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大气运动基本方程组的解析解*

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(周哲玮推荐)

摘要: 在已知大气运动基本方程于光滑函数类中具有最好的稳定性前提下, 讨论了它的局部解的解空间构造. 根据它的解空间构造, 分析了这个方程具有代表性和应用性的第三初值问题, 在解析函数类中给出了适定的第三初值问题的解析解的计算方法以及具体的关系表达式, 在局部解意义下完整的解决了这一点初值问题的解析解所涉及的理论与计算问题. 指出其它类型定解问题都可以仿照文中的计算方法和步骤, 求出所需要的稳定的解析解.

关 键 词: 大气运动基本方程组; 解空间构造; 解析解
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1 问题的提出

在大气动力学的应用研究中, 通常先要确定所论问题的尺度和相应的理论模式[1-3]. 由于这些理论模式是一组偏微分方程, 因此在实施具体的计算之前, 应该对此模式在指定意义下的稳定性如何有个交代. 在考虑湍流摩擦与耗散时, 一般情况下这是一个二阶偏微分方程组, 因此最合理的“指定意义下的稳定性”应该是 $C^k(k \geqslant 2)$ 稳定性. 由于经尺度分析后的简化模式都基于大气运动基本方程组, 因此大气运动基本方程组的 $C^k(k \geqslant 2)$ 稳定性研究就应该是大气动力学理论与应用研究的重要问题之一[1]. “大气运动基本方程组的稳定性分析”[4] 一文中, 已经证明了当方程组中的所有参数函数是无穷可微时, 局地直角坐标系中的大气运动基本方程组是 C^∞ 稳定方程. 于是, 接下来要做的是如何对适定问题求解. 虽然现在我们尚无一般方法去求方程组适定问题的 C^k 解, 但是在问题允许的条件下, 最理想的解应该是解析解, 因为这种性质的解可以为各种形式的近似解提供最好的评价(比如精度、误差估计等)方法. 怎样才能求出这个稳定的解析解呢? 本文将就此问题给出回答. 分析过程中使用的是局地直角坐标系中大气运动基本方程组[1-3]:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + fv - fw + F_1,\quad \frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{1}{\rho}\frac{\partial P}{\partial y} - fu + F_2,$$

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$$\frac{\mathrm{d}w}{\mathrm{d}t} = -\frac{1}{\rho}\frac{\partial P}{\partial z} + fu - g + F_3, \quad \frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0,$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = F_\theta + S_5, \quad \frac{\mathrm{d}q_l}{\mathrm{d}t} = F_l + S_l, \qquad l = 6, 7, 8,$$

$$\frac{\mathrm{d}\chi_r}{\mathrm{d}t} = F_r + S_r, \qquad r = 8+1, \cdots, 8+m,$$

状态方程 $P = \rho RT$, 位温 θ 与 T 之间由下式联系:

$$\theta = T(P_{00}/P)^{R/c_p}.$$

方程中的各种符号含义均按大气动力学的约定, 此外, q_l 分别代表固态、液态、汽态水的比湿, χ_r 代表除水汽以外的各种气溶胶; F_1、F_2、F_3、F_θ 代表分子粘性力与湍流粘性力及耗散之和, F_l、F_r 分别代表由于湍流运动引起的湍流粘性耗散等附加项, S_5、S_l、S_r 代表汇、源项, 它们将按照约定取不同表达式.

设 $$(x, y, z, t) = (x_1, x_2, x_3, x_4) \in R^4 = V,$$

未知函数

$$(u, v, w, \rho, \theta, q_l, \chi_r) = (u_1, \cdots, u_8, \cdots, u_{8+m}) \in R^3 \times (R_+^*)^{5+m} = Z,$$

将上述方程组简记为 D.

按照半经验理论公式[2-3], 湍流粘性力 $\boldsymbol{F} = (F_1, F_2, F_3)$

$$F_1 = \frac{1}{\rho}\left[\frac{\partial}{\partial x_1}\left(\rho k_L \frac{\partial u_1}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\rho k_L \frac{\partial u_1}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(\rho k_H \frac{\partial u_1}{\partial x_3}\right)\right],$$

$$F_2 = \frac{1}{\rho}\left[\frac{\partial}{\partial x_1}\left(\rho k_L \frac{\partial u_2}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\rho k_L \frac{\partial u_2}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(\rho k_H \frac{\partial u_2}{\partial x_3}\right)\right],$$

$$F_3 = \frac{1}{\rho}\left[\frac{\partial}{\partial x_1}\left(\rho k_L \frac{\partial u_3}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\rho k_L \frac{\partial u_3}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(\rho k_H \frac{\partial u_3}{\partial x_3}\right)\right],$$

湍流粘性耗散

$$F_\theta = F_5 = \frac{1}{\rho}\left[\frac{\partial}{\partial x_1}\left(\rho k_L \frac{\partial u_5}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\rho k_L \frac{\partial u_5}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(\rho k_H \frac{\partial u_5}{\partial x_3}\right)\right],$$

$$F_l = \frac{1}{\rho}\left[\frac{\partial}{\partial x_1}\left(\rho k_{1,l} \frac{\partial u_l}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\rho k_{1,l} \frac{\partial u_l}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(\rho k_{2,l} \frac{\partial u_l}{\partial x_3}\right)\right], \qquad l = 6, 7, 8,$$

$$F_r = \frac{1}{\rho}\left[\frac{\partial}{\partial x_1}\left(\rho k_{1,r} \frac{\partial u_r}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\rho k_{1,r} \frac{\partial u_r}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(\rho k_{2,r} \frac{\partial u_r}{\partial x_3}\right)\right],$$

$$r = 8+1, \cdots, 8+m,$$

各粘性系数严格大于 0, 在本文中且均假设为 $(x, y, z, t) = (x_1, x_2, x_3, x_4)$ 和未知函数的已知并充分光滑的函数. 作为理论分析的需要, 此处还假设了

$$S_n = g_n(x_j, u_i, \partial u_i) \in C^\infty,$$

$$j = 1, 2, 3, 4; \; i = 1, 2, \cdots, 8+m; \; n = 5, \cdots, 8+m.$$

由于在实际应用中常忽略分子粘性力的作用, 因此本文在分析与计算解析解时也不计分子粘性力. 但如果将分子粘性力考虑在内并且各种湍流参数还依赖于某些指定的未知函数的一阶偏导数[2-3], 所有定理的结论均不改变, 因而本文涉及的定理的分析以及所有的计算方法与过程也将适用.

D 在 Ehresmann 空间 $J^2(V, Z)$ 的局部坐标表示下为

$$f_1: \; k_L(p^1_{11} + p^1_{22}) + k_H p^1_{33} + \Phi_1 = 0,$$

$$f_2:\ k_L(p_{11}^2+p_{22}^2)+k_H p_{33}^2+\Phi_2=0,$$
$$f_3:\ k_L(p_{11}^3+p_{22}^3)+k_H p_{33}^3+\Phi_3=0,$$
$$f_4:\ p_4^4+u_1p_1^4+u_2p_2^4+u_3p_3^4+u_4(p_1^1+p_2^2+p_3^3)=0,$$
$$f_5:\ \bar{k}_L(p_{11}^5+p_{22}^5)+\bar{k}_H p_{33}^5+\Phi_5=0,$$
$$f_l:\ k_{1,l}(p_{11}^l+p_{22}^l)+k_{2,l}p_{33}^l+\Phi_l=0,$$
$$f_r:\ k_{1,r}(p_{11}^r+p_{22}^r)+k_{2,r}p_{33}^r+\Phi_r=0,$$
$$l=6,7,8;\ r=8+1,\cdots,8+m.$$

Φ_i 是原来方程中微分阶数小于 2 的所有项之和在 $J^2(V,Z)$ 的局部坐标之下的表达式(见附录).

将上述各式的左侧依次记为 f_i,

$$f_i:\ J^2(V,Z)\to R,\qquad i=1,2,\cdots,8+m,$$

则方程组可表示为

$$\mathrm{D}=V(f_1,f_2,\cdots,f_{8+m})\subset J^2(V,Z),$$

即对应 f_i 的公共零点. $f_i\in C^\infty$.

2 大气运动基本方程组的源空间与(局部解)解空间构造

本文将使用以下定理:

• 设 C^∞ 超曲面 $\Sigma_3:\left\{x_3=g_3(x_1,x_2,x_4)\right\}\subset V$. 第三类初值问题

$$\begin{cases}\mathrm{D}\\ u_i\mid_{\Sigma_3}=u_i^0(Q)\end{cases}\qquad (Q\in\Sigma_3,\ g_3\in C^\infty,\ u_i^0\in C^\infty,\ i=1,2,\cdots,8+m).$$

适定的充要条件是

$$u_3^0-u_1^0\frac{\partial g_3}{\partial x_1}-u_2^0\frac{\partial g_3}{\partial x_2}-\frac{\partial g_3}{\partial x_4}\neq 0.$$

于是,有关于大气运动基本方程组的源空间与解空间定理如下:

定理 1(源空间构造定理) 设 $x^0=(x_1^0,x_2^0,x_3^0,x_4^0)\in R^4=V$ 是任意给定的一点,方程组 D 中的所有参数都是其变元的解析函数,$U=(u,v,w,\rho,\theta,q_l,\kappa_r)=(u_1,\cdots,u_5,u_l,u_r)\in Z(l=6,7,8;r=8+1,\cdots,8+m)$ 是 D 的任意一组解,并在 x^0 解析. 那么,过 x^0 在 $R^4=V$ 内可以找到两个充分光滑的超曲面 $S\subset V$ 和 $S\subset V$,使得 U 在 S 或 S 上的限制所构成的定解问题是适定的,并且由这一限制作为定解条件得到的解析解就是原来的 U.

证明 根据第三类初值问题适定的充分必要条件

$$u_3^0-u_1^0\frac{\partial g_3}{\partial x_1}-u_2^0\frac{\partial g_3}{\partial x_2}-\frac{\partial g_3}{\partial x_4}\neq 0. \tag{1}$$

可知,如果 $u_3^0(x^0)=u_3(x^0)=0$,那么取 $S=\left\{x_1^0+\zeta_1,x_2^0+\zeta_2,x_3^0+\zeta_4,x_4^0+\zeta_4\right\}$,即 $S=\Sigma_3:\left\{x_3=g_3(x_1,x_2,x_4)=x_3^0+x_4\right\}$,因此 $\partial g_3/\partial x_1=\partial g_3/\partial x_2=0$,$\partial g_3/\partial x_4=1$,于是由 $U\mid_S$ 构成的定解问题是适定的. 如果 $u_3^0(x^0)=u_3(x^0)\neq 0$,则取 $S=\Sigma_3:\left\{x_3=g_3(x_1,x_2,x_4)=x_3^0+0\right\}$,则由 $U\mid_S$ 构成的定解问题是适定的. 因此,无论是哪种情形,我们都可根据分层时所得到的代数方程组[5-6] 逐步计算出所对应的适定问题的解析解,而这个解就是 U. 定理证毕.

定理 2(局部解析解的解空间构造定理) 设 $x^0=(x_1^0,x_2^0,x_3^0,x_4^0)\in R^4=V$ 是任意给定的

一点, $G_i(x)(i=1,\cdots,15+2m)$ 是任意给定的一组 $15+2m$ 个函数, 它们在 x^0 解析, 并且 $G_4(x^0)>0$, $G_5(x^0)>0$, $G_{i'}(x^0)\geqslant 0$, $i'=6,\cdots,8+m$. 那么, 根据这 $15+2m$ 个函数可构造大气运动基本方程组 D 的一个适定问题, 使得此问题在 x^0 附近存在的那个唯一、稳定的解析解, 由这组函数 $G_i(x)(i=1,\cdots,15+2m)$ 所完全决定.

证明 假设 D 中所有的参数都是其变元的解析函数. 下面根据所给的 $15+2m$ 个函数 $G_i(x)$ 构造 D 的一个适定的第三初值问题: 对初始条件中的 u_i^0 和 p_j^i 作如下选择:

$$\begin{cases} u_i^0=G_i(x(\zeta)), & i=1,\cdots,8+m,\\ p_3^i=G_{8+m+i}(x(\zeta)), & i=1,2,\cdots,8+m,\ i\neq 4,\\ p_3^4=\dfrac{G}{G_\delta}, \end{cases} \tag{2}$$

其中

$$G_\delta=G_3-G_1\delta_1-G_2\delta_2-\delta_4,$$
$$G=G_4(\delta_1G_{9+m}+\delta_2G_{10+m}-G_{11+m})-\varphi_G,$$
$$\varphi_G=G_1\frac{\partial G_4}{\partial x_1}+G_2\frac{\partial G_4}{\partial x_2}+\frac{\partial G_4}{\partial x_4}+G_4\left(\frac{\partial G_1}{\partial x_1}+\frac{\partial G_2}{\partial x_2}\right).$$

其余的 p_j^i 由下面的关系式决定[6]:

$$p_1^i=\frac{\partial u_i^0}{\partial x_1}-\delta_1p_3^i,\quad p_2^i=\frac{\partial u_i^0}{\partial x_2}-\delta_2p_3^i,\quad p_4^i=\frac{\partial u_i^0}{\partial x_4}-\delta_4p_3^i,$$
$$i=1,2,\cdots,8+m, \tag{3}$$

这里 $x(\zeta)=(x_1(\zeta),x_2(\zeta),x_3(\zeta),x_4(\zeta))\in\Sigma_3\subset R^4$, $\delta_l(l=1,2,4)$ 则由超曲面 Σ_3 的表达式决定, 比如定理 1 中的 $\Sigma_3=S$ 或 $\Sigma_3=S$.

根据定理 1, 不论 $u_3^0(x^0)=G_3(x(0))=G_3(x^0)$ 等于 0 或不等于 0, 我们都可以选取适当的超曲面 Σ_3, 使得由此构成的初值问题是适定的. 然后即可逐步计算出这个适定问题唯一稳定的解析解. 定理证毕.

注 (2) 式来源于 D 的一阶本方程 D_1.

3 稳定解析解的计算

下面以适定的第三初值问题为例, 给出这个问题以收敛幂级数表示的解析解的计算方法与过程.

为简单起见, 设 $x^0=\mathbf{0}=(0,0,0,0)\in R^4=V$. 我们将根据初始条件定义一组解析函数

$$H_{j^\alpha}^i(\zeta),\quad \zeta=(\zeta_1,\zeta_2,\zeta_3)\in R^3,$$
$$(1\leqslant i\leqslant 8+m,\ j^\alpha=j_1^{\alpha_1}j_2^{\alpha_2}j_3^{\alpha_3}j_4^{\alpha_4},\ |\alpha|=\alpha_1+\alpha_2+\alpha_3+\alpha_4=k,\ k=0,1,\cdots),$$

这些解析函数在原点 $(0,0,0)\in R^3$ 的值, 就完全确定了以绝对收敛幂级数表示的解析函数

$$u_i(x)=\sum_\alpha\frac{1}{\alpha!}H_{j^\alpha}^i(0)x^\alpha, \tag{4}$$
$$x=(x_1,x_2,x_3,x_4)\in R^4,\quad x^\alpha=x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3}x_4^{\alpha_4},\qquad i=1,2,\cdots,8+m,$$

而这些 $u_i(x)$ 就是 D 在原点 $\mathbf{0}\in R^4$ 邻域稳定的解析解, 即

$$H_{j^\alpha}^i(\zeta)=p_{j^\alpha}^i(\zeta)=\frac{\partial^{|\alpha|}u_i}{\partial x^\alpha}(\zeta), \tag{5}$$

这里 $$\frac{\partial^{|\alpha|}u_i}{\partial x^\alpha}=\frac{\partial^{|\alpha|}u_i}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\partial x_3^{\alpha_3}\partial x_4^{\alpha_4}}.$$

下面的工作是根据初始条件计算 $H^i_{j^\alpha}(\zeta)$.

* * $H^i_{j^\alpha}(\zeta)$ 的计算公式

分两种情形:

1) $u_3^0(0)=G_3(x(0))=G_3(0)=0$.

取 $\Sigma_3=S$, 即 $\Sigma_3:\{0+\zeta_1, 0+\zeta_2, 0+\zeta_4, 0+\zeta_4\}$, $\zeta=(\zeta_1,\zeta_2,\zeta_3)\in R^3$, 或直接写成

$$\Sigma_3=S:\{x_3=g_3(x_1,x_2,x_4)=x_4\},\tag{6}$$

且 $x^0=\mathbf{0}=(0,0,0,0)\in\Sigma_3\subset R^4$.

这时 $\delta_j=\partial g_3/\partial x_j(j=1,2,4)$, 于是有 $\delta_1=\delta_2=0$, $\delta_4=1$.

根据(2)式和(3)式, 则有

$k=0$:

$$H^i_0(\zeta)=G_i(x(\zeta)),\qquad i=1,2,\cdots,8+m;$$

$k=1$:

$$H^i_3(\zeta)=G_{8+m+i}(x(\zeta)),\qquad i=1,2,\cdots,8+m,\ i\neq 4;$$

$$H^4_3(\zeta)=G(\zeta),$$

$$H^i_1(\zeta)=\frac{\partial G_i}{\partial\zeta_1},\ H^i_2(\zeta)=\frac{\partial G_i}{\partial\zeta_2},\ H^i_4(\zeta)=\frac{\partial G_i}{\partial\zeta_4}-H^i_3(\zeta),\qquad i=1,2,\cdots,8+m,$$

这里

$$G(\zeta)=G_1\frac{\partial G_4}{\partial\zeta_1}+G_2\frac{\partial G_4}{\partial\zeta_2}+\frac{\partial G_4}{\partial\zeta_4}+G_4\left(\frac{\partial G_1}{\partial\zeta_1}+\frac{\partial G_2}{\partial\zeta_2}\right)+G_4G_{11+m};$$

$k\geqslant 2$:

根据 D 的 k 阶本方程 D_k, $H^i_{3^k}=Z_i$ 是以下代数方程组 $(*)_k$ 的唯一解:

$$k_HZ_1=\omega_{1,k-2},$$
$$k_HZ_2=\omega_{2,k-2},$$
$$k_HZ_3=\omega_{3,k-2},$$
$$u_4Z_3-Z_4=\omega_{4,k-2},$$
$$k_HZ_5=\omega_{5,k-2},$$
$$k_{2,l}Z_l=\omega_{l,k-2},\qquad l=6,7,8,$$
$$k_{2,r}Z_r=\omega_{r,k-2},\qquad r=8+1,\cdots,8+m,$$

即

$$H^1_{3^k}=\frac{1}{k_H}\omega_{1,k-2},\ H^2_{3^k}=\frac{1}{k_H}\omega_{2,k-2},\ H^3_{3^k}=\frac{1}{k_H}\omega_{3,k-2},$$

$$H^4_{3^k}=\frac{G_4}{k_H}\omega_{3,k-2}-\omega_{4,k-2},\ H^5_{3^k}=\frac{1}{k_H}\omega_{5,k-2},$$

$$H^l_{3^k}=\frac{1}{k_{2,l}}\omega_{l,k-2},\qquad l=6,7,8,$$

$$H^r_{3^k}=\frac{1}{k_{2,r}}\omega_{r,k-2},\qquad r=8+1,\cdots,8+m,$$

$$H^i_{j\alpha_1} = \frac{\partial}{\partial \zeta_1} H^i_{j\alpha},\ H^i_{j\alpha_2} = \frac{\partial}{\partial \zeta_2} H^i_{j\alpha},\ H^i_{j\alpha_4} = \frac{\partial}{\partial \zeta_4} H^i_{j\alpha} - H^i_{j\alpha_3},$$

$$i = 1, 2, \cdots, 8 + m,\ |\alpha| = k - 1.$$

$\omega_{i,k-2}$ 的计算公式参看附录.

为了直观起见, 下面给出 $k = 2$ 时的完整计算. 同时为了简化计算, 这里只考虑一种水汽状态而且也不计气溶胶对大气运动的影响.

根据 D 的 2 阶本方程[6] D_2 的表达式, 此时有代数方程组

$$k_H p^1_{3^2} = \omega_{1,0},\ k_H p^2_{3^2} = \omega_{2,0},\ k_H p^3_{3^2} = \omega_{3,0},$$

$$u_4 p^3_{3^2} - p^4_{3^2} = \omega_{4,0},\ \bar{k}_H p^5_{3^2} = \omega_{5,0},\ k_{2,6} p^6_{3^2} = \omega_{6,0},$$

于是有

$$H^1_{3^2} = \frac{1}{k_H}\omega_{1,0},\ H^2_{3^2} = \frac{1}{k_H}\omega_{2,0},\ H^3_{3^2} = \frac{1}{k_H}\omega_{3,0},$$

$$H^4_{3^2} = \frac{u_4}{k_H}\omega_{3,0} - \omega_{4,0},\ H^5_{3^2} = \frac{1}{\bar{k}_H}\omega_{5,0},\ H^6_{3^2} = \frac{1}{k_H}\omega_{6,0},$$

$$H^i_{j1} = \frac{\partial}{\partial \zeta_1}(H^i_j),\ H^i_{j2} = \frac{\partial}{\partial \zeta_2}(H^i_j),\ H^i_{j4} = \frac{\partial}{\partial \zeta_4}(H^i_j) - H^i_{j3},$$

$$i = 1, \cdots, 6;\ j = 1, 2, 3, 4.$$

$\omega_{i,0}$ 如下:

$$\omega_{i,0} = -\Phi_i - k_L(p^i_1(1) + p^i_2(2)),\qquad i = 1, 2, 3,$$

$$\omega_{4,0} = -\Phi_4 - [p^4_3(4) + u_1 p^4_3(1) + u_2 p^4_3(2) + u_4 p^1_3(1) + u_4 p^2_3(2)],$$

$$\omega_{5,0} = -\Phi_5 - \bar{k}_L(p^5_1(1) + p^5_2(2)),$$

$$\omega_{6,0} = -\Phi_6 - k_{1,6}(p^6_1(1) + p^6_2(2)),$$

式中 $p^i_j(l) = \dfrac{\partial p^i_j}{\partial \zeta_l},\ l = 1, 2, 4.$

而 Φ_i 的表达式为:

$$\Phi_i = p^i_4 + u_1 p^i_1 + u_2 p^i_2 + u_3 p^i_3 + \frac{1}{u_4}(P_1 p^4_i + P_2 p^4_i) + (-1)^i \delta_{i2} f u_1 + (-1)^i \delta_{i3} f u_1 +$$
$$(-1)^i \delta_{i3} f u_2 + (-1)^{i-1} \delta_{i1} f u_3 + (-1)^{i-1} \delta_{i3} g - \bar{\Phi}_i,\qquad i = 1, 2, 3, \tag{7}$$

$$\Phi_4 = p^1_3 p^4_1 + p^2_3 p^4_2 + p^3_3 p^4_3 + p^4_3(p^1_1 + p^2_2 + p^3_3), \tag{8}$$

$$\Phi_h = S_h - (p^h_4 + u_1 p^h_1 + u_2 p^h_2 + u_3 p^h_3) + \bar{\Phi}_h,\qquad h = 5, 6, \tag{9}$$

其中

$P_1 = \dfrac{\partial P}{\partial \rho},\ P_2 = \dfrac{\partial P}{\partial \theta},\ \delta_{ij}$ 是 Kronecker 符号

$$\bar{\Phi}_i = \frac{k_L}{u_4}(p^i_1 p^4_1 + p^i_2 p^4_2) + \frac{k_H}{u_4} p^i_3 p^4_3 + \left(p^i_1 \frac{\partial k_L}{\partial x_1} + p^i_2 \frac{\partial k_L}{\partial x_2} + p^i_3 \frac{\partial k_H}{\partial x_3}\right),$$
$$i = 1, 2, 3, \tag{10}$$

$$\bar{\Phi}_5 = \frac{\bar{k}_L}{u_4}(p^5_1 p^4_1 + p^5_2 p^4_2) + \frac{\bar{k}_H}{u_4} p^5_3 p^4_3 + \left(p^5_1 \frac{\partial \bar{k}_L}{\partial x_1} + p^5_2 \frac{\partial \bar{k}_L}{\partial x_2} + p^5_3 \frac{\partial \bar{k}_H}{\partial x_3}\right), \tag{11}$$

$$\bar{\Phi}_6 = \frac{k_{1,6}}{u_4}(p^6_1 p^4_1 + p^6_2 p^4_2) + \frac{k_{2,6}}{u_4} p^6_3 p^4_3 + \left(p^6_1 \frac{\partial k_{1,6}}{\partial x_1} + p^6_2 \frac{\partial k_{1,6}}{\partial x_2} + p^6_3 \frac{\partial k_{2,6}}{\partial x_3}\right). \tag{12}$$

上述各式中出现的 u_i 和 p^i_j 以及 $p^i_j(l)$ 为

$$u_i(\zeta) = G_i(\zeta), \qquad i = 1, 2, \cdots, 6,$$
$$p_3^i(\zeta) = G_{6+i}(\zeta), \qquad i = 1, 2, \cdots, 6;\ i \neq 4,$$
$$p_3^4(\zeta) = G(\zeta),$$
$$p_1^i(\zeta) = \frac{\partial G_i}{\partial \zeta_1},\ p_2^i(\zeta) = \frac{\partial G_i}{\partial \zeta_2},\ p_4^i(\zeta) = \frac{\partial G_i}{\partial \zeta_4} - p_3^i(\zeta), \qquad i = 1, 2, \cdots, 6,$$
$$p_j^i(l) = \frac{\partial}{\partial \zeta_l}(p_j^i),$$
$$G(\zeta) = G_1 \frac{\partial G_4}{\partial \zeta_1} + G_2 \frac{\partial G_4}{\partial \zeta_2} + \frac{\partial G_4}{\partial \zeta_4} + G_4\left(\frac{\partial G_1}{\partial \zeta_1} + \frac{\partial G_2}{\partial \zeta_2}\right) + G_4 G_{11}.$$

于是我们就有了级数(4) $k = 0, 1, 2$ 时所有系数的计算公式.

对于 $k \geqslant 3$, 类似的表达式可如上一样计算出来.

2) $u_3^0(0) = G_3(x(0)) = G_3(0) \neq 0$.

取 $\Sigma_3 = S$, 即 Σ_3: $\{0 + \zeta_1, 0 + \zeta_2, 0 + 0, 0 + \zeta_4\}$, $(\zeta_1, \zeta_2, \zeta_4) \in R^3$, 或直接写成

$$\Sigma_3 = \{x_3 = g_3(x_1, x_2, x_4) = 0\}, \tag{13}$$

且 $x^0 = \mathbf{0} = (0, 0, 0, 0) \in \Sigma_3 \subset R^4$. 这时 $\delta_l = \partial g_3/\partial x_l = 0$, $l = 1, 2, 4$.

于是有

$k = 0$:

$$H_0^i(\zeta) = G_i(x(\zeta)), \qquad i = 1, 2, \cdots, 8 + m;$$

$k = 1$:

$$H_3^i(\zeta) = G_{8+m+i}(x(\zeta)), \qquad i = 1, 2, \cdots, 8 + m;\ i \neq 4;$$
$$H_3^4(\zeta) = \frac{-G_4 G_{11+m} - G}{G_3},$$
$$H_1^i(\zeta) = \frac{\partial G_i}{\partial \zeta_1},\ H_2^i(\zeta) = \frac{\partial G_i}{\partial \zeta_2},\ H_4^i(\zeta) = \frac{\partial G_i}{\partial \zeta_4}, \qquad i = 1, 2, \cdots, 8 + m,$$

这里

$$G(\zeta) = G_1 \frac{\partial G_4}{\partial \zeta_1} + G_2 \frac{\partial G_4}{\partial \zeta_2} + \frac{\partial G_4}{\partial \zeta_4} + G_4\left(\frac{\partial G_1}{\partial \zeta_1} + \frac{\partial G_2}{\partial \zeta_2}\right) + G_4 G_{11+m};$$

$k \geqslant 2$:

$H_{3^k}^i = Z_i$ 是以下代数方程组的唯一解:

$$k_H Z_1 = \omega_{1, k-2},$$
$$k_H Z_2 = \omega_{2, k-2},$$
$$k_H Z_3 = \omega_{3, k-2},$$
$$G_4 Z_3 + G_3 Z_4 = \omega_{4, k-2},$$
$$k_H Z_5 = \omega_{5, k-2},$$
$$k_{2, l} Z_l = \omega_{l, k-2}, \qquad l = 6, 7, 8,$$
$$k_{2, r} Z_r = \omega_{r, k-2}, \qquad r = 8 + 1, \cdots, 8 + m,$$

即

$$H_{3^k}^1 = \frac{1}{k_H}\omega_{1, k-2},\ H_{3^k}^2 = \frac{1}{k_H}\omega_{2, k-2},\ H_{3^k}^3 = \frac{1}{k_H}\omega_{3, k-2},$$

$$H^4_{3^k}=\frac{1}{G_3}\left(\omega_{4,k-2}-\frac{G_4}{k_H}\omega_{3,k-2}\right),\quad H^5_{3^k}=\frac{1}{\hat{k}_H}\omega_{5,k-2},$$

$$H^l_{3^k}=\frac{1}{k_{2,l}}\omega_{l,k-2},\qquad l=6,7,8,$$

$$H^r_{3^k}=\frac{1}{k_{2,r}}\omega_{r,k-2},\qquad r=8+1,\cdots,8+m,$$

$$H^i_{j^\alpha 1}=\frac{\partial}{\partial\zeta_1}H^i_{j^\alpha},\quad H^i_{j^\alpha 2}=\frac{\partial}{\partial\zeta_2}H^i_{j^\alpha},\quad H^i_{j^\alpha 4}=\frac{\partial}{\partial\zeta_4}(H^i_{j^\alpha}),$$

$$i=1,2,\cdots,8+m;\ |\alpha|=k-1.$$

$\omega_{i,k-2}$ 的计算公式参看附录.

说明 使问题适定的超曲面的选择不是唯一的, 但是, 有此超曲面参与的定解条件必须满足适定的充要条件(1).

1) 其它类型适定的初值问题, 其解析解的计算方法与本文给出的第三初值适定问题解析解的计算方法类似, 详细过程可参看文献[5]和文献[6].

2) 本文所涉及的局部解虽然指的是在"一点附近", 但是根据大气运动基本方程组的源空间构造定理可知, 如果在一个给定的超曲面上的任一点, 所论问题均适定, 那么就存在此超曲面的一个"管状邻域"[5-7], 使得问题的解析解在此邻域中完全被确定, 这就是与应用问题有关的局部解的真正含义.

附录

1) $\omega^{(3)}_{i,k-2}$ 的一般计算公式

$$\omega^{(3)}_{i,k-2}=-\Phi_{i,k-2}+k_L[\delta_1 p^i_{3^{k-1}}(1)+\delta_2 p^i_{3^{k-1}}(2)-p^i_{13^{k-2}}(1)-p^i_{23^{k-2}}(2)],\qquad i=1,2,3,$$

$$\omega^{(3)}_{4,k-2}=-\Phi_{4,k-2}-[p^4_{3^{k-1}}(4)+u_1p^4_{3^{k-1}}(1)+u_2p^4_{3^{k-1}}(2)+u_4p^1_{3^{k-1}}(1)+u_4p^1_{3^{k-1}}(2)],$$

$$\omega^{(3)}_{5,k-2}=-\Phi_{5,k-2}+\hat{k}_L[\delta_1 p^5_{3^{k-1}}(1)+\delta_2 p^5_{3^{k-1}}(2)-p^5_{13^{k-2}}(1)-p^5_{23^{k-2}}(2)],$$

$$\omega^{(3)}_{l,k-2}=-\Phi_{l,k-2}+k_{1,l}[\delta_1 p^l_{3^{k-1}}(1)+\delta_2 p^l_{3^{k-1}}(2)-p^l_{13^{k-2}}(1)-p^l_{23^{k-2}}(2)],\qquad l=6,7,8,$$

$$\omega^{(3)}_{r,k-2}=-\Phi_{r,k-2}+k_{1,r}[\delta_1 p^r_{3^{k-1}}(1)+\delta_2 p^r_{3^{k-1}}(2)-p^r_{13^{k-2}}(1)-p^r_{23^{k-2}}(2)],$$

$$r=8+1,\cdots,8+m.$$

其中 $\Phi_{i,k-2}$ 的计算公式如下

$$\Phi_{i,0}=\Phi_i=p^i_4+u_1p^i_1+u_2p^i_2+u_3p^i_3+\frac{1}{u_4}(P_1p^4_i+P_2p^4_i)+(-1)^i\delta_{i2}fu_1+(-1)^i\delta_{i3}fu_1+$$
$$(-1)^i\delta_{i3}fu_2+(-1)^{i-1}\delta_{i1}fu_3+(-1)^{i-3}\delta_{i3}g-\Phi_i,\qquad i=1,2,3,$$

$$\Phi_{4,0}=\Phi_4=p^1_3p^4_1+p^2_3p^4_2+p^3_3p^4_3+p^4_3(p^1_1+p^2_2+p^3_3),$$

$$\Phi_{5,0}=\Phi_5=S_5-[p^5_4+u_1p^5_1+u_2p^5_2+u_3p^5_3]+\frac{1}{u_4}[\hat{k}_L(p^4_1p^5_1+p^4_2p^5_2)+\hat{k}_Hp^4_3p^5_3]+$$
$$\left[p^5_1\left(\frac{\partial\hat{k}_L}{\partial x_1}\right)+p^5_2\left(\frac{\partial\hat{k}_L}{\partial x_2}\right)+p^5_3\left(\frac{\partial\hat{k}_H}{\partial x_3}\right)\right],$$

$$\Phi_{l,0}=\Phi_l=S_l-[p^l_4+u_1p^l_1+u_2p^l_2+u_3p^l_3]+\frac{1}{u_4}[k_{1,l}(p^4_1p^l_1+p^4_2p^l_2)+k_{2,l}p^4_3p^l_3]+$$
$$\left[p^l_1\left(\frac{\partial k_{1,l}}{\partial x_1}\right)+p^l_2\left(\frac{\partial k_{1,l}}{\partial x_2}\right)+p^l_3\left(\frac{\partial k_{2,l}}{\partial x_3}\right)\right],\qquad l=6,7,8,$$

$$\Phi_{r,0}=\Phi_r=S_r-(p^r_4+u_1p^r_1+u_2p^r_2+u_3p^r_3)+\frac{1}{u_4}[k_{1,r}(p^4_1p^r_1+p^4_2p^r_2)+k_{2,r}p^4_3p^r_3]+$$
$$\left[p^r_1\left(\frac{\partial k_{1,r}}{\partial x_1}\right)+p^r_2\left(\frac{\partial k_{1,r}}{\partial x_2}\right)+p^r_3\left(\frac{\partial k_{2,r}}{\partial x_3}\right)\right],\qquad r=8+1,\cdots,8+m,$$

式中

$P_1 = \dfrac{\partial P}{\partial \rho}$, $P_2 = \dfrac{\partial P}{\partial \theta}$, δ_{ij} 是 Kronecker 符号.

$k \geqslant 3$,

$$\Phi_{i,k-2} = C^1_{k-2}\left[\left(p^i_{123^{k-3}} + p^i_{223^{k-3}}\right)\frac{\partial k_L}{\partial x_3} + p^i_{3^{k-1}}\frac{\partial k_H}{\partial x_3}\right] +$$

$$C^2_{k-2}\left[\left(p^i_{123^{k-4}} + p^i_{223^{k-4}}\right)\frac{\partial^2 k_L}{\partial x_3^2} + p^i_{3^{k-2}}\frac{\partial^2 k_H}{\partial x_3^2}\right] + \cdots +$$

$$C^n_{k-2}\left[\left(p^i_{123^{k-2-n}} + p^i_{223^{k-2-n}}\right)\frac{\partial^n k_L}{\partial x_3^n} + p^i_{3^{k-n}}\frac{\partial^n k_H}{\partial x_3^n}\right] + \cdots +$$

$$C^{k-3}_{k-2}\left[\left(p^i_{12} + p^i_{22}\right)\frac{\partial^{k-3} k_L}{\partial x_3^{k-3}} + p^i_{32}\frac{\partial^{k-3} k_H}{\partial x_3^{k-3}}\right] + \frac{\partial^{k-2}\Phi_{i,0}}{\partial x_3^{k-2}}, \qquad i = 1, 2, 3,$$

$$\Phi_{4,k-2} = p^1_3 p^4_{13^{k-2}} + p^2_3 p^4_{23^{k-2}} + p^3_3 p^4_{3^{k-1}} + p^4_3\left(p^1_{13^{k-2}} + p^2_{23^{k-2}} + p^3_{3^{k-1}}\right) + \frac{\partial \Phi_{4,k-3}}{\partial x_3},$$

$$\Phi_{5,k-2} = C^1_{k-2}\left[\left(p^5_{123^{k-3}} + p^5_{223^{k-3}}\right)\frac{\partial \hat{k}_L}{\partial x_3} + p^5_{3^{k-1}}\frac{\partial \hat{k}_H}{\partial x_3}\right] + C^2_{k-2}\left[\left(p^5_{123^{k-4}} + p^5_{223^{k-4}}\right)\frac{\partial^2 \hat{k}_L}{\partial x_3^2} + p^5_{3^{k-2}}\frac{\partial^2 \hat{k}_H}{\partial x_3^2}\right] + \cdots +$$

$$C^n_{k-2}\left[\left(p^5_{123^{k-2-n}} + p^5_{223^{k-2-n}}\right)\frac{\partial^n \hat{k}_L}{\partial x_3^n} + p^5_{3^{k-n}}\frac{\partial^n \hat{k}_H}{\partial x_3^n}\right] + \cdots +$$

$$C^{k-3}_{k-2}\left[\left(p^5_{12} + p^5_{22}\right)\frac{\partial^{k-3} \hat{k}_L}{\partial x_3^{k-3}} + p^5_{32}\frac{\partial^{k-3} \hat{k}_H}{\partial x_3^{k-3}}\right] + \frac{\partial^{k-2}\Phi_{5,0}}{\partial x_3^{k-2}},$$

$$\Phi_{l,k-2} = C^1_{k-2}\left[\left(p^l_{123^{k-3}} + p^l_{223^{k-3}}\right)\frac{\partial k_{1,l}}{\partial x_3} + p^l_{3^{k-1}}\frac{\partial k_{2,l}}{\partial x_3}\right] +$$

$$C^2_{k-2}\left[\left(p^l_{123^{k-4}} + p^l_{223^{k-4}}\right)\frac{\partial^2 k_{1,l}}{\partial x_3^2} + p^l_{3^{k-2}}\frac{\partial^2 k_{1,l}}{\partial x_3^2}\right] + \cdots +$$

$$C^n_{k-2}\left[\left(p^l_{123^{k-2-n}} + p^l_{223^{k-2-n}}\right)\frac{\partial^n k_{1,l}}{\partial x_3^n} + p^l_{3^{k-n}}\frac{\partial^n k_{2,l}}{\partial x_3^n}\right] + \cdots +$$

$$C^{k-3}_{k-2}\left[\left(p^l_{12} + p^l_{22}\right)\frac{\partial^{k-3} k_{1,l}}{\partial x_3^{k-3}} + p^l_{32}\frac{\partial^{k-3} k_{2,l}}{\partial x_3^{k-3}}\right] + \frac{\partial^{k-2}\Phi_{l,0}}{\partial x_3^{k-2}}, \qquad l = 6, 7, 8,$$

$$\Phi_{r,k-2} = C^1_{k-2}\left[\left(p^r_{123^{k-3}} + p^r_{223^{k-3}}\right)\frac{\partial k_{1,r}}{\partial x_3} + p^r_{3^{k-1}}\frac{\partial k_{2,r}}{\partial x_3}\right] +$$

$$C^2_{k-2}\left[\left(p^r_{123^{k-4}} + p^r_{223^{k-4}}\right)\frac{\partial^2 k_{1,r}}{\partial x_3^2} + p^r_{3^{k-2}}\frac{\partial^2 k_{1,r}}{\partial x_3^2}\right] + \cdots +$$

$$C^n_{k-2}\left[\left(p^r_{123^{k-2-n}} + p^r_{223^{k-2-n}}\right)\frac{\partial^n k_{1,r}}{\partial x_3^n} + p^r_{3^{k-n}}\frac{\partial^n k_{2,r}}{\partial x_3^n}\right] + \cdots +$$

$$C^{k-3}_{k-2}\left[\left(p^r_{12} + p^r_{22}\right)\frac{\partial^{k-3} k_{1,r}}{\partial x_3^{k-3}} + p^r_{32}\frac{\partial^{k-3} k_{2,r}}{\partial x_3^{k-3}}\right] + \frac{\partial^{k-2}\Phi_{r,0}}{\partial x_3^{k-2}}, \qquad r = 8+1, \cdots, 8+m.$$

2) $\omega_{i,k-2}$ 与 $\omega_{i,k-2}$ 的计算公式

- 在 $\omega^{(3)}_{i,k-2}$ 的表达式中, 取 $\delta_1 = \delta_2 = 0$, $\delta_4 = 1$ 及对应的初始值, 则有

$$\omega_{i,k-2} = \omega^{(3)}_{i,k-2}, \qquad i = 1, 2, \cdots, 8+m,$$

- 在 $\omega^{(3)}_{i,k-2}$ 的表达式中, 取 $\delta_l = 0 (l = 1, 2, 4)$ 及对应的初始值, 则有

$$\omega_{i,k-2} = \omega^{(3)}_{i,k-2}, \qquad i = 1, 2, \cdots, 8+m.$$

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Analytical Solution of the Basic Equations Set of Atmospheric Motion

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Abstract: In the smooth function classes, the basic equations set of atmospheric motion possesses the best stability, under this condition, its structure of solution space for local analytical solution was discussed, by which the third-class initial value problem with typicality and application was also analyzed. And in the analytic function classes, the calculational method and concrete expressions of analytical solution about the well-posed initial value problem of the third kind were given. In the meaning of local solution, near a appointed point, the relevant theoretical and computational problems about analytical solution of initial value problem were solved completely. Meanwhile, it is pointed out here that, with other types of problems for determining solution, the computational method and process of their stable analytical solution can be obtained in a similar way if needed.

Key words: basic equations set of atmospheric motion; structure of solution space; analytical solution