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压电材料Ⅰ型裂纹动态问题的对偶方程组及其求解*

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(我刊编委王彪来稿)

摘要: 首先引入势函数, 用势函数表示压电材料的基本微分方程, 并采用 Laplace 变换、半无限对称 Fourier 正弦变换和 Fourier 余弦变换, 对微分方程进行变换和初步求解; 然后通过 Fourier 反演和引入边界条件, 建立了二维压电材料动态裂纹问题的对偶方程组; 再根据 Bessel 函数性质, 利用 Abel 型积分方程及其反演, 将对偶方程组化为第二类 Fredholm 积分方程组. 结果表明, 方法是可行的, 可以成为研究此类问题的一种有效方法.

关 键 词: 压电材料; 动态裂纹; 势函数; 对偶积分方程组; 积分变换

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引 言

压电材料的静态损伤和断裂行为研究已经取得了丰硕的成果, Wang[1-2]、Zhou[3] 和 Pak[4] 等人的研究可推为这方面的代表. 随着研究的深入, 压电介质动态断裂分析又成为一个新近兴起的研究领域. Khutoryansky 和 Sosa[5] 率先给出了压电材料电弹性瞬态问题的控制方程和基本解; 侯[6] 和 Chen[7] 等人研究了压电介质中反平面裂纹的瞬态响应问题. 但对于压电材料中的三维或平面动态断裂力学问题, 由于材料的各向异性, 无法进行控制方程的直接解耦分析. 为此, 本文通过引入位移和电势的势函数, 研究了压电材料二维动态问题的基本方程, 对偶积分方程组及其求解方法.

1 压电材料的动态方程及势函数表示

取 xOy 平面为各向异性面, 忽略体积力和体电荷影响, 只考虑惯性力作用. 压电方程为

$$\sigma_{ij} = c_{ijkl}\gamma_{kl} - e_{kij}E_k, \tag{1}$$

$$D_i = e_{ikl}\gamma_{kl} + \varepsilon_{ik}E_k, \tag{2}$$

式中 c_{ij}、e_{ij}、ε_{ij} 分别是材料的弹性、压电和介电常数. 而 σ_{ij}、γ_{ij} 分别是应力和应变; D_i、E_i 分别是电位移和电场强度. 几何方程(这里将电场场强与电势的关系方程也称为几何方程)为

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$$\gamma_{ij}=\frac{1}{2}(u_{j,i}+u_{i,j}),\quad E_i=-\varphi_{,i},\tag{3}$$

$u_i=u_i(x,y,t)$ 是位移函数; $\varphi=\varphi(x,y,t)$ 是电势函数, 它们均为 x、y 和时间 t 的函数. 运动方程为

$$\sigma_{ji,i}=\rho\ddot{u}_j,\quad D_{j,j}=0.\tag{4}$$

将式(1)、(2)和(3)代入到式(4), 整理得

$$\begin{cases}c_{11}\dfrac{\partial^2u_x}{\partial x^2}+c_{44}\dfrac{\partial^2u_x}{\partial y^2}+(c_{13}+c_{44})\dfrac{\partial^2u_y}{\partial x\partial y}+(e_{15}+e_{31})\dfrac{\partial^2\varphi}{\partial x\partial y}=\rho\ddot{u}_x,\\ c_{33}\dfrac{\partial^2u_y}{\partial y^2}+c_{44}\dfrac{\partial^2u_y}{\partial x^2}+(c_{13}+c_{44})\dfrac{\partial^2u_x}{\partial x\partial y}+e_{15}\dfrac{\partial^2\varphi}{\partial x^2}+e_{33}\dfrac{\partial^2\varphi}{\partial y^2}=\rho\ddot{u}_y,\\ e_{33}\dfrac{\partial^2u_y}{\partial y^2}+e_{15}\dfrac{\partial^2u_y}{\partial x^2}+(e_{15}+e_{31})\dfrac{\partial^2u_x}{\partial x\partial y}-\varepsilon_{11}\dfrac{\partial^2\varphi}{\partial x^2}-\varepsilon_{33}\dfrac{\partial^2\varphi}{\partial y^2}=0.\end{cases}\tag{5}$$

上式即为以弹性位移 $u_j=u_j(x,y,t)$ 和电势函数 $\varphi=\varphi(x,y,t)$ 为基本未知量表示的横观各向同性压电材料二维问题的耦合场方程.

引进势函数 $\Phi_j(x,y,t)$ 及 $X_j(x,y,t)$, 即设

$$u_x=\frac{\partial\Phi_1}{\partial x}+\frac{\partial\Phi_2}{\partial y},\quad u_y=\frac{\partial\Phi_1}{\partial y}-\frac{\partial\Phi_2}{\partial x},\quad \varphi=\frac{\partial X_1}{\partial y}-\frac{\partial X_2}{\partial x}.\tag{6}$$

将式(6)代入到式(5)并整理得到关于 $\Phi_j(x,y,t)$ 和 $X_j(x,y,t)$ 的方程,

$$A_j\frac{\partial^2\Phi_j}{\partial x^2}+B_j\frac{\partial^2\Phi_j}{\partial y^2}=C_j\ddot{\Phi}_j,\qquad j=1,2,\tag{7a}$$

$$\varepsilon_{11}\frac{\partial^2X_j}{\partial x^2}+\varepsilon_{33}\frac{\partial^2X_j}{\partial y^2}=U_j\frac{\partial^2\Phi_j}{\partial x^2}+V_j\frac{\partial^2\Phi_j}{\partial y^2},\qquad j=1,2,\tag{7b}$$

式中

$$\begin{aligned}&A_1=c_{11}(e_{33}\varepsilon_{11}-e_{15}\varepsilon_{33})-(e_{15}+e_{31})[(2c_{44}+c_{13})\varepsilon_{11}+e_{15}(2e_{15}+e_{31})],\\&A_2=(e_{33}\varepsilon_{11}-e_{15}\varepsilon_{33})(c_{11}-c_{13}-c_{44})-(e_{15}+e_{31})(c_{44}\varepsilon_{33}+e_{15}e_{33}),\\&B_1=(2c_{44}+c_{13})(e_{33}\varepsilon_{11}-e_{15}\varepsilon_{33})-(e_{15}+e_{31})(c_{33}\varepsilon_{11}+e_{33}e_{15}),\\&B_2=c_{44}(e_{33}\varepsilon_{11}-e_{15}\varepsilon_{33})-(e_{15}+e_{31})[(c_{33}-c_{13}-c_{44})\varepsilon_{33}+\\&\qquad(e_{33}-e_{15}-e_{31})e_{33}],\\&C_1=[(e_{33}-e_{15}-e_{31})\varepsilon_{11}-e_{15}\varepsilon_{33}]\rho,\quad C_2=[e_{33}\varepsilon_{11}-(2e_{15}+e_{31})\varepsilon_{33}]\rho,\\&U_1=2e_{15}+e_{31},\quad V_1=e_{33},\quad U_2=e_{15},\quad V_2=e_{33}-e_{15}-e_{31}.\end{aligned}$$

对一般的横观各向同性压电材料二维问题, 方程(7a)中的系数 A_1、B_1、C_1 有相同的符号; 同样, 系数 A_2、B_2、C_2 也具有相同的符号. 因此, 方程(7a)具有波动方程的形式.

将式(6)代入到式(3)和(1)、(2)有对应的应变、电场强度分量和应力、电位移分量

$$\gamma_{xx}=\frac{\partial^2\Phi_1}{\partial x^2}+\frac{\partial^2\Phi_2}{\partial x\partial y},\quad \gamma_{yy}=\frac{\partial^2\Phi_1}{\partial y^2}-\frac{\partial^2\Phi_2}{\partial x\partial y},\quad \gamma_{xy}=2\frac{\partial^2\Phi_1}{\partial x\partial y}+\frac{\partial^2\Phi_2}{\partial y^2}-\frac{\partial^2\Phi_2}{\partial x^2};\tag{8}$$

$$E_x=-\frac{\partial^2X_1}{\partial x\partial y}+\frac{\partial^2X_2}{\partial x^2},\quad E_y=-\frac{\partial^2X_1}{\partial y^2}+\frac{\partial^2X_2}{\partial x\partial y};\tag{9}$$

$$\begin{cases}\sigma_{xx}=c_{11}\left(\dfrac{\partial^2\Phi_1}{\partial x^2}+\dfrac{\partial^2\Phi_2}{\partial x\partial y}\right)+c_{13}\left(\dfrac{\partial^2\Phi_1}{\partial y^2}-\dfrac{\partial^2\Phi_2}{\partial x\partial y}\right)+e_{31}\left(\dfrac{\partial^2X_1}{\partial y^2}-\dfrac{\partial^2X_2}{\partial x\partial y}\right),\\ \sigma_{yy}=c_{13}\left(\dfrac{\partial^2\Phi_1}{\partial x^2}+\dfrac{\partial^2\Phi_2}{\partial x\partial y}\right)+c_{33}\left(\dfrac{\partial^2\Phi_1}{\partial y^2}-\dfrac{\partial^2\Phi_2}{\partial x\partial y}\right)+e_{33}\left(\dfrac{\partial^2X_1}{\partial y^2}-\dfrac{\partial^2X_2}{\partial x\partial y}\right),\\ \sigma_{xy}=c_{44}\left(2\dfrac{\partial^2\Phi_1}{\partial x\partial y}+\dfrac{\partial^2\Phi_2}{\partial y^2}-\dfrac{\partial^2\Phi_2}{\partial x^2}\right)+e_{15}\left(\dfrac{\partial^2X_1}{\partial x\partial y}-\dfrac{\partial^2X_2}{\partial x^2}\right);\end{cases}\tag{10}$$

$$\begin{cases} D_x = e_{15}\left(2\dfrac{\partial^2 \Phi_1}{\partial x \partial y} + \dfrac{\partial^2 \Phi_2}{\partial y^2} - \dfrac{\partial^2 \Phi_2}{\partial x^2}\right) - \varepsilon_{11}\left(\dfrac{\partial^2 X_1}{\partial x \partial y} - \dfrac{\partial^2 X_2}{\partial x^2}\right), \\ D_y = e_{31}\left(\dfrac{\partial^2 \Phi_1}{\partial x^2} + \dfrac{\partial^2 \Phi_2}{\partial x \partial y}\right) + e_{33}\left(\dfrac{\partial^2 \Phi_1}{\partial y^2} - \dfrac{\partial^2 \Phi_2}{\partial x \partial y}\right) - \varepsilon_{33}\left(\dfrac{\partial^2 X_1}{\partial y^2} - \dfrac{\partial^2 X_2}{\partial x \partial y}\right). \end{cases} \tag{11}$$

综上所述, 横观各向同性压电材料二维动态问题的求解, 归结为确定满足微分方程式(7)的 $\Phi_j(x, y, t)$、$X_j(x, y, t)$, 再由式(6)、(8) ~ (11) 既可求得耦合场各分量． 当然, 对实际问题尚需考虑初始条件与边界条件． 因而, 式(7) 与式(6) 结合起来就构成了横观各向同性压电材料二维动态问题的所谓势函数通解． 由满足式(7) 的 $\Phi_j(x, y, t)$、$X_j(x, y, t)$, 通过式(6) 而求得的弹性位移 $u_j = u_j(x, y, t)$ 和电势函数 $\varphi = \varphi(x, y, t)$ 必使微分方程组(5) 得到满足.

2 Ⅰ 型裂纹问题的对偶积分方程组

作为上述方法的应用, 现在研究含Ⅰ 型裂纹的压电材料问题. 设裂纹位于 x 轴上, 长 $L = 2a$, 中心在坐标原点. 假设介质在无限远处不受载荷作用, 而在裂纹上下表面同时受冲击载荷作用. 由于对称, 只需研究 1/4 平面即可. 其边界条件可写为

$$\sigma_{yy}(x, 0, t) = -\sigma_0 \mathrm{H}(t),\quad \sigma_{xy}(x, 0, t) = 0, \qquad 0 < x < a,\ t > 0, \tag{12a}$$

$$D_y(x, 0, t) = -D_0 \mathrm{H}(t), \qquad 0 < x < a,\ t > 0; \tag{12b}$$

$$u_y(x, 0, t) = 0,\quad \sigma_{xy}(x, 0, t) = 0, \qquad x > a,\ t > 0, \tag{13a}$$

$$\varphi(x, 0, t) = 0, \qquad x > a,\ t > 0; \tag{13b}$$

$$\sigma_{ij}(x, y, t) = 0, \qquad \text{在无限远处: } t > 0, \tag{14a}$$

$$D_i(x, y, t) = 0, \qquad \text{在无限远处: } t > 0; \tag{14b}$$

式中, $\mathrm{H}(t)$ 为 Heaviside 单位阶跃函数. 并假定压电介质的初始条件为

$$u_j(x, y, 0) = \ddot{u}_j(x, y, 0) = 0, \tag{15a}$$

$$\varphi(x, y, 0) = \ddot{\varphi}(x, y, 0) = 0. \tag{15b}$$

引入势函数 $\Phi_j(x, y, t)$、$X_j(x, y, t)$ 的 Laplace 变换及其反演, 对微分方程组(7) 作 Laplace 变换, 并考虑初始条件(15) 得

$$A_j \frac{\partial^2 \Phi_j^*}{\partial x^2} + B_j \frac{\partial^2 \Phi_j^*}{\partial y^2} = C_j p^2 \Phi_j^*, \qquad j = 1, 2, \tag{16a}$$

$$\varepsilon_{11} \frac{\partial^2 X_j^*}{\partial x^2} + \varepsilon_{33} \frac{\partial^2 X_j^*}{\partial y^2} = U_j \frac{\partial^2 \Phi_j^*}{\partial x^2} + V_j \frac{\partial^2 \Phi_j^*}{\partial y^2}, \qquad j = 1, 2. \tag{16b}$$

考虑到问题的对称性以及电势函数 φ 为标量函数, 则势函数具有如下性质: $\Phi_1(x, y, t) = \Phi_1(-x, y, t)$, $\Phi_2(x, y, t) = -\Phi_2(-x, y, t)$, $X_1(x, y, t) = X_1(-x, y, t)$, $X_2(x, y, t) = -X_2(-x, y, t)$. 引入Fourier 正弦变换、余弦变换及其反演, 即对方程组(16) 中的 $j = 1$ 式进行 Fourier 余弦变换, $j = 2$ 式作 Fourier 正弦变换得

$$B_j \frac{\partial^2 \Phi_j^*}{\partial y^2} - (A_j s^2 + C_j p^2)\Phi_j^* = 0, \qquad j = 1, 2, \tag{17a}$$

$$\varepsilon_{33} \frac{\partial^2 X_j^*}{\partial y^2} - \varepsilon_{11} s^2 X_j^* = \frac{(A_j V_j - B_j U_j) s^2 + C_j V_j p^2}{B_j} \Phi_j^*, \qquad j = 1, 2. \tag{17b}$$

解微分方程组(17) , 并考虑其解应满足无限远处边界条件, 得 $\Phi_j^*(s, y, p)$、$X_j^*(s, y, p)$,

$$\begin{cases} \Phi_j^*(s, y, p) = F_j(s, p) \mathrm{e}^{-\omega_j y}, \\ X_j^*(s, y, p) = G(s, p) \mathrm{e}^{-\lambda y} + H_j(s, p) F_j(s, p) \mathrm{e}^{-\omega_j y}, \end{cases} \qquad j = 1, 2, \tag{18}$$

式中 $\omega_j=\sqrt{(A_js^2+C_jp^2)/B_j}$, $\lambda=\sqrt{\varepsilon_{11}s^2/\varepsilon_{33}}$, $H_j(s,p)=(V_j\omega_j^2-U_js^2)/(\varepsilon_{33}\omega_j^2-\varepsilon_{11}s^2)$, 未知函数 $F_j(s,p)$、$G(s,p)$ 由其余的边界条件确定.

对式(18)中的象函数 $\Phi_1^*(s,y,p)$、$X_1^*(s,y,p)$ 作 Fourier 余弦反演, 对象函数 $\Phi_2^*(s,y,p)$、$X_2^*(s,y,p)$ 作 Fourier 正弦反演分别得

$$\begin{cases}\Phi_1^*(x,y,p)=\dfrac{2}{\pi}\displaystyle\int_0^\infty F_1(s,p)\mathrm{e}^{-\omega_1 y}\cos(sx)\mathrm{d}s,\\ \Phi_2^*(x,y,p)=\dfrac{2}{\pi}\displaystyle\int_0^\infty F_2(s,p)\mathrm{e}^{-\omega_2 y}\sin(sx)\mathrm{d}s;\end{cases}\tag{19}$$

$$\begin{cases}X_1^*(x,y,p)=\dfrac{2}{\pi}\displaystyle\int_0^\infty[G(s,p)\mathrm{e}^{-\lambda y}+H_1(s,p)F_1(s,p)\mathrm{e}^{-\omega_1 y}]\cos(sx)\mathrm{d}s,\\ X_2^*(x,y,p)=\dfrac{2}{\pi}\displaystyle\int_0^\infty[G(s,p)\mathrm{e}^{-\lambda y}+H_2(s,p)F_2(s,p)\mathrm{e}^{-\omega_2 y}]\sin(sx)\mathrm{d}s.\end{cases}\tag{20}$$

对边界条件(12)、(13)作 Laplace 变换得

$$\sigma_{yy}^*(x,0,p)=-\sigma_0/p,\quad \sigma_{xy}^*(x,0,p)=0,\qquad 0<x<a,\tag{21a}$$

$$D_y^*(x,0,p)=-D_0/p,\qquad 0<x<a;\tag{21b}$$

$$u_y^*(x,0,p)=0,\quad \sigma_{xy}^*(x,0,p)=0,\qquad x>a,\tag{22a}$$

$$\varphi^*(x,0,p)=0,\qquad x>a.\tag{22b}$$

对式(6)、(8)~(11)作 Laplace 变换并将式(19)、(20)代入, 考虑对称性, 由边界条件 $\sigma_{xy}^*(x,0,p)=0,\ -\infty<x<+\infty$ 整理得

$$G(s,p)=-\frac{\omega_1H_1(s,p)F_1(s,p)+sH_2(s,p)F_2(s,p)}{\lambda+s}-\frac{c_{44}[2s\omega_1F_1(s,p)+(\omega_2^2+s^2)F_2(s,p)]}{e_{15}(s\lambda+s^2)}.\tag{23}$$

由边界条件 $u_y^*(x,0,p)=0$、$\varphi^*(x,0,p)=0$, $x>a$ 得

$$\int_0^\infty[\omega_1F_1(s,p)+sF_2(s,p)]\cos(sx)\mathrm{d}s=0,\qquad x>a,\tag{24a}$$

$$\int_0^\infty[(\lambda+s)G(s,p)+\omega_1H_1(s,p)F_1(s,p)+sH_2(s,p)F_2(s,p)]\cos(sx)\mathrm{d}s=0,\qquad x>a.\tag{24b}$$

将式(23)代入式(24b), 并考虑式(24a), 整理得

$$\int_0^\infty[\omega_1F_1(s,p)+\omega_2^2s^{-1}F_2(s,p)]\cos(sx)\mathrm{d}s=0,\qquad x>a.\tag{25}$$

由边界条件 $\sigma_{yy}^*(x,0,p)=-\sigma_0/p$、$D_y^*(x,0,p)=-D_0/p$, $0<x<a$, 并考虑式(23), 整理得

$$\begin{cases}\displaystyle\int_0^\infty[T_{11}(s,p)F_1(s,p)+T_{12}(s,p)F_2(s,p)]\cos(sx)\mathrm{d}s=\frac{\pi\sigma_0}{2p},\\ \displaystyle\int_0^\infty[T_{21}(s,p)F_1(s,p)+T_{22}(s,p)F_2(s,p)]\cos(sx)\mathrm{d}s=\frac{\pi D_0}{2p},\end{cases}\quad 0<x<a,\tag{26}$$

式中

$$T_{11}(s,p)=(c_{13}s^2-c_{33}\omega_1^2)-e_{33}(\omega_1^2-\omega_1\lambda)H_1(s,p)+2\sqrt{\varepsilon_{11}\varepsilon_{33}^{-1}}c_{44}e_{33}e_{15}^{-1}s\omega_1,$$

$$T_{12}(s,p)=(c_{13}-c_{33})s\omega_2-e_{33}s(\omega_2-\lambda)H_2(s,p)+\sqrt{\varepsilon_{11}\varepsilon_{33}^{-1}}c_{44}e_{33}e_{15}^{-1}(\omega_2^2+s^2),$$

$$T_{21}(s,p) = (e_{31}s^2 - e_{33}\omega_1^2) + e_{33}(\omega_1^2 - \omega_1\lambda)H_1(s,p) - 2\sqrt{\varepsilon_{11}\varepsilon_{33}}c_{44}e_{15}^{-1}s\omega_1,$$
$$T_{22}(s,p) = (e_{31} - e_{33})s\omega_2 + \varepsilon_{33}s(\omega_2 - \lambda)H_2(s,p) - \sqrt{\varepsilon_{11}\varepsilon_{33}}c_{44}e_{15}^{-1}(\omega_2^2 + s^2).$$

至此,问题的求解归结为确定满足对偶积分方程组(24a)、(25)和(26)的 $F_j(s,p)$.

3 对偶积分方程组的求解

设

$$\begin{cases} g_1(s,p) = \omega_1F_1(s,p) + sF_2(s,p), \\ g_2(s,p) = \omega_1F_1(s,p) + \omega_2^2s^{-1}F_2(s,p), \end{cases} \tag{27}$$

有

$$F_1(s,p) = \frac{s^2g_2(s,p) - \omega_2^2g_1(s,p)}{\omega_1(s^2 - \omega_2^2)},\quad F_2(s,p) = \frac{sg_1(s,p) - sg_2(s,p)}{(s^2 - \omega_2^2)}. \tag{28}$$

将 $\cos(sx) = \sqrt{\pi xs/2}\mathrm{J}_{-1/2}(sx)$ 和式(27)、(28)代入到(24a)、(25)和(26)得如下对偶积分方程组

$$\begin{cases} \int_0^\infty \sqrt{s}g_1(s,p)\mathrm{J}_{-1/2}(sx)\mathrm{d}s = 0, \\ \int_0^\infty \sqrt{s}g_2(s,p)\mathrm{J}_{-1/2}(sx)\mathrm{d}s = 0, \end{cases}\quad x > a; \tag{29}$$

$$\begin{cases} \int_0^\infty s^2[m_{11}g_1(s,p) + m_{12}g_2(s,p)]\mathrm{J}_{-1/2}(sx)\mathrm{d}s = T_{10}(x), \\ \int_0^\infty s^2[m_{21}g_1(s,p) + m_{22}g_2(s,p)]\mathrm{J}_{-1/2}(sx)\mathrm{d}s = T_{20}(x), \end{cases}\quad 0 < x < a; \tag{30}$$

式中 $\mathrm{J}_\nu(sx)$ 是 ν 阶 Bessel 函数; 而

$$T_{10}(x) = \frac{\sqrt{\pi}\sigma_0}{\sqrt{2xp}},\quad T_{20}(x) = \frac{\sqrt{\pi}D_0}{\sqrt{2xp}},\quad m_{11} = m_{11}(s,p) = \frac{T_{12}(s,p)\omega_1s - T_{11}(s,p)\omega_2^2}{\omega_1s(s^2 - \omega_2^2)\sqrt{s}},$$
$$m_{12}(s,p) = \frac{T_{11}(s,p)s - T_{12}(s,p)\omega_1}{\omega_1(s^2 - \omega_2^2)\sqrt{s}},\quad m_{21}(s,p) = \frac{T_{22}(s,p)\omega_1s - T_{21}(s,p)\omega_2^2}{\omega_1s(s^2 - \omega_2^2)\sqrt{s}},$$
$$m_{22}(s,p) = \frac{T_{21}(s,p)s - T_{22}(s,p)\omega_1}{\omega_1(s^2 - \omega_2^2)\sqrt{s}}.$$

令

$$\begin{cases} t_1(x) = T_{10}(x) + \int_0^\infty s^2\left[\left(\left(\chi\left(\frac{C_1}{A_1}\right)^{1/4}\sqrt{p}\right)^{-1} - m_{11}\right)g_1(s,p) - m_{12}g_2(s,p)\right]\mathrm{J}_{-1/2}(sx)\mathrm{d}s, \\ t_2(x) = T_{20}(x) + \int_0^\infty s^2\left[\left(\left(\chi\left(\frac{C_2}{A_2}\right)^{1/4}\sqrt{p}\right)^{-1} - m_{22}\right)g_2(s,p) - m_{21}g_1(s,p)\right]\mathrm{J}_{-1/2}(sx)\mathrm{d}s, \end{cases} \tag{31}$$

χ 为一常系数(大于零即可),将式(31)代入到式(30)得

$$\begin{cases} \int_0^\infty s^2g_1(s,p)\mathrm{J}_{-1/2}(sx)\mathrm{d}s = t_1(x)\chi\left(\frac{C_1}{A_1}\right)^{1/4}\sqrt{p}, \\ \int_0^\infty s^2g_2(s,p)\mathrm{J}_{-1/2}(sx)\mathrm{d}s = t_2(x)\chi\left(\frac{C_2}{A_2}\right)^{1/4}\sqrt{p}, \end{cases}\quad 0 < x < a. \tag{32}$$

将未知函数 $g_1(s,p)$、$g_2(s,p)$ 表示成新的未知函数 $\phi_1(\xi,p)$、$\phi_2(\xi,p)$ 的积分

$$g_j(s,p) = \frac{\pi a^2}{2p}s^{-1/4}\int_0^1 \sqrt{\xi}\,\phi_j(\xi,p)\,J_{1/4}(sa\xi)\,d\xi, \tag{33}$$

式中函数 $\phi_1(\xi,p)$、$\phi_2(\xi,p)$ 应满足

$$\lim_{\xi\to 0^+}\xi^{-1}\phi_j(\xi,p) = 0. \tag{34}$$

对 Bessel 函数, 有如下间断积分公式[8]

$$\int_0^\infty J_\lambda(r\xi)J_\mu(b\xi)\,\xi^{1+\mu-\lambda}d\xi = 0, \qquad 0 < r < b, \tag{35a}$$

$$\int_0^\infty J_\lambda(r\xi)J_\mu(b\xi)\,\xi^{1+\mu-\lambda}d\xi = \frac{b^\mu(r^2-b^2)^{\lambda-\mu-1}}{2^{\lambda-\mu-1}r^\lambda\Gamma(\lambda-\mu)}, \qquad 0 < b < r, \tag{35b}$$

式中, $\Gamma(z)$ 为 Γ 函数, 并要求 $\lambda > \mu > -1$. 将式(33)代入到式(29)并且交换积分次序, 得

$$\int_0^\infty \sqrt{s}\,g_j(s,p)J_{-1/2}(sx)\,ds = \frac{\pi a^2}{2p}\int_0^1\sqrt{\xi}\,\phi_j(\xi,p)\int_0^\infty s^{1/4}J_{1/4}(sa\xi)J_{-1/2}(sx)\,ds\,d\xi.$$

考虑式(35)的间断公式, 知上式为零. 即式(29)自动满足. 根据 Bessel 函数的微分公式[9]

$$\frac{d}{dz}[z^{-\nu}J_\nu(z)] = -z^{-\nu}J_{\nu+1}(z).$$

将式(33)的右端进行分部积分, 考虑式(34)得

$$g_j(s,p) = -\frac{\pi a}{2s^{5/4}p}\Big\{\phi_j(1,p)J_{-3/4}(as) - \int_0^1(as\xi)^{3/4}J_{-3/4}(as\xi)\frac{d}{d\xi}\Big[\frac{\sqrt{\xi}\,\phi_j(\xi,p)}{(as\xi)^{3/4}}\Big]d\xi\Big\}.$$

将上式代入到式(32), 并考虑式(35)的间断公式, 得到如下 Abel 型的积分方程得

$$t_j(x) = \frac{(aA_j)^{1/4}\pi\sqrt{x}}{(2C_j)^{1/4}\chi p^{3/2}\Gamma(1/4)}\int_0^{x/a}\frac{1}{(x^2-a^2\xi^2)^{3/4}}\frac{d}{d\xi}[\xi^{-1/4}\phi_j(\xi,p)]\,d\xi, \qquad 0 < \xi < x/a. \tag{36}$$

利用 Abel 型积分方程及其反演公式[10], 令

$$h(x) = \frac{(2C_j)^{1/4}\chi p^{3/2}\Gamma(1/4)}{(aA_j)^{1/4}\pi\sqrt{x}}t_j(x) = \int_0^{x/a}\frac{1}{(x^2-a^2\xi^2)^{3/4}}\frac{d}{d\xi}[\xi^{-1/4}\phi_j(\xi,p)]\,d\xi,$$

那么,

$$r(\xi) = \frac{-2\sin(-3\pi/4)}{\pi}\frac{d}{d\xi}\int_0^{a\xi}(a^2\xi^2-x^2)^{-1/4}x\,h(x)\,dx = \frac{d}{d\xi}[\xi^{-1/4}\phi_j(\xi,p)],$$

于是得

$$\phi_j(\xi,p) = \chi\left(\frac{\xi C_j}{aA_j}\right)^{1/4}\frac{2^{3/4}p^{3/2}\Gamma(1/2)}{\pi^2}\int_0^{a\xi}\frac{\sqrt{x}\,t_j(x)}{[(a\xi)^2-x^2]^{1/4}}dx. \tag{37}$$

将式(31)代入到式(37), 并令

$$\beta_1 = \chi\left(\frac{\xi C_j}{aA_j}\right)^{1/4}\frac{2^{3/4}p^{3/2}\Gamma(1/2)}{\pi^2},$$

得

$$\phi_1(\xi,p) = \beta_1\int_0^{a\xi}\Big\{\Big[\sqrt{x}\Big(T_{10}(x) + \int_0^\infty s^2\Big[\Big(\Big(\chi\Big(\frac{C_1}{A_1}\Big)^{1/4}\sqrt{p}\Big)^{-1} - m_{11}\Big)g_1 -$$

$$m_{12}g_2\Big] J_{-1/2}(sx)\,ds\Big)\Big]\Big\backslash\left[(a\xi)^2-x^2\right]^{1/4}\Big\}dx. \tag{38a}$$

对(38a)式右侧进行积分、分部积分等运算, 利用恒等式

$$\int_0^t x^{\nu+1}J_\nu(sx)\frac{dx}{(t^2-x^2)^{1-\alpha}}=2^{\alpha-1}\Gamma(\alpha)s^{-\alpha}t^{\alpha+\nu}J_{\alpha+\nu}(st),\qquad \Gamma(1/2)=\sqrt{\pi}, \tag{38b}$$

于是得如下的第二类 Fredholm 积分方程组

$$\begin{cases}\dfrac{1}{\chi\sqrt{p}}\left(\dfrac{A_1}{C_1}\right)^{1/4}\phi_1(\xi,p)+\displaystyle\int_0^1[K_{11}\phi_1(\eta,p)+K_{12}\phi_2(\eta,p)]d\eta=0.454a^{1/4}\xi^{3/4}\sigma_0,\\ \dfrac{1}{\chi\sqrt{p}}\left(\dfrac{A_2}{C_2}\right)^{1/4}\phi_2(\xi,p)+\displaystyle\int_0^1[K_{21}\phi_1(\eta,p)+K_{22}\phi_2(\eta,p)]d\eta=0.454a^{1/4}\xi^{3/4}D_0,\end{cases} \tag{39}$$

式中, 积分方程的核 K_{ij} 分别为

$$K_{11}(\xi,\eta,p)=0.489a^2\sqrt{\xi\eta}\int_0^\infty s[m_{11}(s,p)-(\chi(C_1/A_1)^{1/4}\sqrt{p})^{-1}]J_{1/4}(sa\xi)J_{1/4}(sa\eta)ds,$$

$$K_{12}(\xi,\eta,p)=0.489a^2\sqrt{\xi\eta}\int_0^\infty sm_{12}(s,p)J_{1/4}(sa\xi)J_{1/4}(sa\eta)ds,$$

$$K_{21}(\xi,\eta,p)=0.489a^2\sqrt{\xi\eta}\int_0^\infty sm_{21}(s,p)J_{1/4}(sa\xi)J_{1/4}(sa\eta)ds,$$

$$K_{22}(\xi,\eta,p)=0.489a^2\sqrt{\xi\eta}\int_0^\infty s[m_{22}(s,p)-(\chi(C_2/A_2)^{1/4}\sqrt{p})^{-1}]J_{1/4}(sa\xi)J_{1/4}(sa\eta)ds.$$

将方程组(39)化成代数方程组, 用 MatLab 等语言工具进行编程, 即可得到函数 ϕ_1 和 ϕ_2, 详细求解过程请参见文献[11].

4 结　语

文献[12]指出了针对普通材料的动态裂纹积分变换法中存在的数学问题. 在本文的研究中, 文献[12]提出的问题得到了成功的解决, 而且也为这类问题的解决提供了一个方法.

本文针对压电材料的动态问题, 通过引入 3 个势函数, 并用其表示出了压电材料动态问题的基本方程, 通过 Laplace 变换和 Fourier 变换对微分方程进行求解, 数学理论和方法成熟、缜密; 针对含Ⅰ型裂纹的压电材料受冲击载荷的这一具体问题, 得到了求解该问题的对偶积分方程组, 并详细地研究了将对偶积分方程组转化为第二类 Fredholm 积分方程组的方法. 本文方法是可行的, 可以成为研究压电材料动态裂纹问题的一种方法.

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Dual Equations and Solutions of Ⅰ-Type Crack of Dynamic Problems in Piezoelectric Materials

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Abstract: Firstly, basic differential equations of piezoelectric materials expressed in terms of the potential functions, which are introduced in the very beginning, were worked out. Then these equations were primarily solved through Laplace transformation, semi-infinite Fourier sine transformation and cosine transformation. After that, the dual equations of dynamic cracks problem in the 2D piezoelectric materials were founded with the help of Fourier reverse transformation and the introduction of boundary conditions. Finally, according to the character of the Bessel function and by making ful use of Abel integral equation and its reverse transform, the dual equations were changed into the second type of Fredholm integral equations. The investigation indicates that the study approach taken is feasible and has potential to be an effective method to do research on issues of this kind.

Key words: piezoelectric material; dynamic crack; potential function; coupled integral equations; integral transformation