形(即交换行和列)后写成矩阵形式

$$\begin{Bmatrix} \boldsymbol{P}_1 \\ \boldsymbol{P}_2 \end{Bmatrix} = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{12}^{\mathrm{T}} & \Gamma_{22} \end{bmatrix} \begin{Bmatrix} \boldsymbol{D}_1 \\ \boldsymbol{D}_2 \end{Bmatrix} + \begin{Bmatrix} \gamma_1 T \\ \gamma_2 T \end{Bmatrix}, \tag{4}$$

式中

$$\boldsymbol{D}_2 = [u_{,x}\quad v_{,y}\quad u_{,y} + v_{,x}\quad \phi_{,x}\quad \phi_{,y}]^{\mathrm{T}},\quad \boldsymbol{D}_1 = [w_{,x} + u_{,z}\quad w_{,y} + v_{,z}\quad w_{,z}\quad \phi_{,z}]^{\mathrm{T}},$$

$$\boldsymbol{P}_1 = [\sigma_{xz}\quad \sigma_{yz}\quad \sigma_{zz}\quad d_z]^{\mathrm{T}},\quad \boldsymbol{P}_2 = [\sigma_{xx}\quad \sigma_{yy}\quad \sigma_{xy}\quad d_x\quad d_y]^{\mathrm{T}},$$

$$\gamma_1 = [-\lambda_{xz}\quad -\lambda_{yz}\quad -\lambda_{zz}\quad \gamma_z]^{\mathrm{T}},\quad \gamma_2 = [-\lambda_{xx}\quad -\lambda_{yy}\quad -\lambda_{xy}\quad \gamma_x\quad \gamma_y]^{\mathrm{T}}.$$

将(4)式中的 $\boldsymbol{D}_1$、$\boldsymbol{P}_2$ 作为未知量求出有

$$\boldsymbol{D}_1 = \Phi_{11}\boldsymbol{P}_1 + \Phi_{11}\boldsymbol{D}_2 + \alpha_1 T,\quad \boldsymbol{P}_2 = \Phi_{21}\boldsymbol{P}_1 + \Phi_{22}\boldsymbol{D}_2 + \alpha_2 T, \tag{5}$$

式中

$$\Phi_{11} = \Phi_{11}^{\mathrm{T}} = \Gamma_{11}^{-1},\quad \Phi_{12} = -\Phi_{21}^{\mathrm{T}} = -\Phi_{11}\Gamma_{12},\quad \Phi_{22} = \Phi_{22}^{\mathrm{T}} = \Gamma_{22} + \Gamma_{21}\Phi_{12},$$

$$\alpha_1 = -\Phi_{11}\gamma_1,\quad \alpha_2 = \gamma_2 - \Gamma_{21}\Phi_{11}\gamma_1.$$

根据文献[13_15],压电热弹性体的广义H_R 变分原理可表达为

$$\Pi = \iiint_V L_{\mathrm{R}}\,\mathrm{d}V - \iint_{S_\sigma} \boldsymbol{T}^{\mathrm{T}}\bar{\boldsymbol{Q}}\,\mathrm{d}S_\sigma - \int_{S_u} \boldsymbol{T}^{\mathrm{T}}(\boldsymbol{Q} - \bar{\boldsymbol{Q}})\,\mathrm{d}S_u, \tag{6}$$

其中 L_{R} 是广义的Reissner 能密度函数; $\boldsymbol{Q} = [u\quad v\quad w\quad \phi]^{\mathrm{T}}$, $\bar{\boldsymbol{Q}} = [\bar{u}\quad \bar{v}\quad \bar{w}\quad \bar{\phi}]^{\mathrm{T}}$, $\boldsymbol{T} = [T_{xx}\quad T_{yy}\quad T_{zz}\quad T_q]^{\mathrm{T}}$.

用(5)式中的 $\boldsymbol{D}_1$ 和 $\boldsymbol{P}_2$ 消去(6)式 L_{R} 项中的 $\boldsymbol{D}_1^{\mathrm{T}}$、$\boldsymbol{P}_2$ 和 $\boldsymbol{P}_2^{\mathrm{T}}$. 将应变_位移关系写成向量形式

$$\boldsymbol{D}_1 = \boldsymbol{Q}_{,z} + \boldsymbol{G}_1\boldsymbol{Q},\quad \boldsymbol{D}_2 = \boldsymbol{G}_2\boldsymbol{Q}, \tag{7}$$

其中

$$\boldsymbol{G}_1 = \begin{bmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\quad \boldsymbol{G}_2 = \begin{bmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ \beta & \alpha & 0 & 0 \\ 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & \beta \end{bmatrix}$$

是微分算子矩阵, $\alpha = \partial/\partial x$, $\beta = \partial/\partial y$. 所以有

$$L_R = \boldsymbol{P}_1^T\boldsymbol{Q}_{,z} + \boldsymbol{P}_1^T((\boldsymbol{G}_1\boldsymbol{Q}) + \Phi_{21}^T(\boldsymbol{G}_2\boldsymbol{Q})) + \frac{1}{2}(\boldsymbol{G}_2\boldsymbol{Q})^T\Phi_{22}(\boldsymbol{G}_2\boldsymbol{Q}) - \frac{1}{2}\boldsymbol{P}_1^T\Phi_{11}\boldsymbol{P}_1 + \alpha_2^T(\boldsymbol{G}_2\boldsymbol{Q})T - \alpha_1^T\boldsymbol{P}_1T, \tag{8}$$

式(8)可写成 $L_R = \boldsymbol{P}_1^T\boldsymbol{Q}_{,z} - H$, 则式(6)可写成广义的 Hamilton 变分原理[13]

$$\Pi = \iiint_V(\boldsymbol{P}_1^T\boldsymbol{Q}_{,z} - H)\,dV - \iint_{S_\sigma}\boldsymbol{T}^T\boldsymbol{Q}\,dS_\sigma - \int_{S_u}\boldsymbol{T}^T(\boldsymbol{Q} - \boldsymbol{Q})\,dS_u, \tag{9}$$

式中 H 是 Hamilton 函数.

假设广义的应力边界条件和位移边界条件满足 $\boldsymbol{T} = \boldsymbol{T}$ 和 $\boldsymbol{Q} = \boldsymbol{Q}$, 对式(9)进行变分并分部积分可得 Hamilton 正则方程

$$\frac{d}{dz}\begin{Bmatrix} \boldsymbol{P} \\ \boldsymbol{Q} \end{Bmatrix} = \begin{bmatrix} \boldsymbol{G}_1^T + \boldsymbol{G}_2^T\Phi_{21} & \boldsymbol{G}_2^T\Phi_{22}\boldsymbol{G}_2 \\ \Phi_{11} & -(\boldsymbol{G}_1 + \Phi_{21}^T\boldsymbol{G}_2) \end{bmatrix}\begin{Bmatrix} \boldsymbol{P} \\ \boldsymbol{Q} \end{Bmatrix} + \begin{Bmatrix} \boldsymbol{G}_2^T\alpha_2 \\ \alpha_1 \end{Bmatrix}T, \tag{10}$$

式中 $\boldsymbol{P} = \boldsymbol{P}_1$, $\boldsymbol{P}$ 和 $\boldsymbol{Q}$ 为对偶的列向量.

事实上, 很多文献[8_11]应用不同的方法导出了类似于方程(10)的非齐次方程, 通常用它来求解耦合的三维问题还要联立导热方程和热平衡方程

$$\begin{cases} p_x = -k_{11}T_{,x},\ p_y = -k_{22}T_{,y},\ p_z = -k_{33}T_{,z}, & (p_i = -k_{ii}T_{,i}), \\ p_{x,x} + p_{y,y} + p_{z,z} = 0 & (p_{i,i} = 0). \end{cases} \tag{11}$$

因此, 文献[8_11]以方程(11)为基础导出二阶微分方程

$$\partial^2T(x,y,z)/\partial z^2 - CT(x,y,z) = 0. \tag{12}$$

实际上, 从方程(11)可导出下面两个简明的关系式

$$\partial p_z/\partial z = (k_{11}\alpha^2 + k_{22}\beta^2)T,\quad \partial T/\partial z = -1/k_{33}p_z. \tag{13}$$

从理论上可以证明方程(13)中的热流量 $p_z(x,y,z)$ 和温度增量 $T(x,y,z)$ 是对偶变量[13_14].

简要的证明　将热流量分量 p_i 类比为应力分量, $T_{,i}$ 类比为应变分量, 则温度增量 T 是位移. 那么, 梯度方程 $p_i = -k_{ii}T_{,i}$ 就是本构关系, k_{ii} 是刚度系数. 根据文献[13_14]中关于对偶变量的描述, 可知方程(13b)中热流量 $p_z(x,y,z)$ 和温度增量 $T(x,y,z)$ 是对偶变量.

因此, 结合方程(13), 扩展方程(10)中的 $\boldsymbol{P}$ 和 $\boldsymbol{Q}$ 成统一的对偶列向量, 我们得到关于压电热弹性体的齐次对偶方程

$$d\boldsymbol{R}/dz = \boldsymbol{K}\boldsymbol{R}, \tag{14}$$

式中

$$\boldsymbol{R} = [\sigma_{xz}\quad \sigma_{yz}\quad \sigma_{zz}\quad d_z\quad p_z\quad u\quad v\quad w\quad \phi\quad T]^T,$$

$$
\boldsymbol{K} = \begin{bmatrix} \boldsymbol{G}_1^{\mathrm{T}} + \boldsymbol{G}_2^{\mathrm{T}}\Phi_{21} & \boldsymbol{0} & \boldsymbol{G}_2^{\mathrm{T}}\Phi_{22}\boldsymbol{G}_2 & \boldsymbol{G}_2^{\mathrm{T}}\alpha_2 \\ \boldsymbol{0}^{\mathrm{T}} & 0 & \boldsymbol{0}^{\mathrm{T}} & k_{11}\alpha^2 + k_{22}\beta^2 \\ \Phi_{11} & \boldsymbol{0} & -(\boldsymbol{G}_1 + \Phi_{21}^{\mathrm{T}}\boldsymbol{G}_2) & \alpha_1 \\ \boldsymbol{0}^{\mathrm{T}} & -1/k_{33} & \boldsymbol{0}^{\mathrm{T}} & 0 \end{bmatrix},
$$

$\boldsymbol{0} = [0 \quad 0 \quad 0 \quad 0]^{\mathrm{T}}$.

很明显,方程(14)包含了机械变形、温度场和电场完全耦合的关系.方程(14)是能独立求解压电热弹性体三维问题的控制方程.

这里的推导过程与增维方法[16]有相似之处,即通过增加方程的维数来将非齐次方程转化为齐次方程.但本文的理论依据是热通量 $p_z(x, y, z)$ 和温度增量 $T(x, y, z)$ 是对偶变量.

3 简支混合层板的精确解

对于一般正交异性压电热弹性体:$c_{ij} = c_{ji} = 0\ (i = 1, 2, 3; j = 4, 5, 6)$;$c_{4j} = c_{j4} = 0\ (j = 5, 6)$;$c_{56} = c_{65} = 0$,$e_{1j} = e_{16} = 0\ (j = 1, 2, 3, 4)$;$e_{2j} = e_{25} = e_{26} = 0\ (j = 1, 2, 3)$;$e_{3j} = 0\ (j = 4, 5, 6)$;$r_1 = r_2 = 0$;式(10)中的各项为:

$$
\Phi_{11} = \begin{bmatrix} s_1 & 0 & 0 & 0 \\ 0 & s_2 & 0 & 0 \\ 0 & 0 & s_3 & s_4 \\ 0 & 0 & s_4 & s_5 \end{bmatrix}, \quad \boldsymbol{G}_1^{\mathrm{T}} + \boldsymbol{G}_2^{\mathrm{T}}\Phi_{21} = \begin{bmatrix} 0 & 0 & s_8\alpha & s_{10}\alpha \\ 0 & 0 & s_9\beta & s_{11}\beta \\ -\alpha & -\beta & 0 & 0 \\ s_6\alpha & s_7\beta & 0 & 0 \end{bmatrix}, \quad \boldsymbol{G}_2^{\mathrm{T}}\alpha_2 = \begin{Bmatrix} -s_{18}\alpha \\ -s_{19}\beta \\ 0 \\ 0 \end{Bmatrix},
$$

$$
\boldsymbol{G}_2^{\mathrm{T}}\Phi_{22}\boldsymbol{G}_2 = \begin{bmatrix} -s_{12}\alpha^2 - s_{15}\beta^2 & -s_{13}\alpha\beta - s_{15}\beta\alpha & 0 & 0 \\ -s_{13}\alpha\beta - s_{15}\beta\alpha & -s_{14}\beta^2 - s_{15}\alpha^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -s_{16}\alpha^2 - s_{17}\beta^2 \end{bmatrix}, \quad \alpha_1 = \begin{Bmatrix} s_{20} \\ s_{21} \\ 0 \\ 0 \end{Bmatrix},
$$

式中 $s_i\ (i = 1, 2, 3, \cdots, 21)$ 是与材料参数相关的常数.

考虑四边简支矩形混合层合板[10]

$$
\begin{cases} \sigma_{xx} = w = v = \phi = T = 0 \quad (x = 0, a), \\ \sigma_{yy} = w = u = \phi = T = 0 \quad (y = 0, b). \end{cases} \tag{15}
$$

设层合板任意一层满足边界条件(15)式的级数解形式如下:

$$
\begin{cases} (\sigma_{xz}, u) = \sum_m^{\infty}\sum_n^{\infty}(\sigma_{xz}(z), u(z))\cos(\eta x)\sin(\zeta y), \\ (\sigma_{yz}, v) = \sum_m^{\infty}\sum_n^{\infty}(\sigma_{yz}(z), v(z))\sin(\eta x)\cos(\zeta y), \\ (\sigma_{zz}, w, p_z) = \sum_m^{\infty}\sum_n^{\infty}(\sigma_{zz}(z), w(z), p_z(z))\sin(\eta x)\sin(\zeta y), \\ (d_z, \phi, T) = \sum_m^{\infty}\sum_n^{\infty}(d_z(z), \phi(z), T(z))\sin(\eta x)\sin(\zeta y), \end{cases} \tag{16}
$$

其中 $\eta = m\pi/a$, $\xi = n\pi/b$.

将式(16)代入方程(14),则任意层的控制微分方程可表示为

$$
\mathrm{d}\boldsymbol{R}^{mn}/\mathrm{d}z = \boldsymbol{K}\boldsymbol{R}^{mn}, \tag{17}
$$

其中 $\boldsymbol{K}$ 是系数矩阵.

方程(17)的精确解为

$$\boldsymbol{R}^{mn}(z)=\mathrm{e}^{\boldsymbol{K}z}\boldsymbol{R}^{mn}(0),\qquad z\in[0,h].\tag{18}$$

对于 n 层的板来说,可令式(18)中的 $z=h_j$ 对应于第 j 层的厚度,则有

$$\boldsymbol{R}_j^{mn}(h_j)=\mathrm{e}^{\boldsymbol{K}(h_j)}\boldsymbol{R}_j^{mn}(h_{j-1}).\tag{19}$$

根据层间的连续条件,有下列的递推关系

$$\boldsymbol{R}_n^{mn}(h_j)=\boldsymbol{T}\boldsymbol{R}_1^{mn}(0),\tag{20}$$

其中 $\boldsymbol{T}=\mathrm{e}^{\boldsymbol{K}(h_n)}\mathrm{e}^{\boldsymbol{K}(h_{n-1})}\cdots\mathrm{e}^{\boldsymbol{K}(h_1)}$, $\boldsymbol{R}_1^{mn}(0)$ 是第一层外表面($z=0$)的广义位移和广义应力.

例题 1 考虑一混合层板[0/0/90/0/0/90/0p][10],层合板厚度 $h=1$ m, $a=b=50$ m.压电热弹性材料 PZT_5A 层的厚度, $h_{\mathrm{p}}=0.1h$,弹性材料层的厚度相同 $h_{\mathrm{s}}=0.9h/6$. PZT_5A 参数:$(Y_1, Y_2, Y_3, G_{12}, G_{23}, G_{31})=(61.0, 61.0, 53.2, 22.6, 21.1, 21.1)$ GPa;$(\nu_{12}, \nu_{23}, \nu_{13})=(0.35, 0.38, 0.38)$;$(\eta_1, \eta_2, \eta_3)=(1.53, 1.53, 1.50)\times10^{-8}$ F/m;$(d_1, d_2, d_3, d_4, d_5)=(-171, -171, 374, 584, 584)\times10^{-2}$ m/V;$(\alpha_1, \alpha_2, \alpha_3)=(1.5, 1.5, 2.0)\times10^{-6}\ \mathrm{K}^{-1}$;$k_1=k_2=k_3=1.8$ W•m^{-1}•K^{-1};$r_3=0.0007$ C•m^{-2}•K^{-1}. 弹性材料参数:$(Y_{\mathrm{L}}, Y_{\mathrm{T}}, G_{\mathrm{LT}}, G_{\mathrm{TT}})=(181, 10.3, 7.17, 2.87)$ GPa;$(\nu_{\mathrm{LT}}, \nu_{\mathrm{TT}})=(0.28, 0.33)$;$\eta_{\mathrm{L}}=\eta_{\mathrm{T}}=1.53\times10^{-8}$ F/m;$(\alpha_{\mathrm{L}}, \alpha_{\mathrm{T}})=(0.02, 2.25)\times10^{-6}\ \mathrm{K}^{-1}$;$(k_{\mathrm{L}}, k_{\mathrm{T}})=(1.5, 0.5)$ W•m^{-1}•K^{-1};$r_3=0$;$d_i=0\ (i=1, 2, \cdots, 5)$.

工况 1 $\sigma_{zz1}=0$, $\sigma_{zz2}=\sin(\eta x)\sin(\zeta y)$, $\phi_1=\phi_2=0$, $T_1=T_2=0$, $m=n=1$(下标 1 和 2 分别标记层合板上下表面).

工况 2 $\sigma_{zz1}=\sigma_{zz2}=0$, $\phi_1=\phi_2=0$, $T_1=0$, $T_2=\sin(\eta x)\sin(\zeta y)$, $m=n=1$.

表 1 结果与比较(无量纲参数: $K_1=(a/h)/10.3\times10^9$, $K_2=(a/h)\times22.5\times10^{-6}$)

	工况 1		工况 2	
	$w(a/2, b/2, 0)/K_1$	$u(0, b/2, 0)/K_1$	$10\times w(a/2, b/2, 0)/K_2$	$p_2(a/2, b/2, 0)/K_2$
本文	−641.342	−20.044	−3.309	481.211
文献[10]	−641.404	−20.742	−3.291	481.432

例题 2 考虑一混合层合板[90/0/90/0p],两种材料的参数与例 1 相同,板的总厚度从 0.02 m 到 0.2 m 变化.设每一层的厚度 h_{s} 相同, $a=b=1$ m,考虑工况:1. $\sigma_{zz1}=0$, $\sigma_{zz2}=\sin(\eta x)\sin(\zeta y)$, $\phi_1=\phi_2=0$, $T_1=T_2=0$;2. $\sigma_{zz1}=0$, $\sigma_{zz2}=\sin(\eta x)\sin(\zeta y)$, $\phi_1=0$, $\phi_2=\sin(\eta x)\sin(\zeta y)$, $T_1=T_2=0$;3. $\sigma_{zz1}=0$, $\sigma_{zz2}=\sin(\eta x)\sin(\zeta y)$, $\phi_1=0$, $\phi_2=\sin(\eta x)\sin(\zeta y)$, $p_{z1}=0$, $p_{z2}=\sin(\eta x)\sin(\zeta y)$.

从图 1 可以看出:若 3 种工况不变,层合板表面中央 z 方向的位移(图 1a)和温度(图 1b)随总厚度的增加而减少.工况 2(图 1a)说明了电势载荷对变形的抑制作用;工况 3(图 1a)说明热通量对变形的影响作用相当明显.

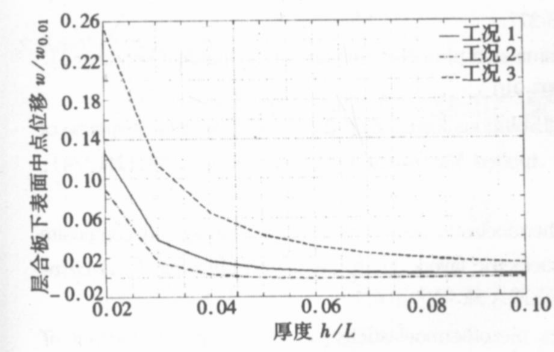
4 Hamilton 等参元的列式

把梯度关系 $p_i=-k_{ii}T_{,i}$ 看作本构关系,将其与本构关系(1)中前两式写在一起

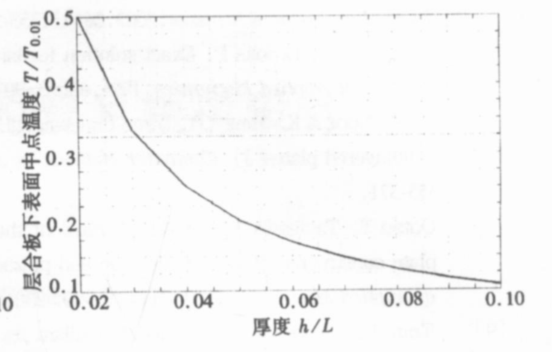
$$\boldsymbol{\sigma}=\boldsymbol{c}\boldsymbol{\varepsilon}-\boldsymbol{e}^{\mathrm{T}}\boldsymbol{E}-\boldsymbol{\lambda}T,\quad \boldsymbol{d}=\boldsymbol{e}\boldsymbol{\varepsilon}+\boldsymbol{\kappa}\boldsymbol{E}+\boldsymbol{r}T,\quad \boldsymbol{p}=-\boldsymbol{k}\boldsymbol{\tau},\tag{21}$$

式中 $\boldsymbol{\tau}=[T_{,1}\quad T_{,2}\quad T_{,3}]^{\mathrm{T}}$.

设 $\boldsymbol{P}=[\sigma_{xz}\quad\sigma_{yz}\quad\sigma_{zz}\quad d_z\quad p_z]^{\mathrm{T}}$, $\boldsymbol{Q}=[u\quad v\quad w\quad\phi\quad T]^{\mathrm{T}}$ 并将其代入方程(6).四结点



(a)



(b)

图 1 层合板下表面中点的位移和温度

Hamilton 等参元的形函数和场函数为

$$N_i(\xi,\eta) = (1+\xi_i\xi)(1+\eta_i\eta)/4, \qquad i = 1,2,3,4, \tag{22}$$

$$\begin{cases} u = [\boldsymbol{N}(x,y)]\{u(z)\}, \ v = [\boldsymbol{N}(x,y)]\{v(z)\}, \ w = [\boldsymbol{N}(x,y)]\{w(z)\}, \\ \phi = [\boldsymbol{N}(x,y)]\{\phi(z)\}, \ T = [\boldsymbol{N}(x,y)]\{T(z)\}, \\ \sigma_{xz} = [\boldsymbol{N}(x,y)]\{\sigma_{xz}(z)\}, \ \sigma_{yz} = [\boldsymbol{N}(x,y)]\{\sigma_{yz}(z)\}, \ \sigma_{zz} = [\boldsymbol{N}(x,y)]\{\sigma_{zz}(z)\}, \\ d_z = [\boldsymbol{N}(x,y)]\{d_z(z)\}, \ p_z = [\boldsymbol{N}(x,y)]\{p_z(z)\}. \end{cases} \tag{23}$$

将(23)式代入(6)式并执行变分可得到齐次的 Hamilton 等参元列式

$$\iint \begin{bmatrix} \boldsymbol{M}^e & 0 \\ 0 & \boldsymbol{M}^e \end{bmatrix} |\boldsymbol{J}| \, d\xi d\eta \frac{d}{dz} \begin{Bmatrix} \boldsymbol{P}^e(z) \\ \boldsymbol{Q}^e(z) \end{Bmatrix} = \iint \begin{bmatrix} \boldsymbol{K}_{11}^e & \boldsymbol{K}_{12}^e \\ \boldsymbol{K}_{21}^e & \boldsymbol{K}_{22}^e \end{bmatrix} |\boldsymbol{J}| \, d\xi d\eta \begin{Bmatrix} \boldsymbol{P}^e(0) \\ \boldsymbol{Q}^e(0) \end{Bmatrix}, \tag{24}$$

式中 $\boldsymbol{J}$ 是 Jacobi 矩阵, $\boldsymbol{K}_{ij}^e (i,j = 1,2)$ 是与材料参数相关的等效刚度矩阵, 其维数是 20×20.

5 结 论

本文建立了压电热弹性体广义的 Hamilton 变分原理和相应的非齐次 Hamilton 正则方程. 结合热平衡方程与温度场中对偶变量的关系, 通过增维方法将非齐次 Hamilton 正则方程转化为齐次控制方程. 本文的齐次控制方程大大简化了求解过程.

齐次的 Hamilton 等参元列式为复杂边界条件、不规则几何边界和复杂表面载荷的混合层合板的半解析法提供了理论基础.

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Variation Principle of Piezothermoelastic Bodies, Canonical Equation and Homogeneous Equation

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Abstract: Combining the symplectic variations theory, the homogeneous control equation and isoparametric element homogeneous formulations for piezothermoelastic hybrid laminates problems were deduced. Firstly, based on the generalized Hamilton variation principle, the non_homogeneous Hamilton canonical equation for piezothermoelastic bodies was derived. Then the symplectic relationship of variations in the thermal equilibrium formulations and gradient equations was considered. The non_homogeneous canonical equation was transformed to homogeneous control equation for solving independently the coupling problem of piezothermoelastic bodies by the incensement of dimensions of the canonical equation. For the convenience of deriving Hamilton isoparametric element formulations with four nodes, one can consider the temperature gradient equation as constitutive relation and reconstruct new variation principle. The homogeneous equation simplifies greatly the solution programs which are often performed to solve non_homogeneous equation and second order differential equation on the thermal equilibrium and gradient relationship.

Key words: piezothermoelasticity; Hamilton principle; Hamilton canonical equation; symplectic variables; homogeneous equation; homogeneous isoparametric element formulation