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蒸汽沉淀化学反应方程混合元法的离散格式研究*

罗振东[1,3], 周艳杰[2], 朱 江[3]

(1. 北京交通大学 理学院, 北京 100044;
2. 北京工商大学 数理系, 北京 100037;
3. 中国科学院 大气物理研究所, 北京 100029)

(林建忠推荐)

摘要: 蒸汽沉淀化学反应过程有着极其广泛的应用, 其数学模型归结为一个包含流速场, 温度场, 压力场和气体溶质场的非线性偏微分方程组. 用混合有限元方法研究蒸汽沉淀化学反应方程组, 导出其半离散化和全离散化的混合元格式, 并证明这些格式的解的存在性和收敛性(误差估计). 用混合元法处理究蒸汽沉淀化学反应方程组, 可以同时求出流速场, 温度场, 压力场和气体溶质场的数值解. 因此该研究既具有重要的理论意义, 又具有广泛的应用前景.

关 键 词: 化学蒸汽沉淀反应方程; 混合元方法; 半离散化格式; 全离散化格式

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引 言

蒸汽沉淀化学反应过程有着极其广泛的应用, 其数学模型归结为一个包含流速场, 温度场, 压力场和气体溶质场的非线性偏微分方程组(参见文献[1]):

$$\begin{cases} \nabla\cdot(\rho_0\boldsymbol{u}) = 0, \\ \dfrac{\partial\boldsymbol{u}}{\partial t} + \rho_0(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\nabla p + \nabla\cdot\sigma + Tg\boldsymbol{j}, \\ \sigma + \operatorname{tr}\sigma\boldsymbol{I} = 2\mu_{\mathrm{T}}\varepsilon(\boldsymbol{u}), \\ \dfrac{\partial T}{\partial t} + c_2\rho\boldsymbol{u}\cdot\nabla T = \lambda_{\mathrm{T}}\nabla\cdot(\nabla T), \\ \dfrac{\partial C}{\partial t} + \boldsymbol{u}\cdot\nabla C = D_{\mathrm{T}}\nabla\cdot(\nabla C), \end{cases} \tag{1}$$

其中 $\boldsymbol{u} = (u_1, u_2)$ 是速度向量, T 是温度, C 是TMGa的质量级分, p 是压力, μ_{T} 是输运气体的黏性系数, g 是重力加速度, $\boldsymbol{j} = (1, 0)$, $Tg\boldsymbol{j}$ 项看作为由重力和温度T 引起的自然对流影响,

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作者简介: 罗振东(1958—), 男, 广西桂平人, 教授, 博士生导师(联系人. Tel: +86-10-51684751; Fax: +86-10-68902789; E-mail: zhdluo@bjtu.edu.cn).

$$\varepsilon(\boldsymbol{u}) = (\varepsilon_{ij})_{2\times 2},\quad \varepsilon_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right],$$

ρ_0 是气体的密度(是常量), c_2 是热吸收率, λ_T 是输运气体的热传导率, D_T 是在输运气体中的 TMGa 的扩散系数．方程组(1) 的初边值条件定为:

$$\begin{cases} \boldsymbol{u}(0) = \boldsymbol{0},\ T(0) = C(0) = 0,\ C|_{\partial\Omega} = C_0, \\ T|_{\partial\Omega} = T_0,\ \ \boldsymbol{u}|_{\partial\Omega} = \boldsymbol{0},\ \ \sigma|_{\partial\Omega} = \boldsymbol{0}. \end{cases} \tag{2}$$

据我们所知, 虽然过去有过一些用有限差分方法去求方程组(1) 的数值解(参考文献[1]及其当中的参考文献), 但没有对其理论分析．本文用混合有限元方法研究蒸汽沉淀化学反应方程, 导出其半离散化和全离散化的混合元格式, 并证明这些格式的解的存在性和收敛性(误差估计)．用混合元法处理方程组(1), 可以同时求出流速场, 温度场, 压力场和气体溶质场的数值解．因此该研究既具有重要的理论意义, 又具有广泛的应用前景．在另文将把特征投影分解与混合元方法结合起来处理蒸汽沉淀化学反应方程．

1　半离散化的混合有限元解的存在性和收敛性

方程组(1) 的变分形式为:

问题(Ⅰ)　求 $\boldsymbol{u}\in L^2(0,t;X)$, $p\in L^2(0,t;M)$, $\sigma\in H$, $T\in L^2(0,t;W)$, $C\in L^2(0,t;W)$ 满足

$$\begin{cases} b(q,\boldsymbol{u}) = 0, & \forall q\in M, \\ (\boldsymbol{u}_t,\boldsymbol{v}) + \rho_0 a(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}) + b(p,\boldsymbol{v}) + (\sigma,\varepsilon(\boldsymbol{v})) = (Tg\boldsymbol{j},\boldsymbol{v}), & \forall \boldsymbol{v}\in X, \\ (\sigma,\tau) + (\operatorname{tr}\sigma,\operatorname{tr}\tau) - 2\mu_T(\varepsilon(\boldsymbol{u}),\tau) = 0, & \forall\tau\in H, \\ (T_t,\varphi) + \lambda_T(\nabla T,\nabla\varphi) + c_2 a_1(\boldsymbol{u},T,\varphi) = 0, & \forall\varphi\in W_0, \\ (C_t,\psi) + D_T(\nabla C,\nabla\psi) + a_1(\boldsymbol{u},C,\psi) = 0, & \forall\psi\in W_0, \\ \boldsymbol{u}(0) = \boldsymbol{0},\ T(0) = C(0) = 0, \\ C|_{\partial\Omega} = C_0;\ T|_{\partial\Omega} = T_0;\ \boldsymbol{u}|_{\partial\Omega} = \boldsymbol{0};\ \sigma|_{\partial\Omega} = \boldsymbol{0}, \end{cases}$$

其中

$$X = H_0^1(\Omega)^2,\quad V = \left\{\boldsymbol{v}\in X;\ \operatorname{div}\boldsymbol{v} = 0\right\},$$

$$M = L_0^2(\Omega) = \left\{\varphi\in L^2(\Omega);\ \int_\Omega \varphi\,\mathrm{d}x = 0\right\},$$

$$H = \left\{\tau = (\tau_{ij})_{2\times 2}\in L^2(\Omega)^{2\times 2};\ \tau_{12} = \tau_{21}\right\},\quad W = H^1(\Omega),$$

$$W_0 = H_0^1(\Omega),\quad b(q,\boldsymbol{u}) = (q,\operatorname{div}\boldsymbol{v}),$$

$$a(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) = \int_\Omega(\boldsymbol{u}\cdot\nabla\boldsymbol{v})\boldsymbol{w}\,\mathrm{d}x = \frac{1}{2}\left[\int_\Omega\sum_{i,j=1}^2 u_i\frac{\partial v_j}{\partial x_i}w_j\,\mathrm{d}x - \int_\Omega\sum_{i,j=1}^2 u_i\frac{\partial w_j}{\partial x_i}v_j\,\mathrm{d}x\right],$$

$$a_1(\boldsymbol{u},T,\varphi) = \int_\Omega(\boldsymbol{u}\cdot\nabla T)\varphi\,\mathrm{d}x = \frac{1}{2}\left[\int_\Omega\sum_{i=1}^2 u_i\frac{\partial T}{\partial x_i}\varphi\,\mathrm{d}x - \int_\Omega\sum_{i=1}^2 u_i\frac{\partial\varphi}{\partial x_i}T\,\mathrm{d}x\right].$$

由 Sobolev 空间的性质或泛函分析可知, $a(\cdot,\cdot,\cdot)$ 有下面假设成立($a_1(\cdot,\cdot,\cdot)$ 同样, 可参考文献[2]至文献[7]等)．

(A_1)　$\forall\boldsymbol{u}\in X$, $\operatorname{div}\boldsymbol{u} = 0$, $\forall\boldsymbol{v},\boldsymbol{w}\in H^1(\Omega)^2$ 有 $a(\boldsymbol{u},\boldsymbol{v},\boldsymbol{v}) = 0$ 和 $a(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) = -a(\boldsymbol{u},\boldsymbol{w},\boldsymbol{v})$.

(A_2)　$\forall\boldsymbol{u}\in V$, $\boldsymbol{w},\boldsymbol{v}\in X$ 都有

$$|a(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})| \leqslant \|\boldsymbol{u}\|_{0,\infty}\|\boldsymbol{v}\|_0\|\nabla\boldsymbol{w}\|_0,$$

$$|a(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})| \leqslant \|\nabla \boldsymbol{u}\|_0 \|\boldsymbol{v}\|_0 \|\boldsymbol{w}\|_{0,\infty},$$
$$|a(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})| \leqslant \|\boldsymbol{u}\|_0 \|\nabla \boldsymbol{v}\|_{0,\infty} \|\boldsymbol{w}\|_0,$$
$$|a(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})| \leqslant \|\boldsymbol{u}\|_0 \|\nabla \boldsymbol{v}\|_0 \|\boldsymbol{w}\|_{0,\infty}.$$

(A_3) 假定 $\partial\Omega \in C^{k,\alpha}(k \geqslant 1, \alpha \geqslant 1)$，则对于 $T_0, C_0 \in C^{k,\alpha}(\partial\Omega)$ 存在 $C_0^{k,\alpha}(R^2)$ 中的拓广(仍记为 T_0, C_0)，使

$$\|T_0\|_{k,q} \leqslant \delta, \quad \|C_0\|_{k,q} \leqslant \delta, \qquad k \geqslant 1; 1 \leqslant q \leqslant \infty,$$

其中 δ 是任意小可以选择的正常数.

(A_4) $\mathcal{T}_h$ 为 Ω 的正规三角形剖分(可参见文献[3]、文献[6]、文献[7] 等)，或者为 Ω 的拟一致四边形剖分(可参见文献[6])，$H_h \subset H$(为分片 $m-1$ 次多项式的函数空间)，$X_h \subset X \cap C^0(\Omega)$(至少是分片 m 次多项式的向量函数空间)，$M_h \subset M$(为分片 $m-1$ 次多项式的函数空间)，$W_h \subset W \cap C^0(\Omega)$(为分片 m 次多项式的函数空间)，$W_{0h} = W_h \cap H_0^1(\Omega)$ 为混合有限元空间，满足 $\forall v_h \in X_h$ 有 $\varepsilon(v_h) \in H_h$，且 X_h 和 M_h 满足离散的 B.-B. 条件:

$$\sup_{\phi_h \in X_h} \frac{b(q_h, \phi_h)}{\|\nabla \phi_h\|_0} \geqslant \beta \|q_h\|_0, \qquad \forall \phi_h \in M_h,$$

其中，$m \geqslant 1$ 为整数，β 与 h 无关为正的常数． 满足离散的 B.-B. 条件的空间 X_h 和 M_h 可以取 Bernardi-Raugel 元或其他元(构造参照文献[6]，那里给出了很多).

利用 Gronwall 引理和迭代法，类似文献[2]的定理 6.1 的证明可证得下面的结论.

定理 1.1 设(A_1) ~ (A_3) 满足，则问题(Ⅰ)存在唯一的解.

由正则性知，当 $\partial\Omega \in C^{k,\alpha}(k \geqslant 1, \alpha \geqslant 1)$，$T_0, C_0 \in C_0^{k,\alpha}(R^2)$ 时，问题(Ⅰ)的解$(\boldsymbol{u}, \sigma, T, C, p) \in C^{k+2}(\Omega)^2 \times C^{k+1}(\Omega)^{2\times 2} \times C^{k+2}(\Omega) \times C^{k+2}(\Omega) \times C^{k+1}(\Omega)$.

问题(Ⅰ)的半离散化(即空间变量离散化)的混合有限元格式为:

问题(Ⅱ) 求

$$(\boldsymbol{u}_h, p_h, \sigma_h, T_h, C_h) \in [H^1(0,t_1;V) \cap L^2(0,t_1;X_h)] \times$$
$$L^2(0,t_1;M_h) \times H(0,t_1;H_h) \cap H^1(0,t_1;W_h) \times H^1(0,t_1;W_h),$$

使得 $T_h|_{\partial\Omega} = T_0, C_h|_{\partial\Omega} = C_0$，满足

$$\begin{cases} b(\phi_h, \boldsymbol{u}_h) = 0, & \forall \phi \in M_h, \\ (\boldsymbol{u}_{th}, \boldsymbol{v}) + \rho_0 a(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}) - b(p_h, \boldsymbol{v}_h) + (\sigma_h, \varepsilon(\boldsymbol{v})) = (T_h g \boldsymbol{j}, \boldsymbol{v}), & \\ & \forall \boldsymbol{v} \in X_h, \\ (\sigma_h, \tau) + (\mathrm{tr}\,\sigma_h, \mathrm{tr}\,\tau) - 2\mu_T(\varepsilon(\boldsymbol{u}_h), \tau) = 0, & \forall \tau \in H_h, \\ (T_{ht}, \varphi) + \lambda_T(\nabla T_h, \nabla \varphi) + c_2 a_1(\boldsymbol{u}_h, T_h, \varphi) = 0, & \forall \varphi \in W_{0h}, \\ (C_{ht}, \psi) + D_T(\nabla C_h, \nabla \psi) + a_1(\boldsymbol{u}_h, C_h, \psi) = 0, & \forall \psi \in W_{0h}, \\ \boldsymbol{u}_h(0) = \boldsymbol{0}, \quad T_h(0) = C_h(0) = 0, & \\ C|_{\partial\Omega} = C_0, \quad T|_{\partial\Omega} = T_0, \quad \boldsymbol{u}|_{\partial\Omega} = \boldsymbol{0}, \quad \sigma|_{\partial\Omega} = \boldsymbol{0}, & \end{cases}$$

其中 $T_h|_{\partial\Omega} = T_0$ 和 $C_h|_{\partial\Omega} = C_0$ 表示满足(A_3) 的 T_0, C_0 在 Ω 上的插值在 $\partial\Omega$ 上的结点满足此式．令

$$V_h = \left\{\phi_h \in X_h;\ b(q_h, \phi_h) = 0, \ \forall q_h \in M_h\right\}.$$

注意: 在一般情况下，V_h D V.

利用 Gronwall 引理，类似文献[2]的定理 6.2 至定理 6.3 可证得下面的两结论.

定理 1.2 设(A_1) ~ (A_4) 满足，则问题(Ⅱ) 存在唯一的解 $(\boldsymbol{u}_h, p_h, \sigma_h, T_h, C_h) \in [H^1(0,$

$t_1; V) \cap L^2(0, t_1; X_h)] \times L^2(0, t_1; M_h) \times H(0, t_1; H_h) \cap H^1(0, t_1; W_h) \times H^1(0, t_1; W_h)$, 而且 $\| \nabla \boldsymbol{u}_h \|_{L^2(L^2)}$, $\| T_h \|_{L^2(L^2)}$, $\| \nabla T_h \|_{L^2(L^2)}$, $\| C_h \|_{L^2(L^2)}$, $\| \nabla C_h \|_{L^2(L^2)}$, $\| \sigma_h \|_{L^2(L^2)}$ 都有界记为 c.

定理 1.3　设 $(\mathrm{A}_1) \sim (\mathrm{A}_4)$ 满足, 而且问题(Ⅱ)的解 $\boldsymbol{u} \in L^2(0, t_1; H^{m+1}(\Omega)^2) \cap W^{1,\infty}(\Omega)^2$; $p \in L^2(0, t_1; H^m(\Omega))$; $\sigma \in L^2(0, t_1; H^m(\Omega)^{2\times 2})$; $T, C \in L^2(0, t_1; H^m(\Omega)) \cap L^2(0, t_1; W^{1,\infty}(\Omega))$, 则有下面的误差估计

$$\begin{aligned} &\| \sigma - \sigma_h \|_{L^2(L^2)} + \| p - p_h \|_{L^2(L^2)} + \| \nabla (C - C_h) \|_{L^2(L^2)} + \| \nabla (\boldsymbol{u} - \boldsymbol{u}_h) \|_{L^2(L^2)} + \\ &\quad \| \nabla (T - T_h) \|_{L^2(L^2)} \leqslant ch^m (\| T \|_{L^2(H^{m+1})} + \\ &\quad \| C \|_{L^2(H^{m+1})} + \| \boldsymbol{u} \|_{L^2(H^{m+1})} + \| p \|_{L^2(H^m)} + \| \sigma \|_{L^2(H^m)}). \end{aligned}$$

2　全离散化的混合有限元解的存在性及其误差分析

下面对时间用差分离散而对空间用有限元离散. 设 L 为正整数, $k = t_1/L$ 为时间步长,

$$t^{(n)} = nk, (\boldsymbol{u}_h^n, p_h^n, \sigma_h^n, T_h^n, C_h^n) \in X_h \times M_h \times H_h \times W_h \times W_h$$

为对应于

$$(\boldsymbol{u}(t^n), p(t^n), \sigma(t^n), T(t^n), C(t^n)) \equiv (\boldsymbol{u}^n, p^n, \sigma^n, T^n, C^n) \qquad (0 \leqslant n \leqslant L)$$

的混合有限元逼近. 则问题(Ⅱ)的全离散化有限元解可描述为:

问题(Ⅲ)　求 $(\boldsymbol{u}_h^n, p_h^n, \sigma_h^n, T_h^n, C_h^n) \in X_h \times M_h \times H_h \times W_h \times W_h$ 使得 $\boldsymbol{u}_h^0 = \mathbf{0}$, $T_h^0 = C_h^0 = 0$, 满足

$$\begin{cases} b(\phi_h, \boldsymbol{u}_h^n) = 0, \qquad \forall \phi \in M_h, \\ (\boldsymbol{u}_h^n, \boldsymbol{v}_h) + kc_1 a(\boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n, \boldsymbol{v}_h) - kb(p_h^n, \boldsymbol{v}_h) + k(\sigma_h^n, \varepsilon(\boldsymbol{v}_h)) = \\ \qquad k(T_h^n g\boldsymbol{j}, \boldsymbol{v}_h) + (\boldsymbol{u}_h^{n-1}, \boldsymbol{v}_h), \qquad \forall \boldsymbol{v}_h \in X_h, \\ (\sigma_h^n, \tau_h) + (\mathrm{tr}\,\sigma_h^n, \mathrm{tr}\,\tau_h) - 2\mu_{\mathrm{T}}(\varepsilon(\boldsymbol{u}_h^n), \tau_h) = 0, \qquad \forall \tau_h \in H_h, \\ (T_h^n, \varphi_h) + k\lambda_{\mathrm{T}}(\nabla T_h^n, \nabla \varphi_h) + kc_2 a_1(\boldsymbol{u}_h^{n-1}, T_h^n, \varphi_h) = (T_h^{n-1}, \varphi_h), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall \varphi_h \in W_{0h}, \\ (C_h^n, \psi_n) + kD_{\mathrm{T}}(\nabla C_h^n, \nabla \psi_n) + ka_1(\boldsymbol{u}_h^{n-1}, C_h^n, \psi_h) = (C_h^{n-1}, \varphi_h), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall \psi_h \in W_{0h}. \end{cases}$$

讨论问题(Ⅲ)的解的存在性和收敛性还要用到下两引理(参见文献[5]和文献[6]等).

引理 2.1　若 $\{a_n\}$、$\{b_n\}$、$\{c_n\}$ 是正数列, $\{c_n\}$ 是单调递增列而且满足

$$a_n + b_n \leqslant c_n + \lambda \sum_{j=0}^{n-1} a_j, \qquad n \geqslant 1,\ \lambda \geqslant 0;\ a_0 + b_0 \leqslant c_0,$$

则 $a_n + b_n \leqslant c_n \exp(\lambda n)$, $n \geqslant 0$.

引理 2.2　存在 $Q_h: H \to H_h$, 使得 $\forall \sigma \in H$ 满足

$$(\sigma - Q_h\sigma, \tau_h) + (\mathrm{tr}(\sigma - Q_h\sigma), \mathrm{tr}\,\tau_h) = 0, \qquad \forall \tau_h \in H_h,$$

且当 $\sigma \in W^{r,q}(\Omega)^{2\times 2}$ $(1 \leqslant q \leqslant \infty)$ 时, 有

$$\| \sigma - Q_h\sigma \|_{-s,q} \leqslant ch^{r+s} \| \sigma \|_{r,q}, \qquad -1 \leqslant s \leqslant m, 1 \leqslant r \leqslant m.$$

定理 2.1　设 $(\mathrm{A}_1) \sim (\mathrm{A}_4)$ 满足, 而且 $(\boldsymbol{u}_h^{n-1}, T_h^{n-1}, C_h^{n-1}) \in X_h \times W_h \times W_h$ 满足 $T_h^{n-1}|_{\partial\Omega} = T_0^{n-1}$, $C_h^{n-1}|_{\partial\Omega} = C_0^{n-1}$ $(1 \leqslant n \leqslant L)$, 则问题(Ⅲ)存在唯一的 $(\boldsymbol{u}_h^n, p_h^n, \sigma_h^n, T_h^n, C_h^n) \in X_h \times M_h \times H_h \times W_h \times W_h$, $T_h^n|_{\partial\Omega} = T_0^n$, $C_h^n|_{\partial\Omega} = C_0^n$ 而且有

$$\|\sigma_h^n\|_0,\ \|\boldsymbol{u}_h^n\|_0,\ \|\nabla\boldsymbol{u}_h^n\|_0,\ \|T_h^n\|_0,\ \|\nabla T_h^n\|_0,\ \|C_h^n\|_0,\ \|\nabla C_h^n\|_0\leqslant M.$$

证明 对给定的$(\boldsymbol{u}_h^{n-1},T_h^{n-1},C_h^{n-1})\in X_h\times W_h\times W_h$，利用标准的有限元法(参见文献[6]、文献[7])容易证明问题(Ⅲ)的最后两式存在唯一解$T_h^n\in W_h$, $T_h^n|_{\partial\Omega}=T_0^n$和$C_h^n\in W_h$, $C_h^n|_{\partial\Omega}=C_0^n$. 再将已知的$\boldsymbol{u}_h^{n-1}$, T_h^n代入问题(Ⅲ)的第2式，并令

$$A(\Phi_h,\Psi_h)=\frac{1}{k}(\boldsymbol{u}_h^n,\boldsymbol{v}_h)+c_1a(\boldsymbol{u}_h^{n-1},\boldsymbol{u}_h^n,\boldsymbol{v}_h)+(\sigma_h^n,\varepsilon(\boldsymbol{v}_h))+(\sigma_h^n,\tau_h)+$$
$$(\mathrm{tr}\,\sigma_h^n,\mathrm{tr}\,\tau_h)-2\mu_{\mathrm{T}}(\varepsilon(\boldsymbol{u}_h^n),\tau_h),$$
$$\Phi_h=(\boldsymbol{u}_h^n,\sigma_h^n),\ \Psi_h=(\boldsymbol{v}_h,\tau_h);\ b(\phi_h,\Phi_h)=b(\phi_h,\boldsymbol{u}_h^n),$$
$$F(\Psi_h)=(T_h^ng\boldsymbol{j},\boldsymbol{v}_h)+\frac{1}{k}(\boldsymbol{u}_h^{n-1},\boldsymbol{v}_h),\quad Z=\left\{\boldsymbol{u};b(\boldsymbol{u},p)=0,\ \forall p\in M\right\},$$

则问题(Ⅲ)前3个式子可表示为

$$\begin{cases}A(\Phi_h,\Psi_h)-b(p_h,\Psi_h)=F(\Psi_h), & \forall\Psi_h\in X_h\times H_h,\\ b(\phi_h,\Phi_h)=0, & \forall\Phi_h\in M_h.\end{cases}\qquad(\text{Ⅲ})'$$

则

$$A(\Phi_h,\Phi_h)=\frac{1}{k}\|\boldsymbol{u}_h^n\|_0^2+\|\sigma_h^n\|_0^2+\|\mathrm{tr}\,\sigma_h^n\|_0^2\geqslant\alpha\|\Phi_h\|_0^2,\qquad\forall\Phi_h\in X_h\times H_h,$$

其中$\alpha=\min\left\{k^{-1},1\right\}$, $\|\Phi_h\|_0^2=\|\boldsymbol{u}_h^n\|_0^2+\|\sigma_h^n\|_0^2$，所以$A(\cdot,\cdot)$在$Z\times H$中正定，同时已知$b(\cdot,\cdot)$满足离散的B.-B条件，根据混合有限元方法的理论(可参见文献[6])知，问题(Ⅲ)′存在唯一的$(\boldsymbol{u}_h^n,\sigma_h^n,p_h^n)\in X_h\times H_h\times M_h$. 于是，问题(Ⅲ)存在唯一的解. 与文献[10]的定理3.1同理可证明得有界性. 定理2.1证毕.

下面对全离散化的混合有限元解进行误差分析.

令$\bar\partial_t\boldsymbol{u}^n=(\boldsymbol{u}^n-\boldsymbol{u}^{n-1})/k$，则问题(Ⅲ)可写成如下形式:

$$\begin{cases}b(\phi_h,\boldsymbol{u}_h^n)=0, & \forall\phi_h\in M_h,\\ (\bar\partial_t\boldsymbol{u}_h^n,\boldsymbol{v}_h)+c_1a(\boldsymbol{u}_h^{n-1},\boldsymbol{u}_h^n,\boldsymbol{v}_h)-b(p_h^n,\boldsymbol{v}_h)+(\sigma_h^n,\varepsilon(\boldsymbol{v}_h))=(T_h^ng\boldsymbol{j},\boldsymbol{v}_h), & \\ \qquad\qquad\qquad\qquad\forall\boldsymbol{v}_h\in X_h, & \\ (\sigma_h^n,\tau_h)+(\mathrm{tr}\,\sigma_h^n,\mathrm{tr}\,\tau_h)-2\mu_{\mathrm{T}}(\varepsilon(\boldsymbol{u}_h^n),\tau_h)=0, & \forall\tau_h\in H_h,\\ (\bar\partial_tT_h^n,\varphi_h)+\lambda_{\mathrm{T}}(\nabla T_h^n,\nabla\varphi_h)+c_2a_1(\boldsymbol{u}_h^{n-1},T_h^n,\varphi_h)=0, & \forall\varphi_h\in W_{0h},\\ (\bar\partial_tC_h^n,\psi_n)+D_{\mathrm{T}}(\nabla C_h^n,\nabla\psi_n)+a_1(\boldsymbol{u}_h^{n-1},C_h^n,\psi_h)=0, & \forall\psi_h\in W_{0h}.\end{cases}\qquad(\text{Ⅲ})^*$$

在问题(Ⅱ)中取$\boldsymbol{v}=v_h$, $\tau=\tau_h$, $\varphi=\varphi_h$, $\psi=\psi_h$, $\phi=\phi_h$并与问题(Ⅲ)*对应相减可得

$$\begin{cases}b(\phi_h,\boldsymbol{u}^n-\boldsymbol{u}_h^n)=0, & \forall\phi_h\in M_h,\\ (\boldsymbol{u}_t^n-\bar\partial_t\boldsymbol{u}_h^n,\boldsymbol{v}_h)+c_1a(\boldsymbol{u}^n,\boldsymbol{u}^n,\boldsymbol{v}_h)-c_1a(\boldsymbol{u}_h^{n-1},\boldsymbol{u}_h^n,\boldsymbol{v}_h)- & \\ \quad b(p^n-p_h^n,\boldsymbol{v}_h)+(\sigma^n-\sigma_h^n,\varepsilon(\boldsymbol{v}_h))=((T^n-T_h^n)g\boldsymbol{j},\boldsymbol{v}_h), & \forall\boldsymbol{v}_h\in X_h,\\ (\sigma^n-\sigma_h^n,\tau_h)+(\mathrm{tr}(\sigma^n-\sigma_h^n),\mathrm{tr}\,\tau_h)-2\mu_{\mathrm{T}}(\varepsilon(\boldsymbol{u}^n-\boldsymbol{u}_h^n),\tau_h)=0, & \forall\tau_h\in H_h,\\ (T_t^n-\bar\partial_tT_h^n,\varphi_h)+\lambda_{\mathrm{T}}(\nabla(T_n-T_h^n),\nabla\varphi_h)+c_2a_1(\boldsymbol{u}^n,T^n,\varphi_h)- & \\ \quad c_2a_1(\boldsymbol{u}_h^{n-1},T_h^n,\varphi_h)=0, & \forall\varphi_h\in W_{0h},\\ (C_t^n-\bar\partial_tC_h^n,\psi_n)+D_{\mathrm{T}}(\nabla(C^n-C_h^n),\nabla\psi_n)+ & \\ \quad a_1(\boldsymbol{u}^n,C^n,\psi_h)-a_1(\boldsymbol{u}_h^{n-1},C_h^n,\psi_h)=0, & \forall\psi_h\in W_{0h}.\end{cases}$$

(3)

定理 2.2　设(A_1)~(A_4)满足，则当 $\boldsymbol{u}\in W^{2,\infty}(0,t_1;L^2(\Omega)^2)\cap W^{1,\infty}(0,t_1;H^m(\Omega)^2)\cap L^\infty(0,t_1;H^{m+1}(\Omega)^2)$, $p\in L^\infty(0,t_1;H^m(\Omega)^2)$, $\sigma\in L^\infty(0,t_1;H^m(\Omega)^{2\times2})$, $T,C\in W^{2,\infty}(0,t_1;L^2(\Omega))\cap W^{1,\infty}(0,t_1;H^m(\Omega))\cap L^\infty(0,t_1;H^{m+1}(\Omega))$ 时，有下面的误差估计

$$\begin{cases}\|\boldsymbol{u}^n-\boldsymbol{u}_h^n\|_0+k^{1/2}\sum_{i=1}^n\|\nabla(\boldsymbol{u}^i-\boldsymbol{u}_h^i)\|_0\leqslant c(h^m+k),\\ \|T^n-T_h^n\|_0+k^{1/2}\sum_{i=0}^n\|\nabla(T^i-T_h^i)\|_0\leqslant c(h^m+k),\\ \|C^n-C_h^n\|_0+k^{1/2}\sum_{i=0}^n\|\nabla(C^i-C_h^i)\|_0\leqslant C(h^m+k),\\ \max_{1\leqslant i\leqslant n}\|\sigma^i-\sigma_h^i\|_0+\|p^n-p_h^n\|_0\leqslant C(h^m+k),\end{cases}\tag{4}$$

其中 c 是与 h 和 k 无关，与 $(\boldsymbol{u},\sigma,T,C,p)$ 有关的常数.

证明　由Hölder不等式有

$$\begin{aligned}&|a(\boldsymbol{u}^n,\boldsymbol{u}^n,\boldsymbol{v}_h)-a(\boldsymbol{u}_h^{n-1},\boldsymbol{u}_h^n,\boldsymbol{v}_h)|=|a(\boldsymbol{u}^n,\boldsymbol{u}^n,\boldsymbol{v}_h)-a(\boldsymbol{u}^{n-1},\boldsymbol{u}^n,\boldsymbol{v}_h)+\\&\quad a(\boldsymbol{u}^{n-1},\boldsymbol{u}^n,\boldsymbol{v}_h)-a(\boldsymbol{u}_h^{n-1},\boldsymbol{u}^n,\boldsymbol{v}_h)+a(\boldsymbol{u}_h^{n-1},\boldsymbol{u}^n,\boldsymbol{v}_h)-a(\boldsymbol{u}_h^{n-1},\boldsymbol{u}_h^n,\boldsymbol{v}_h)|\leqslant\\&\quad|a(\boldsymbol{u}^n-\boldsymbol{u}^{n-1},\boldsymbol{u}^n,\boldsymbol{v}_h)|+|a(\boldsymbol{u}^{n-1}-\boldsymbol{u}_h^{n-1},\boldsymbol{u}^n,\boldsymbol{v}_h)|+|a(\boldsymbol{u}_h^{n-1},\boldsymbol{u}^n-\boldsymbol{u}_h^n,\boldsymbol{v}_h)|\leqslant\\&\quad(c\|\boldsymbol{u}^{n-1}-\boldsymbol{u}_h^{n-1}\|_0+ck\|\nabla\boldsymbol{u}_t^n\|_0+M\|\nabla(\boldsymbol{u}_n-\boldsymbol{u}_h^n)\|_0)\|\nabla\boldsymbol{v}_h\|_0.\end{aligned}\tag{5}$$

特别地，当 $\boldsymbol{v}_h=R_h\boldsymbol{u}^n-\boldsymbol{u}_h^n=\xi^n$（$R_h$ 是 Ritz 投影，参见文献[2]的引理5.5）时，由Hölder不等式和Cauchy不等式有

$$\begin{aligned}&|a(\boldsymbol{u}^n,\boldsymbol{u}^n,\xi^n)-a(\boldsymbol{u}_h^{n-1},\boldsymbol{u}_h^n,\xi^n)|=|a(\boldsymbol{u}^n,\boldsymbol{u}^n,\xi^n)-a(\boldsymbol{u}^{n-1},\boldsymbol{u}^n,\xi^n)+\\&\quad a(\boldsymbol{u}^{n-1},\boldsymbol{u}^n,\xi^n)-a(\boldsymbol{u}_h^{n-1},\boldsymbol{u}^n,\xi^n)+a(\boldsymbol{u}_h^{n-1},\boldsymbol{u}^n,\xi^n)-a(\boldsymbol{u}_h^{n-1},\boldsymbol{u}_h^n,\xi^n)|\leqslant\\&\quad|a(\boldsymbol{u}^n-\boldsymbol{u}^{n-1},\boldsymbol{u}^n,R_h\boldsymbol{u}^n-\boldsymbol{u}_h^n)|+|a(\boldsymbol{u}_h^{n-1},\boldsymbol{u}^n-\boldsymbol{u}_h^n,R_h\boldsymbol{u}^n-\boldsymbol{u}^n+\boldsymbol{u}^n-\boldsymbol{u}_h^n)|+\\&\quad|a(\boldsymbol{u}^{n-1}-\boldsymbol{u}_h^{n-1},\boldsymbol{u}^n,R_h\boldsymbol{u}^n-\boldsymbol{u}^n+\boldsymbol{u}^n-\boldsymbol{u}_h^n)|\leqslant\\&\quad\theta_0\|\nabla(\boldsymbol{u}_n-\boldsymbol{u}_h^n)\|_0^2+c\|\boldsymbol{u}^{n-1}-\boldsymbol{u}_h^{n-1}\|_0^2+\\&\quad ck^2\|\nabla\boldsymbol{u}_t^n\|_0^2+c\|\nabla(\boldsymbol{u}^n-R_h\boldsymbol{u}^n)\|_0^2,\end{aligned}\tag{6}$$

其中 θ_0 是可以任意选取的小正常数. 由

$$(\sigma^n-\sigma_h^n,\tau_h)+(\mathrm{tr}(\sigma^n-\sigma_h^n),\mathrm{tr}\,\tau_h)-2\mu_{\mathrm{T}}(\varepsilon(\boldsymbol{u}^n-\boldsymbol{u}_h^n),\tau_h)=0$$

及算子 R_h 的性质（参见文献[2]的引理5.5），取 $\tau_h=\varepsilon(\boldsymbol{u}^n-\boldsymbol{u}_h^n)$，并由Korn不等式可得

$$\begin{aligned}&\|\nabla(R_h\boldsymbol{u}^n-\boldsymbol{u}_h^n)\|_0^2\leqslant2c\mu_{\mathrm{T}}\|\varepsilon(R_h\boldsymbol{u}^n-\boldsymbol{u}_h^n)\|_0^2\leqslant\\&\quad c\|\sigma^n-\sigma_h^n\|_0\|\nabla(R_h\boldsymbol{u}^n-\boldsymbol{u}_h^n)\|_0^2,\end{aligned}\tag{7}$$

所以由 Q_h 的性质有

$$\|\nabla(R_h\boldsymbol{u}^n-\boldsymbol{u}_h^n)\|_0\leqslant c\|\sigma^n-\sigma_h^n\|_0\leqslant ch^m\|\sigma^n\|_m+c\|Q_h\sigma^n-\sigma_h^n\|_0.\tag{8}$$

由于 $\rho_h\boldsymbol{u}_t^n,\bar\partial_t\boldsymbol{u}_h^n\in V_h$（$\rho_h$ 是文献[2]中的引理6.2中的 L^2 投影），则由文献[2]中的引理6.2和引理5.3及Hölder不等式有

$$\begin{aligned}\|\boldsymbol{u}_t^n-\bar\partial_t\boldsymbol{u}_h^n\|_{-1}&\leqslant\|\boldsymbol{u}_t^n-\rho_h\boldsymbol{u}_t^n\|_{-1}+\|\rho_h\boldsymbol{u}_t^n-\bar\partial_t\boldsymbol{u}_h^n\|_{-1}\leqslant\\&\|\boldsymbol{u}_t^n-\rho_h\boldsymbol{u}_t^n\|_{-1}+\sup_{\boldsymbol{v}\in V}(\rho_h\boldsymbol{u}_t^n-\bar\partial_t\boldsymbol{u}_h^n,\boldsymbol{v})/\|\nabla\boldsymbol{v}\|_0\leqslant\\&\|\boldsymbol{u}_t^n-\rho_h\boldsymbol{u}_t^n\|_{-1}+\sup_{\boldsymbol{v}\in V}(\rho_h\boldsymbol{u}_t^n-\bar\partial_t\boldsymbol{u}_h^n,\rho_h\boldsymbol{v})/\|\nabla\boldsymbol{v}\|_0\leqslant\\&\|\boldsymbol{u}_t^n-\rho_h\boldsymbol{u}_t^n\|_{-1}+\sup_{\boldsymbol{v}\in V}(\boldsymbol{u}_t^n-\bar\partial_t\boldsymbol{u}_h^n,\rho_h\boldsymbol{v})/\|\nabla\boldsymbol{v}\|_0\leqslant\end{aligned}$$

$$
\begin{aligned}
&\| \boldsymbol{u}_t^n - \rho_h \boldsymbol{u}_t^n \|_{-1} + \sup_{\boldsymbol{v}\in V} [- c_1 a(\boldsymbol{u}^n, \boldsymbol{u}^n, \rho_h \boldsymbol{v}) + c_1 a(\boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n, \rho_h \boldsymbol{v}) + \\
&b(p^n - P_h p^n, \rho_h \boldsymbol{v}) + b(P_h p^n - p_h^n, \rho_h \boldsymbol{v}) - (\sigma^n - \sigma_h^n, \varepsilon(\rho_h \boldsymbol{v})) + \\
&((T^n - T_h^n) g \boldsymbol{j}, \rho_h \boldsymbol{v})] / \| \nabla \boldsymbol{v} \|_0 \leqslant \\
&c(h^m + k) + c(\| \nabla (\boldsymbol{u}^n - \boldsymbol{u}_h^n) \|_0 + \| \boldsymbol{u}^{n-1} - \boldsymbol{u}_h^{n-1} \|_0 + \\
&\| \sigma^n - \sigma_h^n \|_0 + \| T^n - T_h^n \|_0),
\end{aligned}
\tag{9}
$$

其中 P_n 为文献[2] 中的 L^2 投影．由离散的 B.–B. 条件，有

$$
\begin{aligned}
&\| p^n - p_h^n \|_0 \leqslant \| p^n - P_h p^n \|_0 + \| P_h p^n - p_h^n \|_0 \leqslant \\
&\quad \| p^n - P_h p^n \|_0 + \beta^{-1} \sup_{\boldsymbol{v}_h \in V_h} b(P_h p^n - p_h^n, v_h) / \| \nabla v_h \|_0 \leqslant \\
&\quad C \| p^n - P_h p^n \|_0 + \beta^{-1} \sup_{\boldsymbol{v}_h \in V_h} b(p^n - p_h^n, v_h) / \| \nabla v_h \|_0 .
\end{aligned}
\tag{10}
$$

由问题(3) 式和 Hölder 不等式有

$$
\begin{aligned}
&b(p^n - p_h^n, \boldsymbol{v}_h) = (\boldsymbol{u}_t^n - \bar{\partial}_t \boldsymbol{u}_h^n, \boldsymbol{v}_h) + c_1 a(\boldsymbol{u}^n, \boldsymbol{u}^n, \boldsymbol{v}_h) - c_1 a(\boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n, \boldsymbol{v}_h) + \\
&\quad (\sigma^n - \sigma_h^n, \varepsilon(\boldsymbol{v}_h)) - ((T^n - T_h^n) g \boldsymbol{j}, \boldsymbol{v}_h) \leqslant \\
&\quad [c(h^m + k) + c(\| \nabla (\boldsymbol{u}^n - \boldsymbol{u}_h^n) \|_0 + \| \boldsymbol{u}^{n-1} - \boldsymbol{u}_h^{n-1} \|_0 + \\
&\quad \| T^n - T_h^n \|_0 + \| \sigma^n - \sigma_h^n \|_0)] \| \nabla \boldsymbol{v}_h \|_0 .
\end{aligned}
\tag{11}
$$

把(11) 式代入到(10) 式可得

$$
\begin{aligned}
&\| p^n - p_h^n \|_0 \leqslant c(h^m + k + \| \nabla (\boldsymbol{u}^n - \boldsymbol{u}_h^n) \|_0 + \| \boldsymbol{u}^{n-1} - \boldsymbol{u}_h^{n-1} \|_0 + \\
&\quad \| T^n - T_h^n \|_0 + \| \sigma^n - \sigma_h^n \|_0) .
\end{aligned}
\tag{12}
$$

由于 $\xi^n = R_h \boldsymbol{u}^n - \boldsymbol{u}_h^n$，根据问题(III)*，引理 2. 2，文献[2] 中的引理 5. 4 和引理 5. 5，并在问题(II) 中取 $\boldsymbol{v} = \xi^n$ 有

$$
\begin{aligned}
&(\bar{\partial}_t \xi^n, \xi^n) + \frac{1}{2\mu_T} [\| Q_h \sigma^n - \sigma_h^n \|_0^2 + \| \mathrm{tr}(Q_h \sigma^n - \sigma_h^n) \|_0^2] = \\
&\quad (\bar{\partial}_t R_h \boldsymbol{u}^n, \xi^n) - (\bar{\partial}_t \boldsymbol{u}_h^n, \xi^n) + (\varepsilon(\boldsymbol{u}^n - \boldsymbol{u}_h^n), Q_h \sigma^n - \sigma_h^n) = \\
&\quad (\bar{\partial}_t R_h \boldsymbol{u}^n, \xi^n) + c_1 a(\boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n, \xi^n) - b(p_h^n, \xi^n) + \\
&\quad (\varepsilon(\boldsymbol{u}^n - \boldsymbol{u}_h^n), Q_h \sigma^n - \sigma_h^n) + (\sigma_h^n, \varepsilon(\xi^n)) - (T_h^n g \boldsymbol{j}, \xi^n) = \\
&\quad (\bar{\partial}_t R_h \boldsymbol{u}^n - \boldsymbol{u}_t^n, \xi^n) + c_1 a(\boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n, \xi^n) - c_1 a(\boldsymbol{u}^n, \boldsymbol{u}^n, \xi^n) + \\
&\quad (\sigma_h^n - \sigma^n, \varepsilon(\xi^n)) + (\varepsilon(\boldsymbol{u}^n - \boldsymbol{u}_h^n), Q_h \sigma^n - \sigma_h^n) - ((T_h^n - T^n) g \boldsymbol{j}, \xi^n) - \\
&\quad b(p_h^n - p^n, R_h \boldsymbol{u}^n - \boldsymbol{u}^n) - b(P_h p^n - p^n, \boldsymbol{u}^n - \boldsymbol{u}_h^n) .
\end{aligned}
\tag{13}
$$

注意到

$$
\begin{aligned}
&(\sigma_h^n - \sigma^n, \varepsilon(\xi^n)) + (\varepsilon(\boldsymbol{u}^n - \boldsymbol{u}_h^n), Q_h \sigma^n - \sigma_h^n) = \\
&\quad - (\sigma^n - \sigma_h^n, \varepsilon(R_h \boldsymbol{u}^n - \boldsymbol{u}_h^n)) + (\varepsilon(\boldsymbol{u}^n - \boldsymbol{u}_h^n), Q_h \sigma^n - \sigma_h^n) = \\
&\quad - (\sigma^n - Q_h \sigma^n, \varepsilon(R_h \boldsymbol{u}^n - \boldsymbol{u}_h^n)) - (Q_h \sigma^n - \sigma_h^n, \varepsilon(R_h \boldsymbol{u}^n - \boldsymbol{u}_h^n)) + \\
&\quad (\varepsilon(\boldsymbol{u}^n - R_h \boldsymbol{u}^n), Q_h \sigma^n - \sigma_h^n) + (\varepsilon(R_h \boldsymbol{u}^n - \boldsymbol{u}_h^n), Q_h \sigma^n - \sigma_h^n) = \\
&\quad (Q_h \sigma^n - \sigma^n, \varepsilon(R_h \boldsymbol{u}^n - \boldsymbol{u}_h^n)) + (\varepsilon(\boldsymbol{u}^n - R_h \boldsymbol{u}^n), Q_h \sigma^n - \sigma_h^n) .
\end{aligned}
\tag{14}
$$

所以将(14) 式代入(13) 式得

$$
\begin{aligned}
&(\bar{\partial}_t \xi^n, \xi^n) + \frac{1}{2\mu_T} [\| Q_h \sigma^n - \sigma_h^n \|_0^2 + \| \mathrm{tr}(Q_h \sigma^n - \sigma_h^n) \|_0^2] = \\
&\quad (\bar{\partial}_t R_h \boldsymbol{u}^n - \boldsymbol{u}_t^n, \xi^n) + c_1 a(\boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n, \xi^n) - c_1 a(\boldsymbol{u}^n, \boldsymbol{u}^n, \xi^n) -
\end{aligned}
$$

$$((T_h^n - T^n)g\boldsymbol{j}, \xi^n) - b(p_h^n - p^n, R_h\boldsymbol{u}^n - \boldsymbol{u}^n) - b(P_hp^n - p^n, \boldsymbol{u}^n - \boldsymbol{u}_h^n) + (Q_h\sigma^n - \sigma^n, \varepsilon(R_h\boldsymbol{u}^n - \boldsymbol{u}_h^n)) + (\varepsilon(\boldsymbol{u}^n - R_h\boldsymbol{u}^n), Q_h\sigma^n - \sigma_h^n). \tag{15}$$

又由于

$$k^{-1}\int_{\Delta t^{(n)}}(s - t^{n-1})\boldsymbol{u}_{tt}\mathrm{d}s = k^{-1}\int_{t^{n-1}}^{t^n} s\boldsymbol{u}_{tt}(s)\mathrm{d}s - k^{-1}\int_{t^{n-1}}^{t^n} t^{n-1}\boldsymbol{u}_{tt}(s)\mathrm{d}s =$$

$$k^{-1}[s\boldsymbol{u}_t(s) - \boldsymbol{u}(s)]\Big|_{t^{n-1}}^{t^n} - k^{-1}t^{n-1}\boldsymbol{u}_t(s)\Big|_{t^{n-1}}^{t^n} =$$

$$k^{-1}t^n\boldsymbol{u}_t(t^n) - k^{-1}(\boldsymbol{u}(t^n) - \boldsymbol{u}(t^{n-1})) - k^{-1}t^{n-1}\boldsymbol{u}_t(t^n) + k^{-1}t^{n-1}\boldsymbol{u}_t(t^{n-1}) =$$

$$k^{-1}(t^n - t^{n-1})\boldsymbol{u}_t(t^n) - k^{-1}(\boldsymbol{u}(t^n) - \boldsymbol{u}(t^{n-1})) = \boldsymbol{u}_t(t^n) - \bar{\partial}_t\boldsymbol{u}^n,$$

所以

$$\|\bar{\partial}_tR_h\boldsymbol{u}^n - \boldsymbol{u}_t^n\|_0 \leqslant \|\bar{\partial}_tR_h\boldsymbol{u}^n - \partial_t\boldsymbol{u}^n\|_0 + \|\partial_t\boldsymbol{u}^n - \boldsymbol{u}_t^n\|_0 =$$

$$k^{-1}\left\|\int_{\Delta t^{(n)}}(R_h\boldsymbol{u}_t - \boldsymbol{u}_t)\mathrm{d}s\right\|_0 + k^{-1}\left\|\int_{\Delta t^{(n)}}(s - t^{n-1})\boldsymbol{u}_{tt}\mathrm{d}s\right\|_0 \leqslant$$

$$C(h^m\|\boldsymbol{u}_t\|_{L^\infty(H^m)} + k\|\boldsymbol{u}_{tt}\|_{L^\infty(L^2)}) \leqslant C(h^m + k). \tag{16}$$

利用 $(a - b)a = [a^2 - b^2 + (a - b)^2]/2$, 由(6)式、(8)式、(15)式至(16)式,Hölder 不等式和 Cauchy 不等式有

$$\frac{1}{2k}(\|\xi^n\|_0^2 - \|\xi^{n-1}\|_0^2 + \|\xi^n - \xi^{n-1}\|_0^2) + \frac{1}{2\mu_T}\|Q_h\sigma^n - \sigma_h^n\|_0^2 \leqslant$$

$$c(\|\nabla(\boldsymbol{u}^n - R_h\boldsymbol{u}^n)\|_0^2 + \|\bar{\partial}_tR_h\boldsymbol{u}^n - \boldsymbol{u}_t^n\|_0^2 + \|P_hp^n - p^n\|_0^2) + \theta\|p_h^n - p^n\|_0^2 +$$

$$c\|\xi^n\|_0^2 + c\theta\|T_h^n - T^n\|_0^2 + \theta\|Q_h\sigma^n - \sigma_h^n\|_0^2 + c\|Q_h\sigma^n - \sigma^n\|_0^2 +$$

$$|c_1a(\boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n, \xi^n) - c_1a(\boldsymbol{u}^n, \boldsymbol{u}^n, \xi^n)| + c\theta\|\nabla(\boldsymbol{u}^n - \boldsymbol{u}_h^n)\|_0^2 \leqslant$$

$$c(h^{2m} + k^2) + c\|\xi^n\|_0^2 + c\theta\|T_h^n - T^n\|_0^2 +$$

$$c\|\xi^{n-1}\|_0^2 + c\theta\|Q_h\sigma^n - \sigma_h^n\|_0^2. \tag{17}$$

于是,当 $\mu_T\theta \leqslant 1/4$ 时,有

$$\|\xi^n\|_0^2 + \frac{k}{2\mu_T}\|Q_h\sigma^n - \sigma_h^n\|_0^2 \leqslant \|\xi^{n-1}\|_0^2 + ck\|\xi^n\|_0^2 + C(h^{2m} + k^2) +$$

$$c\theta k\|T_h^n - T^n\|_0^2 + ck\|\xi^{n-1}\|_0^2. \tag{18}$$

上式两边从 1 到 n 作和而且注意到 $\xi^0 = \boldsymbol{0}$、$\boldsymbol{u}^0 = \boldsymbol{0}$、$\boldsymbol{u}_h^0 = \boldsymbol{0}$ 可得

$$\|\xi^n\|_0^2 + \frac{k}{4\mu_T}\sum_{i=1}^n\|Q\sigma_h^i - \sigma_h^i\|_0^2 \leqslant ckn(h^{2m} + k^2) + ck\sum_{i=1}^n\|\xi^i\|_0^2 +$$

$$ck\theta\sum_{i=1}^n\|T_h^i - T^i\|_0^2 + ck\sum_{i=1}^{n-1}\|\xi^i\|_0^2, \tag{19}$$

则当 $ck \leqslant 1/2$ 时,由上式可得

$$\|\xi^n\|_0^2 + \frac{k}{2\mu_T}\sum_{i=1}^n\|Q_h\sigma^i - \sigma_h^i\|_0^2 \leqslant$$

$$ckn(h^{2m} + k^2) + ck\sum_{i=1}^n\|\xi^i\|_0^2 + ck\theta\sum_{i=1}^n\|T_h^i - T^i\|_0^2. \tag{20}$$

在引理 2.1 中取

$$a_n = \|\xi^n\|_0^2,\quad b_n = \frac{k}{2\mu_T}\sum_{i=1}^n\|Q_h\sigma^i - \sigma_h^i\|_0^2,$$

$$\lambda = ck,\ c_n = ckn(h^{2m} + k^2) + ck\theta \sum_{i=1}^{n} \| T_h^i - T^i \|_0^2 \quad (n = 1, 2, \cdots).$$

显然有 $a_0 + b_0 = c_0$. 由引理 2.1 得

$$\| \xi^n \|_0^2 + \frac{k}{2\mu_T} \sum_{i=1}^{n} \| Q_h \sigma^i - \sigma_h^i \|_0^2 \leqslant$$

$$[ckn(h^{2m} + k^2) + ck\theta \sum_{i=1}^{n} \| T_h^i - T^i \|_0^2] \exp(ckn). \tag{21}$$

从而由引理 2.2 和文献[2]中的引理 5.5 得到

$$\| \boldsymbol{u}^n - \boldsymbol{u}_h^n \|_0^2 + \frac{k}{2\mu_T} \sum_{i=1}^{n} \| \sigma^i - \sigma_h^i \|_0^2 \leqslant$$

$$\| \boldsymbol{u}^n - R_h \boldsymbol{u}^n \|_0^2 + \| \xi^n \|_0^2 + \frac{k}{\mu_T} \sum_{i=1}^{n} (\| \sigma^i - Q_h \sigma^i \|_0^2 + \| Q_h \sigma^i - \sigma_h^i \|_0^2) \leqslant$$

$$ckn(h^{2m} + k^2) + ck\theta \sum_{i=1}^{n} \| T_h^i - T^i \|_0^2. \tag{22}$$

取 $\eta^n = r_h T^n - T_h^n$, 有

$$| a_1(\boldsymbol{u}^n, T^n, \eta^n) - a_1(\boldsymbol{u}_h^{n-1}, T_h^n, \eta^n) | =$$

$$| a_1(\boldsymbol{u}^n, T^n, \eta^n) - a_1(\boldsymbol{u}^{n-1}, T^n, \eta^n) + a_1(\boldsymbol{u}^{n-1}, T^n, \eta^n) - a_1(\boldsymbol{u}_h^{n-1}, T^n, \eta^n) +$$

$$a_1(\boldsymbol{u}_h^{n-1}, T^n, \eta^n) - a_1(\boldsymbol{u}_h^{n-1}, T_h^n, \eta^n) | \leqslant$$

$$| a_1(\boldsymbol{u}_h^{n-1}, T^n - T_h^n, r_h T^n - T^n) | + | a_1(\boldsymbol{u}^n - \boldsymbol{u}^{n-1}, T^n, r_h T^n - T_h^n) | +$$

$$| a_1(\boldsymbol{u}^{n-1} - \boldsymbol{u}_h^{n-1}, T^n, r_h T^n - T_h^n) | \leqslant$$

$$\theta \| \nabla (T^n - T_h^n) \|_0^2 + C \| \nabla (r_h T^n - T^n) \|_0^2 +$$

$$\theta \| \boldsymbol{u}^{n-1} - \boldsymbol{u}_h^{n-1} \|_0^2 + \theta \| \boldsymbol{u}^n - \boldsymbol{u}^{n-1} \|_0^2 + c \| r_h T^n - T_h^n \|_0^2 \leqslant$$

$$C(h^{2m} + k^2) + \theta \| \nabla (T^n - T_h^n) \|_0^2 + c \| r_h T^n - T_h^n \|_0^2 +$$

$$\theta \| \boldsymbol{u}^{n-1} - \boldsymbol{u}_h^{n-1} \|_0^2. \tag{23}$$

又由于

$$\| \bar{\partial}_t r_h T^n - T_t^n \|_0 \leqslant \| \bar{\partial}_t r_h T^n - \bar{\partial}_t T^n \|_0 + \| \bar{\partial}_t T^n - T_t^n \|_0 =$$

$$\| k^{-1} \int_{\Delta t^{(n)}} (r_h T_t - T_t) \mathrm{d}s \|_0 + \| k^{-1} \int_{\Delta t^{(n)}} (s - t^{n-1}) T_{tt} \mathrm{d}s \|_0 \leqslant$$

$$C(h^m \| T_t \|_{L^\infty(H^m)} + k \| T_{tt} \|_{L^\infty(L^2)}) \leqslant C(h^m + k), \tag{24}$$

所以

$$(\bar{\partial}_t \eta^n, \eta^n) + \lambda_T(\nabla (T^n - T_h^n), \nabla (T^n - T_h^n)) = (\bar{\partial}_t r_h T^n - \bar{\partial}_t T_h^n, \eta^n) +$$

$$(\nabla (T^n - T_h^n), \nabla (T^n - T_h^n)) =$$

$$(\bar{\partial}_t r_h T^n, \eta^n) + \lambda_T(\nabla (T^n - T_h^n), \nabla (T^n - T_h^n)) +$$

$$\lambda_T(\nabla T_h^n, \nabla \eta^n) + c_2 a_1(\boldsymbol{u}_h^{n-1}, T_h^n, \eta^n) =$$

$$(\bar{\partial}_t r_h T^n, \eta^n) + \lambda_T(\nabla (T^n - T_h^n), \nabla (T^n - T_h^n)) +$$

$$\lambda_T(\nabla (T_h^n - T^n), \nabla \eta^n) + c_2 a_1(\boldsymbol{u}_h^{n-1}, T_h^n, \eta^n) - c_2 a_1(\boldsymbol{u}^n, T^n, \eta^n) +$$

$$\lambda_T(\nabla T^n, \nabla \eta^n) + c_2 a_1(\boldsymbol{u}^n, T^n, \eta^n) =$$

$$(\bar{\partial}_t r_h T^n - T_t^n, \eta^n) + \lambda_T(\nabla (T^n - r_h T^n), \nabla (T^n - r_h T^n)) +$$

$$c_2 a_1(\boldsymbol{u}_h^{n-1}, T_h^n, \eta^n) - c_2 a_1(\boldsymbol{u}^n, T^n, \eta^n). \tag{25}$$

再利用 $(a-b)a=[a^2-b^2+(a-b)^2]/2$, 由(24)式、(25)式, Hölder 不等式和 Cauchy 不等式有

$$\frac{1}{2k}(\|\eta^n\|_0^2-\|\eta^{n-1}\|_0^2+\|\eta^n-\eta^{n-1}\|_0^2)+\lambda_\Gamma(\nabla(T^n-T_h^n),\nabla(T^n-T_h^n))\leqslant$$
$$c(h^{2m}+k^2)+\theta\|\nabla(T^n-T_h^n)\|_0^2+c\|r_hT^n-T_h^n\|_0^2+$$
$$\theta\|\boldsymbol{u}^{n-1}-\boldsymbol{u}_h^{n-1}\|_0^2, \tag{26}$$

其中 $\theta\leqslant\lambda_\Gamma/2$. 对上式从 1 到 n 作和得

$$\|\eta^n\|_0^2+k\lambda_\Gamma\sum_{i=0}^{n}\|\nabla(T^i-T_h^i)\|_0^2\leqslant ckn(h^{2m}+k^2)+$$
$$ck\sum_{i=0}^{n}\|\eta^i\|_0^2+ck\theta\sum_{i=0}^{n-1}\|\boldsymbol{u}^i-\boldsymbol{u}_h^i\|_0^2. \tag{27}$$

在引理 2.1 中, 取 $a_n=\|\eta^n\|_0^2$, $\lambda=ck$, $b_n=k\lambda_\Gamma\sum_{i=0}^{n}\|\nabla(T^i-T_h^i)\|_0^2$, $c_n=ckn(h^{2m}+k^2)+ck\theta\sum_{i=0}^{n}\|\boldsymbol{u}^i-\boldsymbol{u}_h^i\|_0^2(n=1,2,\cdots)$. 显然有 $a_0+b_0=c_0$, 从而由引理 2.1 有

$$\|\eta^n\|_0^2+k\lambda_\Gamma\sum_{i=0}^{n}\|\nabla(T^i-T_h^i)\|_0^2\leqslant c(h^{2m}+k^2)+ck\theta\sum_{i=0}^{n-1}\|\boldsymbol{u}^i-\boldsymbol{u}_h^i\|_0^2. \tag{28}$$

由上式和文献[2]中的引理 5.4 可得

$$\|T^n-T_h^n\|_0^2+k\lambda_\Gamma\sum_{i=0}^{n}\|\nabla(T^i-T_h^i)\|_0^2\leqslant\|T^n-r_hT^n\|_0^2+\|\eta^n\|_0^2+$$
$$k\lambda_\Gamma\sum_{i=0}^{n}\|\nabla(T^i-T_h^i)\|_0^2\leqslant c(h^{2m}+k^2)+ck\theta\sum_{i=0}^{n-1}\|\boldsymbol{u}^i-\boldsymbol{u}_h^i\|_0^2. \tag{29}$$

将(22)式代入(29)式得

$$\|T^n-T_h^n\|_0^2+k\lambda_\Gamma\sum_{i=0}^{n}\|\nabla(T^i-T_h^i)\|_0^2\leqslant c(h^{2m}+k^2)+$$
$$ck^2\theta^2\sum_{j=1}^{n}\sum_{i=1}^{j-1}\|T^i-T_h^i\|_0^2\leqslant c(h^{2m}+k^2)+cnk^2\theta^2\sum_{i=0}^{n}\|T^i-T_h^i\|_0^2. \tag{30}$$

在引理 2.1 中, 取 $a_n=\|T^n-T_h^n\|_0^2$, $b_n=k\lambda_\Gamma\sum_{i=0}^{n}\|\nabla(T^i-T_h^i)\|_0^2$, $\lambda=cnk^2\theta^2$, $c_n=c(h^{2m}+k^2)$. 显然有 $a_0=b_0=c_0=0$, 从而又引理 2.1 可得

$$\|T^n-T_h^n\|_0^2+k\lambda_\Gamma\sum_{i=0}^{n}\|\nabla(T^i-T_h^i)\|_0^2\leqslant c(h^{2m}+k^2)\exp(cnk^2\theta^2). \tag{31}$$

由此可得

$$\|T^n-T_h^n\|_0+k^{1/2}\sum_{i=0}^{n}\|\nabla(T^i-T_h^i)\|_0\leqslant c(h^m+k). \tag{32}$$

把(32)式代入(29)式得

$$\|\boldsymbol{u}^n-\boldsymbol{u}_h^n\|_0+k^{1/2}\sum_{i=1}^{n}\|\sigma^i-\sigma_h^i\|_0\leqslant c(h^m+k). \tag{33}$$

与(32)式同理可以得到

$$\|C^n-C_h^n\|_0+k^{1/2}\sum_{i=0}^{n}\|\nabla(C^i-C_h^i)\|_0\leqslant C(h^m+k). \tag{34}$$

由(8)式和(33)式, 并用文献[2]中的引理 2.4 可得

$$\| \boldsymbol{u}^n - \boldsymbol{u}_h^n \|_0 + k^{1/2} \sum_{i=1}^{n} \| \nabla(\boldsymbol{u}^i - \boldsymbol{u}_h^i) \|_0 \leqslant c(h^m + k). \tag{35}$$

由(33)式有

$$\max_{1 \leqslant i \leqslant n} \| \sigma^i - \sigma_h^i \|_0 \leqslant c(h^m + k). \tag{36}$$

于是由(32)式、(33)式、(36)式、(8)式和(12)式可得

$$\| p^n - p_h^n \|_0 \leqslant C(h^m + k), \tag{37}$$

即得定理 2.2 的结论. 定理 2.2 证毕.

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Research of Discrete Formulation of Mixed Finite Element Methods for the Vapor Deposition Chemical Reaction Equations

LUO Zhen-dong[1,2], ZHOU Yan-jie[2], ZHU Jiang[3]

(1. School of Science, Beijing Jiaotong University, Beijing 100044, P. R. China;

2. Department of mathematics and Physics, Beijing Technology and Business University, Beijing 100037, P. R. China;

3. Institute of Atmospheric Physics, Chinese Academy of Sciences, Beijing 100029, P. R. China)

Abstract: The vapor deposition chemical reaction processes, which are of extremely extensive applications, can be classified as a mathematical model by the following governing nonlinear partial differential equations containing velocity vector, temperature field, pressure field, and gas mass field. The mixed finite element (MFE) method is employed to study the system of equations for the vapor deposition chemical reaction processes. The semidiscrete and fully discrete MFE formulations are derived. And the existence and convergence (error estimate) of the semidiscrete and fully discrete MFE solutions are demonstrated. By employing MFE method to treat the system of equations for the vapor deposition chemical reaction processes, the numerical solutions of the velocity vector, the temperature field, the pressure field, and the gas mass field can be found out simultaneously. Thus, these researches are not only of important theoretical meaning, but also of extremely extensive applied vistas.

Key words: vapor deposition chemical reaction equation; the mixed finite element method; semidiscrete formulation; fully discrete formulation